The Thursday Colloquium
"The Algebra & Geometry of Modern Physics"

Lecture Notes

CHERN–SIMONS THEORY, QUANTUM KNOT INVARIANTS, AND VOLUME CONJECTURES

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§1. Chern–Simons theory & polynomial inv.

Chern–Simons theory Witten

M 3-mfd

G semi-simple compact gauge group

A $\mathfrak{g}$-valued gauge connection

The action of Chern-Simons theory

$$S_{CS}(M, A) = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A\,dA + \frac{2}{3} A^3 \right)$$

- S_{CS} is metric indep.
- $k \in \mathbb{Z}$ level (inverse of coupling const.)
- eqn of motion: $F_A = 0$ flat connection
- natural metric indep. observable

$$W_R(K) = \mathrm{Tr}_R\, P \exp \oint_K A$$

K a knot

R rep of $\mathfrak{g}$

- Expectation value of $W_R(K)$

$$\langle W_R(K) \rangle = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \; e^{i S_{CS}(M.A)}\, W_R(K)$$

$\mathcal{A}$: space of connections

$\mathcal{G}$: gauge transformations

$\Rightarrow$ knot invariant.

Suppose M has boundary $\partial M = \Sigma$

$$g^*A = g^{-1}dg + g^{-1}Ag$$

$$S_{cs}(g^*A) - S_{cs}(A)$$
$$= \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}(A \wedge g^{-1}dg) - \frac{k}{12\pi}\int_M \mathrm{Tr}(g^{-1}dg)^3$$

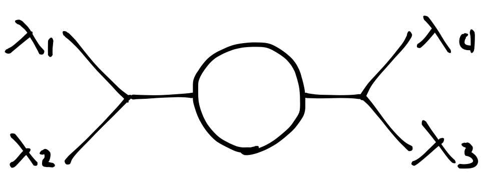
- action of $\hat{\mathfrak{g}}_k$ WZNW model

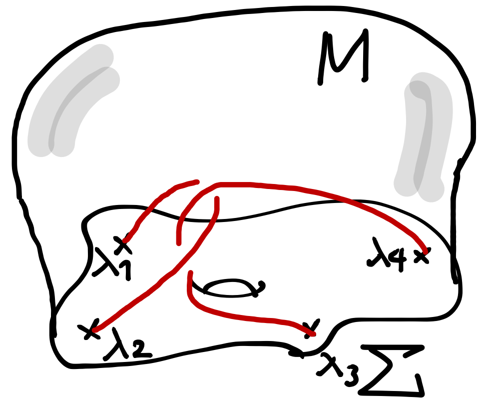
$$S_\Sigma(g) = \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg) - \frac{k}{12\pi}\int_M \mathrm{Tr}(g^{-1}dg)^3$$

The action is indep of the extension M of Σ

- Chern-Simons partition fn. is an element of Hilbert space living on the bdry Σ

$$\mathcal{H}_\Sigma \ni Z^k_{CS}(M) = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\, e^{iS_{cs}(M,A)}$$
$$\shortparallel$$
$$\{\text{space of conformal blocks}\}.$$



$= Z^k_{CS}\Big($ 

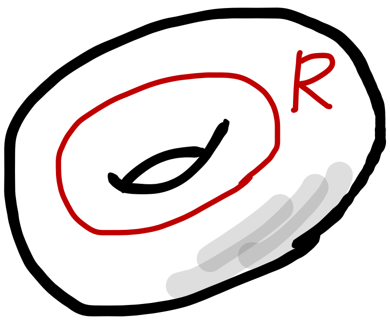
$\Big)$

In particular, $\Sigma = T^2$, $M = S^1 \times D$ (solid Torus)

with Wilson loop colored by R.

$$Z_{CS}(S^1\times D; R) = \chi_R(\tau) = \mathrm{Tr}_R\, q^{L_0 - \frac{c}{24}}$$

character of integrable rep of $\hat{g}_k$



e.g. $g = sl(2)$ $R = \boxed{\cdots}$ less than k

$$\chi_{R_1}(-\tfrac{1}{\tau}) = \sum_{R_2} S_{R_1R_2}\,\chi_{R_2}(\tau)$$

$$\chi_{R_2}(\tau+1) = \sum_{R_2} T_{R_1R_2}\,\chi_{R_2}(\tau) = \exp\left[2\pi i\left(\Delta_R - \tfrac{c}{24}\right)\right]\chi_R(\tau)$$

diagonalizable

$$sl(2):\quad S_{m.n} = \sqrt{\frac{2}{k+2}}\,\sin\frac{(m+1)(n+1)\pi}{k+2}$$

$$\Delta_n = \frac{n(n+2)}{4(k+2)}$$

$$sl(N):\quad S_{\lambda_1\lambda_2} = s_{\lambda_1}(q^{\rho})\, s_{\lambda_2}(q^{\lambda_1+\rho})$$

$s(x)$: Schur fn

Aganagic - Shakirov : Refined CS thy.

$$S_{\lambda_1\lambda_2} = M_{\lambda_1}(q^{\rho})\, M_{\lambda_2}(q^{\rho}t^{\lambda_1})$$

Macdonald poly

Partition fn & Expectation value.

1. $S^3 = (S^1 \times D^2) \cup (S^1 \times D^2)$

$$Z_{cs}(S^3) = \langle 0 | S | 0 \rangle = S_{0,0} \left(= \sin \frac{\pi}{k+2} \right)$$

2. unknot

$$\langle W_R(\bigcirc) \rangle = \frac{Z_{cs}(S^3 ; \bigcirc)}{Z_{cs}(S^3)} = \frac{\langle 0 | S | R \rangle}{\langle 0 | S | 0 \rangle} = \frac{S_{0R}}{S_{00}}$$

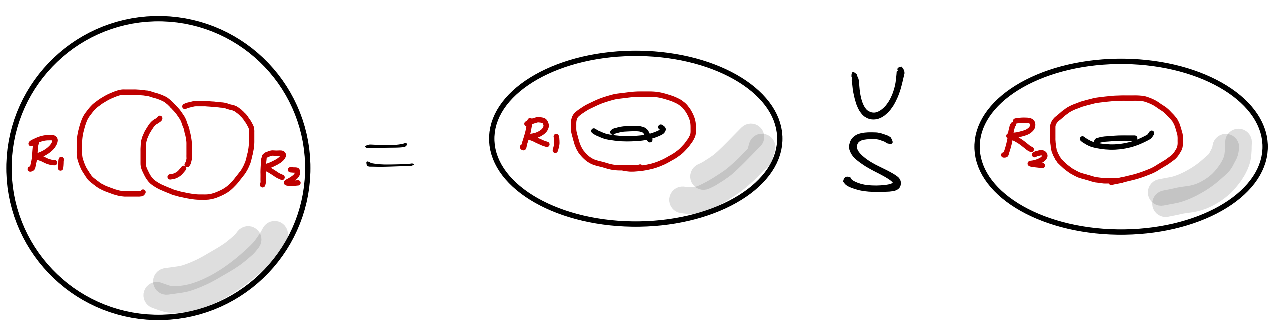
$$= \dim_q R.$$

For $sl(2)$ with $R = \underbrace{\boxed{|\cdots|}}_{n}$

$$\langle W_n(\bigcirc) \rangle = \frac{\sin \frac{(n+1)\pi}{k+2}}{\sin \frac{\pi}{k+2}} = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} = [n+1]$$

where $q = \exp\left(\frac{\pi i}{k+2}\right)$

3. Hopf link



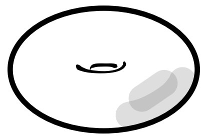
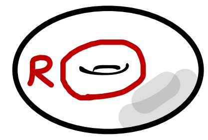
$$\langle W_{R_1 R_2}(\text{Hopf}) \rangle = \frac{\langle R_1 | S | R_2 \rangle}{\langle 0 | S | 0 \rangle} = \frac{S_{R_1 R_2}}{S_{00}}$$

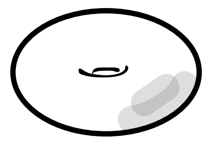
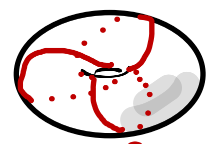
For $sl(2)$. $R_1 = \text{spin}\ \frac{m}{2}$, $R_2 = \text{spin}\ \frac{n}{2}$

$$\langle W_{mn}(\text{Hopf}) \rangle = \frac{q^{(m+1)(n+1)} - q^{-(m+1)(n+1)}}{q - q^{-1}} = [(m+1)(n+1)]$$

4. Torus knots

torus knot operators

$$W_R^{(1,0)} \Big|$$  $$\Big\rangle = \Big|$$  $$\Big\rangle$$

$$W_R^{(2,3)} \Big|$$  $$\Big\rangle = \Big|$$  $$\Big\rangle$$

R

For general (Q,P)-torus knots.

$$W_R^{(Q,0)} = \sum_V C_R^V(Q)\, W_V^{(1,0)}$$ adams operation

$$W^{(Q,P)} = T^{P/Q}\, W_R^{(Q,0)}\, T^{-P/Q}$$

$$T^{P/Q} = \begin{pmatrix} 1 & P/Q \\ 0 & 1 \end{pmatrix}$$

Wilson loop along (Q,P)-torus knot

$$\langle W_R(K(Q,P))\rangle = \frac{\langle 0|\, S\, W_R^{(Q,P)}\, |0\rangle}{\langle 0|\, S\, |0\rangle}$$

$$= (S_{00})^{-1} \langle 0|\, S\, T^{P/Q} \sum_V C_R^V(Q)\, W_V^{(1,0)}\, T^{-P/Q} |0\rangle$$

$$= \sum_V C_R^V(Q)\, q^{\frac{P}{Q}\kappa_V^{q}}\, \dim_q{}_V R$$

Rosso-Jones formula.

5. colored quantum invariants for Non-torus knots are very difficult.

$$\langle W_R(K) \rangle = \overline{J}^{\mathfrak{g}}_R(K; q) \qquad q = \exp \frac{\pi i}{k + h^\vee}$$

$h^\vee$ dual coxeter #

$$\overline{J}^{sl(N)}_R(K;q) \xrightarrow{a = q^N} \overline{P}_R(K;a,q) \text{ colored HOMFLY}$$

$$\overline{J}^{so(N)}_R(K;q) \xrightarrow{\lambda = q^{N-1}} \overline{F}_R(K;\lambda,q) \text{ colored Kauffman}$$

$$\overline{J}^{sp(N)}_R(K;q) \xrightarrow{\lambda = q^{N+1}} \overline{F}_R(K;\lambda,q)$$

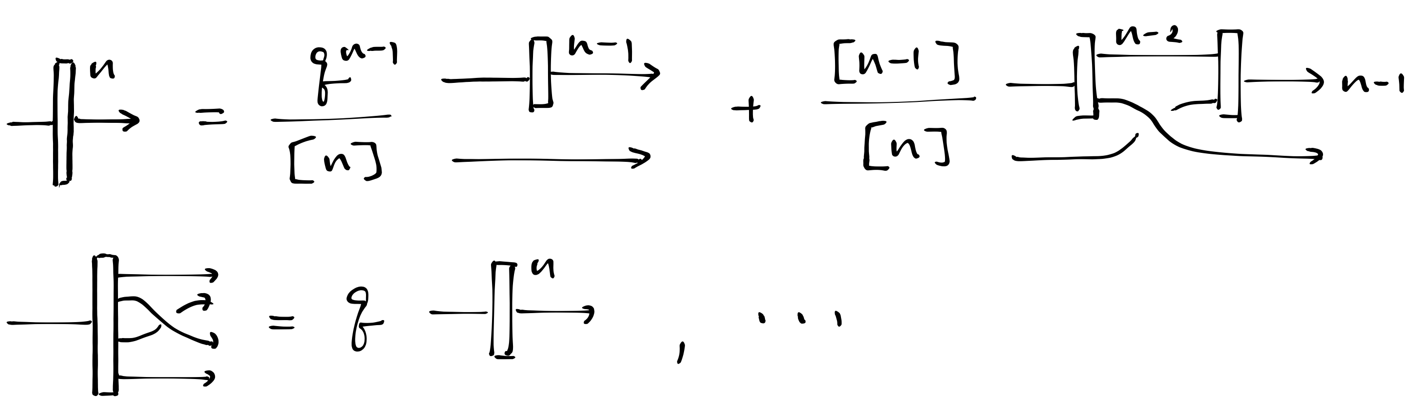
Computational methods

① Reshetikhin–Turaev [ITEP. Itoyama]

$$f : B_n \longrightarrow \mathrm{End}_{U_q(\mathfrak{g})}(V_1 \otimes \cdots \otimes V_m)$$

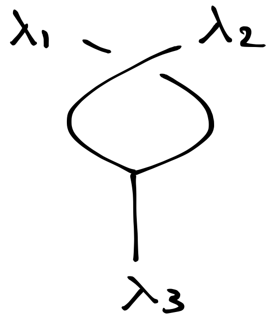
$$\mathrm{Tr}(f) \sim \overline{J}^{\mathfrak{g}}_R(K)$$

② linear Skein relations (Morton, Kawagoe '12)

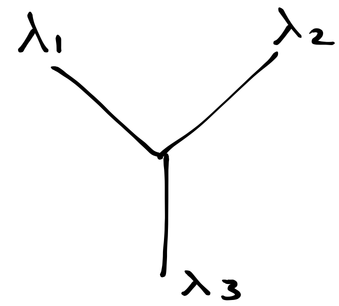


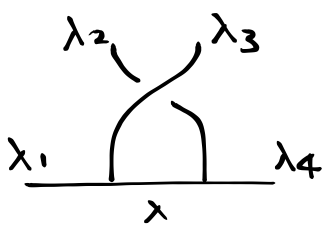
③ Drinfeld - Kohno

Monodromy rep. of Braid group in the space of the KZ eqn is given by Universal R-matrices.

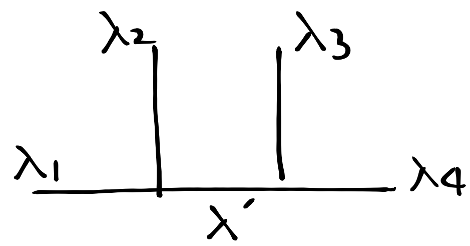


$$= \epsilon\, q^{\kappa_{\lambda_1} - \kappa_{\lambda_2} - \kappa_{\lambda_3}}$$

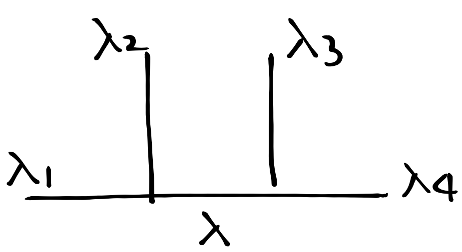




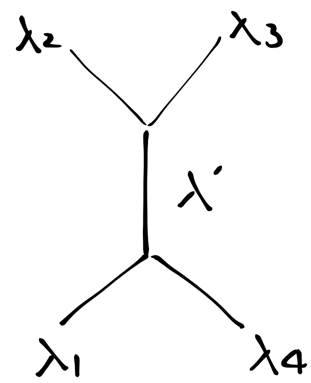
$$= \sum_{\lambda'} B_{\lambda\lambda'}\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$$



Braiding matrix



$$= \sum_{\lambda'} F_{\lambda\lambda'}\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$$



fusion matrix

fusion matrices are often called quantum 6j-symbols

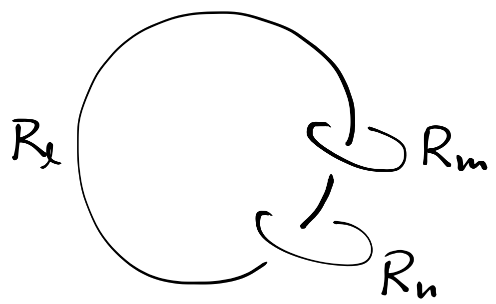
$$F_{\lambda\lambda'}\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \propto \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda \\ \lambda_3 & \lambda_4 & \lambda' \end{Bmatrix}$$

$U_q(sl_2)$: Kirillov - Reshetikhin '88

$U_q(sl_N)$: the simplest class of multiplicity-free q-6j

Nawata - Ramadevi - Zodinmawia

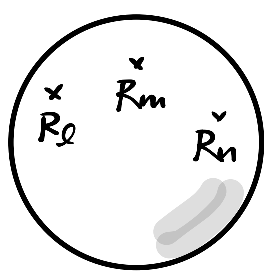
Verlinde formula for SU(2).



$= L$

$$Z_{CS}(S^3; L) = \sum_k S_\ell{}^k \, Z(S^1 \times S^2; R_\ell R_m R_n)$$

$$= \sum_k S_\ell{}^k N_{kmn}$$



$$N_{kmn} := \dim \mathcal{H}_{S^2; R_\ell R_m R_n}.$$

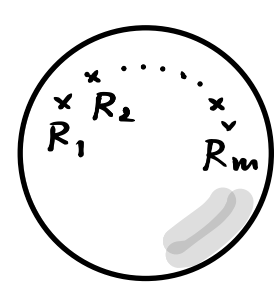
$$Z(S^3; R_\ell) \, Z(S^3; L) = Z(S^3; {}^{R_\ell}\!\bigcirc\!\!\bigcirc^{R_m}) \, Z(S^3; {}_{R_\ell}\!\bigcirc\!\!\bigcirc_{R_n})$$

$$S_{0\ell} \sum_k S_\ell{}^k N_{kmn} = S_{\ell m} S_{\ell n}$$

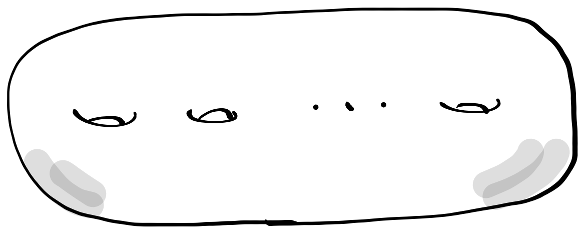
$$\Rightarrow \quad N_{kmn} = \sum_\ell \frac{S_{k\ell} S_{m\ell} S_{n\ell}}{S_{0\ell}}$$

more generally, S^2 with m poncture associated to $(R_1 \cdots R_m)$

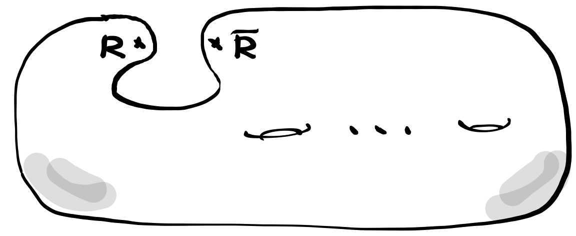
$$\dim \mathcal{H}_{S^2, R_1 \cdots R_m} = \sum_\ell \frac{S_{R_1, \ell} \cdots S_{R_m, \ell}}{(S_{0.\ell})^{m-2}}$$



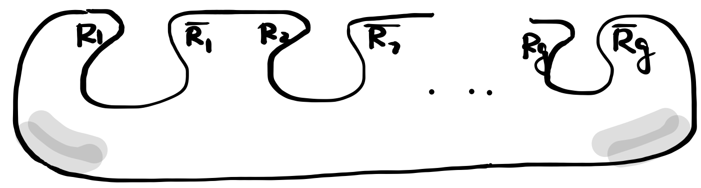
For higher genus Σ_g, one can pinch off a cycle by assigning $(R, \bar{R})$



$= \sum_R$



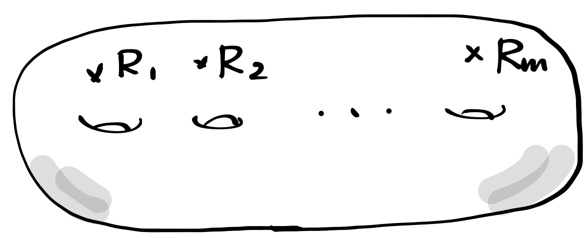
$= \sum_{R_1 \cdots R_g}$



$$\dim \mathcal{H}_{\Sigma_g} = \sum_{R_1 \cdots R_g} \sum_{\ell} \frac{S_{R_1 \ell} S_{\bar{R}_1 \ell} \cdots S_{R_g \ell} S_{\bar{R}_g \ell}}{(S_{0\ell})^{2g-2}}$$

$$= \sum_{\ell} \frac{1}{(S_{0\ell})^{2g-2}}$$

general case : Σ_g with $(R_1 \cdots R_m)$



$$\dim \mathcal{H}_{\Sigma_g, R_1 \cdots R_m} = \sum_{\ell} \frac{S_{R_1 \ell} \cdots S_{R_m \ell}}{(S_{0\ell})^{m+2g-2}}$$

Note that this is the case of $SU(2)$. For higher ranks, it's little complicated.

§2 Categorifications : knot homology

$K \rightsquigarrow \mathcal{H}^{sl(2)}_{ij}(K)$ bi-graded homology (Khovanov)

this is itself knot invariant.

Jones polynomial is q-graded Euler characteristics.

$$J(K;q) = \sum_{i,j} (-1)^j q^i \dim \mathcal{H}^{sl(2)}_{ij}(K)$$

Since homology itself is knot invariant, the Poincare poly is certainly invariant.

$$\mathcal{P}^{sl(2)}_{\square}(K;q,t) = \sum_{i,j} q^i t^j \dim \mathcal{H}^{sl(2)}_{ij}(K)$$

$sl(2)$ homological inv. (Khovanov inv.)

- Status of Categorifications

1. colored $sl(2)$ homology $\left(\mathcal{H}^{sl(2)}_{R}(K)\right)_{ij}$ — Cooper - Krushkal, Frenkel - Stroppel - Sussan, Webster.

$$J^{sl(2)}_{R}(K;q) = \sum_{ij} (-1)^j q^i \dim \left(\mathcal{H}^{sl(2)}_{R}(K)\right)_{ij}$$

2. $sl(N)$ homology $\left(\mathcal{H}^{sl(N)}_{\square}(K)\right)_{ij}$ — Khovanov - Rozansky

$$J^{sl(N)}(K;q) = \sum_{ij} (-1)^j q^i \dim \left(\mathcal{H}^{sl(N)}(K)\right)_{ij}$$

Rasmussen.

3. HOMFLY homology

Khovanov - Rozansky

$(\mathcal{H}^{\mathrm{HOM}}(K))_{ijk}$ triply-graded homology

Poincaré poly (superpolynomial)

$$\mathcal{P}(K;a,q,t) = \sum a^i q^j t^k \dim (\mathcal{H}^{\mathrm{HOM}}(K))_{ijk}$$

$t=-1$

d_N differential

HOMFLY poly $P(K;a,q)$

$\mathfrak{sl}(N)$ homological inv. $\mathcal{P}^{\mathfrak{sl}(N)}(K;q,t)$

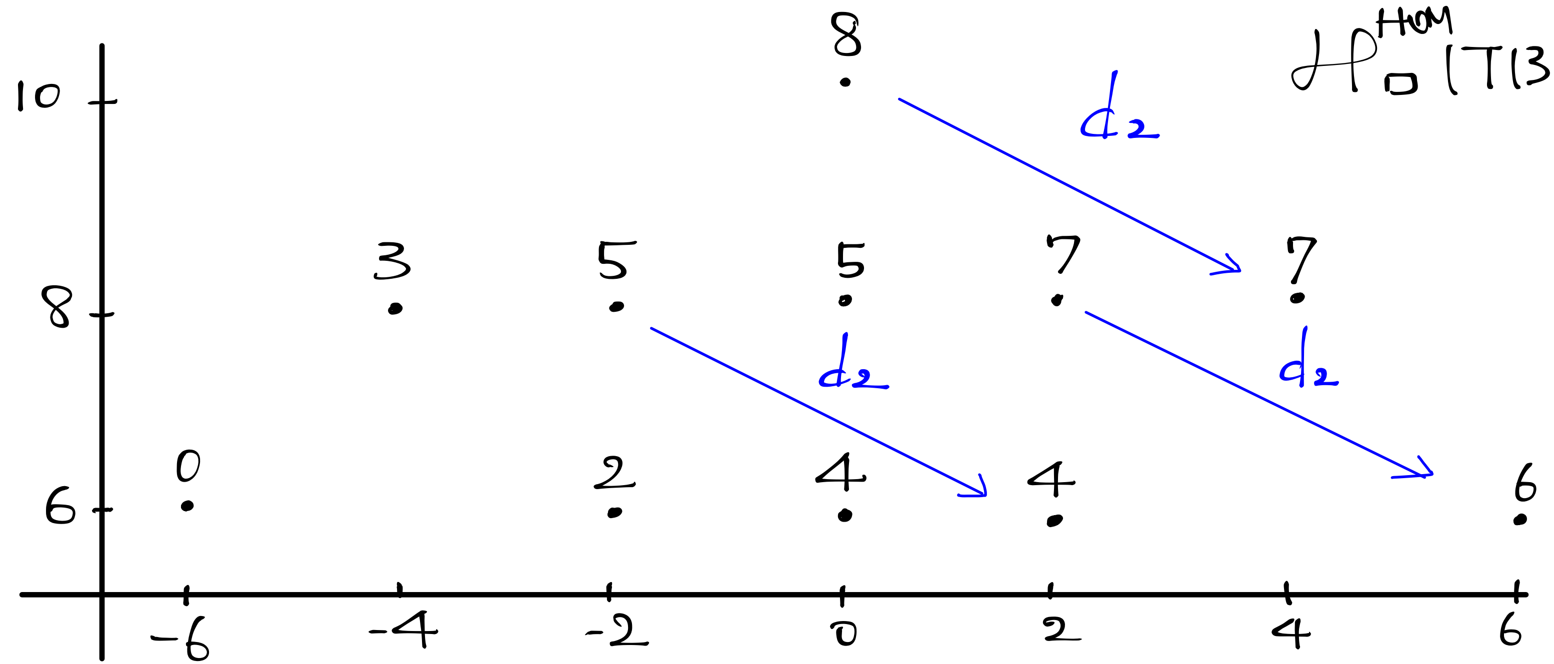
$a=q^N$

$t=-1$

$\mathfrak{sl}(N)$ quantum inv. $J^{\mathfrak{sl}(N)}(K;q)$

$$\mathcal{P}^{\mathfrak{sl}(N)}(K;q,t) = \mathcal{P}(\mathcal{H}^{\mathrm{HOM}}(K), d_N)(a=q^N, q, t)$$

$\mathcal{H}^{\mathrm{HOM}}_{\square}(T(3,4))$



Properties of colored HOMFLY homology.

Gukov - Stosic.

1) $\mathcal{H}^{HOM}_{[r]}(K)$ is finite-dim'l

2) $sl(N)$ differential d_N

$$H^*(\mathcal{H}^{HOM}_{[r]}(K), d_N) \cong \mathcal{H}^{sl(N)}_{[r]}$$

$(a.q.t)$-degree of d_N

$$\deg d_N = (-2, 2N, *)$$

3) Mirror symmetry

$$\left(\mathcal{H}^{HOM}_{[1^r]}(K)\right)_{i,j,k} \cong \left(\mathcal{H}^{HOM}_{[r]}(K)\right)_{i,-j,*}$$

$$* = k - (rS(K) + 2(k-i))$$

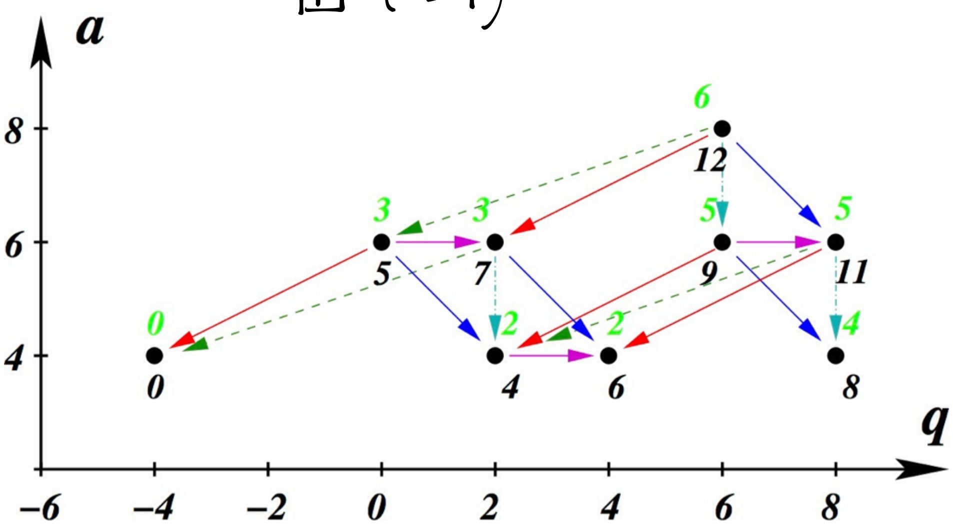
4) Exponential growth properties K: thin knot or torus knot

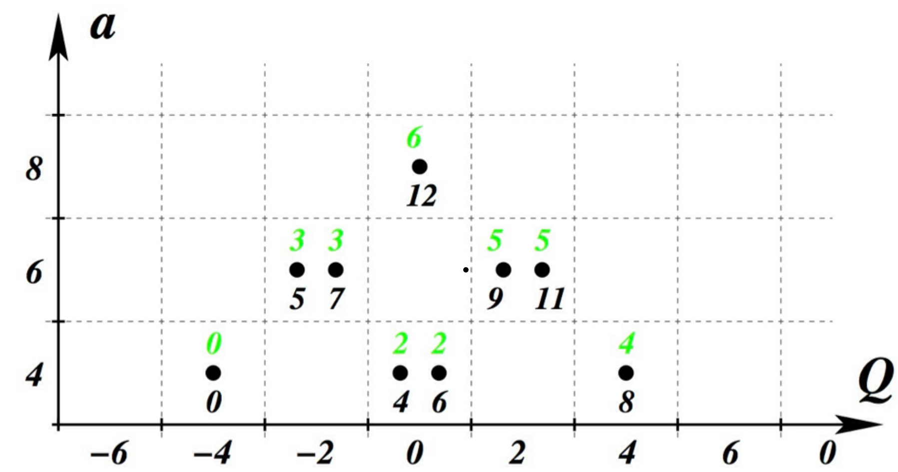
$$\mathcal{P}_{[r]}(K; a.q=1.t) = \left[\mathcal{P}_{\square}(K; a.q=1, t)\right]^r$$

at poly'l level

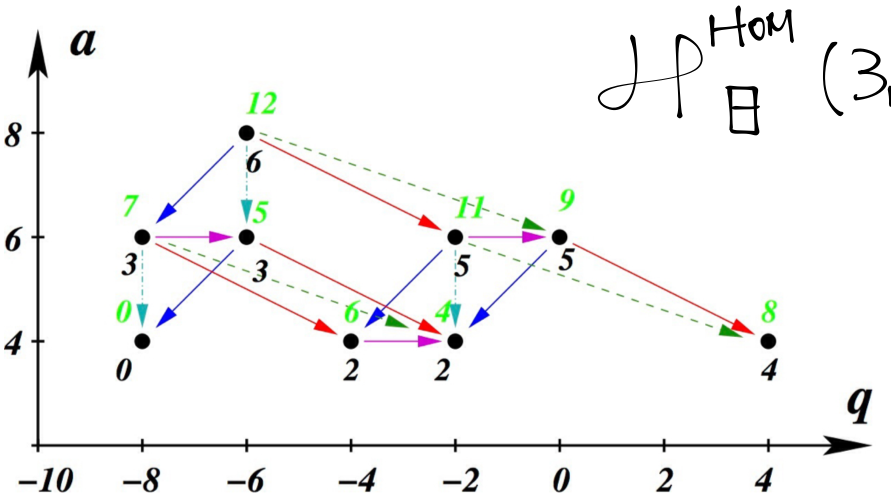
$$P_R(K; a.q=1) = \left[P_{\square}(K; a, q=1)\right]^{|R|}$$

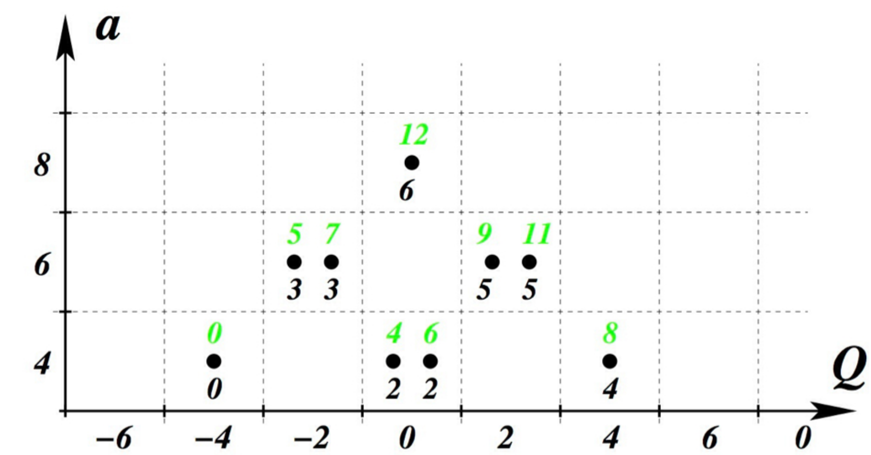
$\mathcal{H}^{HOM}_{\square}(3_1)$





$\mathcal{H}^{HOM}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(3_1)$





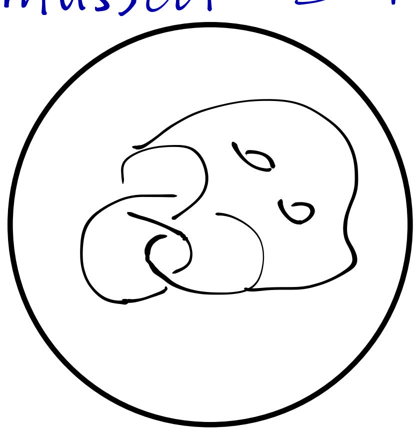
One can introduce 2 homological gradings: T_r- and T_c-gradings

For every element $x \in \mathcal{H}^{HOM}_{[r]}(K_{thin})$, one can assign

δ-grading

$$\delta(x) = a(x) + \frac{q(x)}{2} - \frac{T_r(x)+T_c(x)}{2} = r\,S(K_{thin})$$

$S(K)$: Rasmussen S-invariant.



$g_s(K) \geq \frac{1}{2}|S(K)|.$

↑

Slice (Murasugi) genus.

We introduce an auxiliary grading Q. Gorsky - Gukov - Stošić.

$$(a.Q.t_r.t_c)\text{-grading} \longleftarrow (a.q.t_r,t_c)\text{-grading}$$

$$Q = \frac{q+t_r-t_c}{\rho}$$

We define tilde-version of HOMFLY homology with $(a.Q.t_r.t_c)$

$$\left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.j.k.\ell} := \left(\mathcal{H}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i,\rho j-k+\ell,k,\ell}$$

$$\widetilde{\mathcal{P}}_{[r^\rho]}(K;a.Q.t_r.t_c) = \sum_{i.j.k.\ell} a^i Q^j t_r^k t_c^\ell \dim \widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K).$$

Then the structural properties of HOMFLY homology becomes transparent.

1). Self-symmetry.

$$\left(\mathcal{H}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.j.k.\ell} \cong \left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.-j.k-\rho j.\ell-rj}$$

2). Mirror Symmetry

$$\left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[\rho^r]}(K)\right)_{i.j.k.\ell} \cong \left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.j.\ell.k}$$

$$\cong \left(\widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r^\rho]}(K)\right)_{i.}$$

3) Refined exponential growth property.

$$\widetilde{\mathcal{P}}_{[r^\ell]}(K; a, Q, t_r, t_c=1) = \left[\widetilde{\mathcal{P}}_{[1^\ell]}(K; a, Q, t_r, t_c=1)\right]^r$$

$$\widetilde{\mathcal{P}}_{[r^\ell]}(K; a, Q, t_r=1, t_c) = \left[\widetilde{\mathcal{P}}_{[1^\ell]}(K; a, Q, t_r=1, t_c)\right]^\ell$$

thin or Torus knot.

$$\dim \widetilde{\mathcal{H}}^{HOM}_{[r^\ell]}(K) = \left[\dim \mathcal{H}^{HOM}_{\square}(K)\right]^{r\ell}$$

4) Colored differentials $d^{\pm}_{[r^\ell]\to[\ell^r]}$ & $d^{\pm}_{[r^\ell]\to[r^0]}$

$$H^*\left(\widetilde{\mathcal{H}}^{HOM}_{[r^\ell]}, d^{\pm}_{[r^\ell]\to[\ell^r]}\right) \cong \widetilde{\mathcal{H}}^{HOM}_{[\ell^r]}(K)$$

$$H^*\left(\widetilde{\mathcal{H}}^{HOM}_{[r^\ell]}, d^{\pm}_{[r^\ell]\to[r^0]}\right) \cong \widetilde{\mathcal{H}}^{HOM}_{[r^0]}(K)$$

5) Universal colored differentials.

- Power of refined exp. growth properties.

$$\mathcal{P}_{[r]}(3_1; a, Q, t_r, t_c=1) = \left[a^2Q^{-2}\left(1 + Q^4 t_r^2 (1 + a^2 Q^{-2} t_r)\right)\right]^r$$

$$= a^{2r} Q^{-2r} \sum_{k=0}^{r} Q^{4k} t_r^{2k} \binom{r}{k} \left(1 + a^2 Q^{-2} t_r\right)^k$$

One has to restore t_c-grading carefully.

$$\mathcal{P}_{[r]}(3_1; a, Q, t_r, t_c) = a^{2r} Q^{-2r} \sum_{k=0}^{r} Q^{4k} t_r^{2k} t_c^{2rk} \begin{bmatrix} r \\ k \end{bmatrix}_{t_c^2} (-a^2 Q^{-2} t_r t_c; t_c^2)_k$$

Status of colored HOMFLY homology

- $[r]$-colored homology for $T(2,2p+1)$, twist knots, $T(3,4)$, $T(3,5)$
- $[r,r]$-colored homology for 3_1 & 4_1.

Fuji, Gukov, Sulkowski, Stosic, Nawata, Ramadevi, Zodinmawia.

Remark on Kauffman homology.

Colored Kauffman homology $\widetilde{\mathcal{H}}^{\mathrm{Kauff}}_{[r]}(K)$ holds similar properties.

1) Mirror-Symmetry
2) Refined Exponential growth property for thin knots and torus knots
3) Colored differentials, d_N differentials.

The most interesting fact is that $[r]$-colored Kauffman homology contains $[r]$-colored HOMFLY homology.

More precisely, there exist differentials such that

$$H^*(\widetilde{\mathcal{H}}^{\mathrm{Kauff}}_{[r]}, d^{\mathrm{univ}}_{\rightarrow}) \cong \widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r]}(K)$$

$$H^*(\widetilde{\mathcal{H}}^{\mathrm{Kauff}}_{[r]}, d^{\pm}_{\mathrm{diag}}) \cong \widetilde{\mathcal{H}}^{\mathrm{HOM}}_{[r]}(K)$$

§3. Volume conjectures & generalizations

<u>Volume conjecture</u> Kashaev, Murakami^2

$$2\pi \lim_{r\to\infty} \frac{1}{r} \log \left| J_{[r]}(K; q = e^{\frac{2\pi i}{r}}) \right| = \mathrm{vol}(S^3 \setminus K)$$

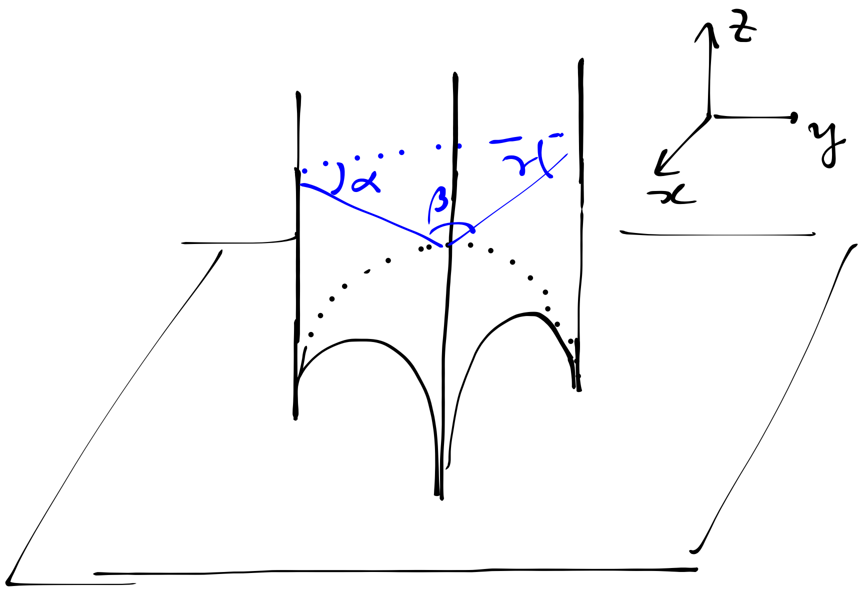
[Thurston]

hyperbolic knot $:=$ $S^3 \setminus K$ admits a hyperbolic str $R_{ij} = -2g_{ij}$

i) geodesically complete

ii) finite volume

iii) Simplicial decomp by ideal tetrahedra $T(\alpha, \beta, \gamma)$



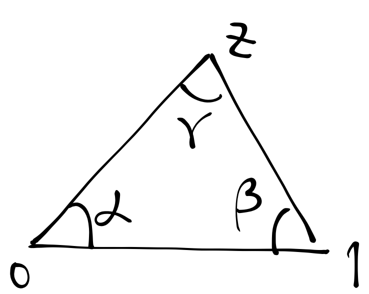
$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

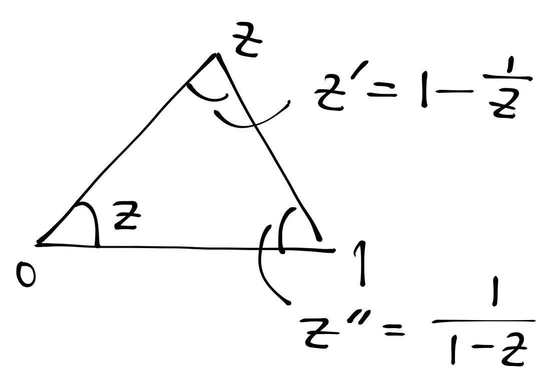
$$\mathrm{Vol}(T(\alpha, \beta, \gamma)) = \int \frac{dx\,dy\,dz}{z}$$

$$= \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma) = D(z)$$

$$\Lambda(\theta) = -\int_0^\theta dt \log|2\sin t| \qquad \text{Lobachevsky}$$

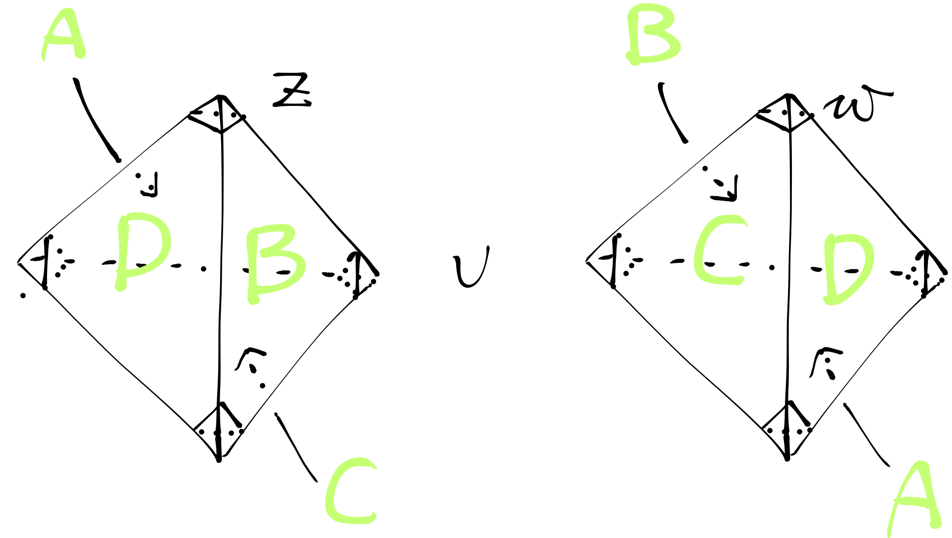
$$D(z) = \mathrm{Im}\, \mathrm{Li}_2(z) + \arg(1-z) \log|z| \qquad \text{Bloch-Wigner}$$



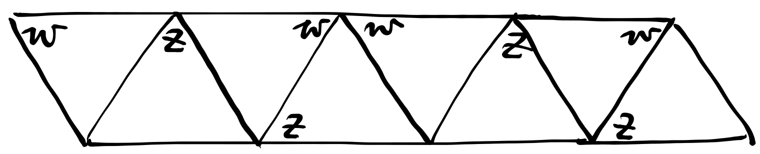


Ex. complement of 4_1

$$S^3 \setminus 4_1 =$$



Developing map



$$\begin{cases} w^2 w'' z' (z'')^2 = 1 & \text{Consistency cond.} \\ w = z & \text{Completeness} \end{cases}$$

$$\Rightarrow w = z = e^{i\frac{\pi}{3}}$$

$$\therefore \mathrm{Vol}(S^3 \setminus 4_1) = 2D(e^{\frac{\pi i}{3}}) = 6\Lambda\left(\frac{\pi}{3}\right)$$

asymptotics of colored Jones poly'l of 4_1

$$\left| J_{[r]}(4_1; e^{2\pi i/r}) \right| \sim e^{\frac{r}{2\pi} \underbrace{\mathrm{Im}\,\mathrm{Li}_2(e^{\frac{\pi i}{3}})}_{\mathrm{Vol}(S^3 \setminus 4_1)}}$$

Volume conjecture & A-polynomials

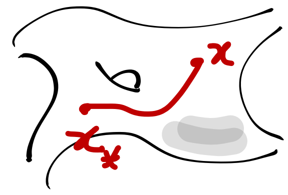
Generalized volume conj. Gukov

$$J_{[r]}(K;q) \xrightarrow[\substack{r\to\infty \\ q^r = x}]{q=e^{\hbar}\to 1} \exp\left(\frac{1}{\hbar}\int_{x_*}^{x} \log y \frac{dx}{x} + \cdots\right)$$

$A(K:x.y(x))=0$

x_* complete hypeblic metric

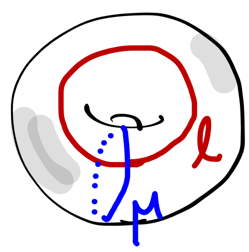
deformation to incomplete hyperbolic metric.



$$\mathcal{M}_{flat}(T^2, SL(2.\mathbb{C})) = \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2} = \left\{ (x.y) \;\middle|\; \mathbb{Z}_2 : \begin{matrix} x \to x^{-1} \\ y \to y^{-1} \end{matrix} \right\}$$

hyper-Kähler manifold.

$$\rho : \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \longrightarrow SL(2.\mathbb{C})$$



$$\mu \longmapsto \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$$

$$l \longmapsto \begin{pmatrix} y & * \\ 0 & y^{-1} \end{pmatrix}$$

A-poly'l determenes the moduli sp. of $SL(2.\mathbb{C})$ flat conn. $/ S^3 \setminus K$.

$$\mathcal{M}_{flat}(S^3\setminus K, SL(2.\mathbb{C})) = \left\{ (x.y) \in \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2} \;\middle|\; A(K:x.y) = 0 \right\}$$

$\cap$ Lagrangian submfd w.r.t. $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$.

$\mathcal{M}_{flat}(T^2, SL(2.\mathbb{C}))$

quantum volume conjecture a.k.a. AJ conjecture.

The q-diff. eqn of minimal order for colored Jones poly

$$\hat{A}(K;\hat{x},\hat{y};q)\,J_{[r]}(K;q)=0$$

where $\hat{x}, \hat{y}$ are defined

$$\hat{x}\,J_{[r]}(K;q) = q^{r}\,J_{[r]}(K;q) \qquad \hat{x}\hat{y} = q\,\hat{y}\hat{x}$$

$$\hat{y}\,J_{[r]}(K;q) = J_{[r+1]}(K;q)$$

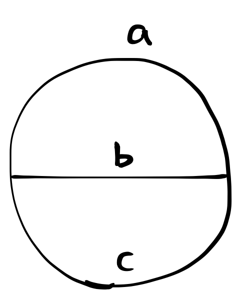
provides quantization of A-polynomial

$$\hat{A}(K;\hat{x},\hat{y},q) \xrightarrow[q\to 1]{} A(K;x,y)$$

Gukov. Garoufalidis.

Volume conjecture for quantum invariants of trivalent graph.

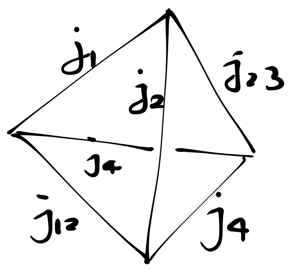
on-going work with Rama Zodih.



$$\longrightarrow (-1)^{m+n+p}\frac{[m+n+p+1]!\,[n]!\,[m]!\,[p]!}{[m+n]!\,[n+p]!\,[p+m]!}$$

$a = m+p$
$b = m+n$
$c = n+p$

theta graph.



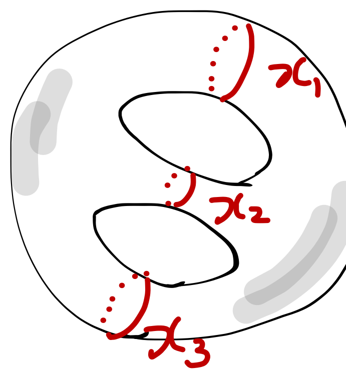
$$\longrightarrow {}_4\varphi_3\left(\begin{matrix} q^{*}\ q^{*}\ q^{*}\ q^{*} \\ q^{*}\ q^{*}\ q^{*} \end{matrix};q\right)$$

basic hypergeom. series

Askey–Wilson poly.

quantum 6j-symbol

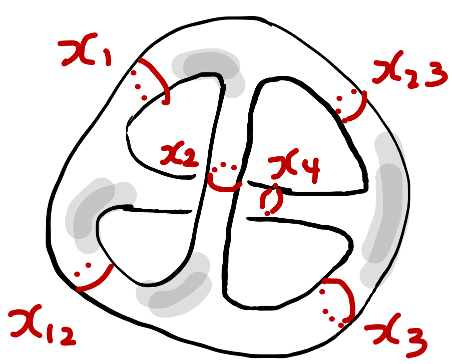
The boundary of tubular neighborhood of Θ-graph or q-$6j$ is a Riemann surface of genus 2 or 3, respectively.



holonomy eigenvalues of these cycles

$$\left.\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right\} \text{ conjugate variables.}$$

spans local coordinate of $\mathcal{M}_{\text{flat}}(\Sigma_{g=2}, SL(2.\mathbb{C}))$



$$\left.\begin{matrix} x_1 . x_2 & x_3 & x_4 & x_{12} & x_{23} \\ y_1 & y_2 & y_3 & y_4 & y_{12} & y_{23} \end{matrix}\right\} \text{ conj. variables}$$

local coordinates $\mathcal{M}_{\text{flat}}(\Sigma_{g=3}, SL(2.\mathbb{C}))$

$$\dim \mathcal{M}_{\text{flat}}(\Sigma_g . SL(2.\mathbb{C})) = \dim \mathcal{M}_{\text{Higgs}}(\Sigma_g) = 6g-6 .$$

For $g>1$, $\mathcal{M}_{\text{flat}}(\Sigma_g . SL(2.\mathbb{C}))$ has non-trivial topology.

Betti numbers are given by Hitchin

$$\sum_{i=1}^{6g-6} b_i t^i = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-2}}{4(1-t^2)(1-t^4)} \times \left\{(1+t^2)^2(1+t)^{2g} - (1+t)^4(1-t)^{2g}\right\}$$
$$-(g-1)t^{4g-3}\frac{(1+t)^{2g-2}}{1-t} + 2^{2g-1}t^{4g-4}\left\{(1+t)^{2g-2} - (1-t)^{2g-2}\right\}$$

$\mathcal{M}_{\text{flat}}(S^3 \setminus \Theta, SL(2.\mathbb{C}))$ is determined by large color asymptotics of quantum invariant of Θ-graph.

$$A_m(y_m, \vec{x}) = (-1 + x_m)(-1 + x_m x_n x_p) + (-1 + x_m x_n)(-1 + x_m x_p) y_m$$
$$A_n(y_n, \vec{x}) = (m \to n)$$
$$A_p(y_p. \vec{x}) = (m \to p).$$

Furthermore, recursion relations provide quantization of

$$\mathcal{M}_{\text{flat}}(\Sigma_{g=2}, SL(2.\mathbb{C}))$$

The same statement holds for q-6j. Since it is expressed by an Askey-Wilson poly'l, it satisfies recursion rel'n

$$A(q)\begin{Bmatrix} j_1+2, & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} + B(q)\begin{Bmatrix} j_1+1, & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} + C(q)\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = 0$$

Something complicated

classical limit.

$$\Rightarrow$$

$$\mathcal{M}_{\text{flat}}(S^3 \setminus H_{g=3}, SL(2.\mathbb{C})) : A_k(\vec{x}, y_k) = 0 \quad k = 1, \cdots, 6$$

$\cap$ Lagrangian submfd

$$\mathcal{M}_{\text{flat}}(\Sigma_{g=3}, SL(2.\mathbb{C}))$$

Volume conjecture for homological invariants.

Super-A-polynomials $\mathcal{A}(K; x, y, a, t)$

large color asymptotic of Poincare poly of $[r]$-colored HOMFLY homology

$$\mathcal{P}_{[r]}(K; q, a, t) \xrightarrow[\substack{q^r = x \\ a, t : \text{fixed.}}]{\substack{r \to \infty \\ q = e^{\hbar} \to 1}} \exp\left(\frac{1}{\hbar} \int_{\mathcal{A}(K : x, y : a, t) = 0} \log y \frac{dx}{x} + \cdots \right)$$

The quantization of super-A-poly'l is given by q-difference eqn.

$$\hat{\mathcal{A}}(K; \hat{x}, \hat{y} : a, q, t)\, \mathcal{P}_{[r]}(K : a, q, t) = 0.$$

where $\hat{\mathcal{A}}(K : \hat{x}, \hat{y} : a, q, t) \xrightarrow{q \to 1} \mathcal{A}(K : x, y : a, t)$

$$\underset{\text{super-A-poly.}}{\mathcal{A}(K : x, y : a, t)} \xrightarrow[t = -1]{} A^{a\text{-def}}(K : x, y : a) \longrightarrow \overset{\text{ordinary A-poly}}{A(K : x, y)}$$

$A^{a\text{-def}}$ || augumentation polynomial in knot contact homology

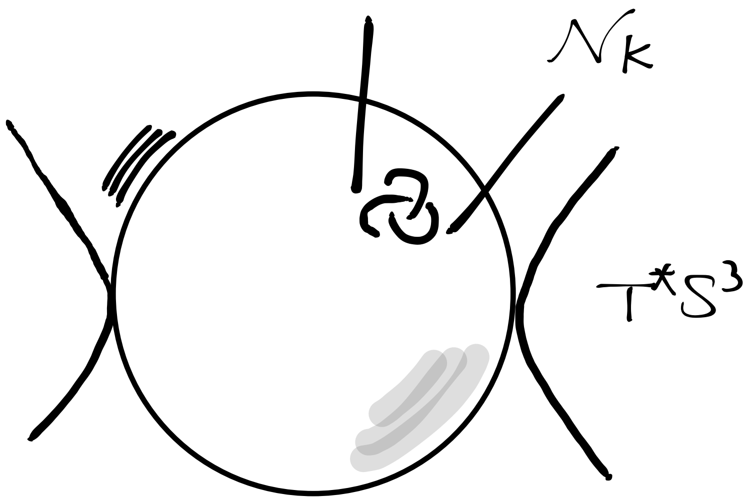
Aganagic-Vafa. Ng.

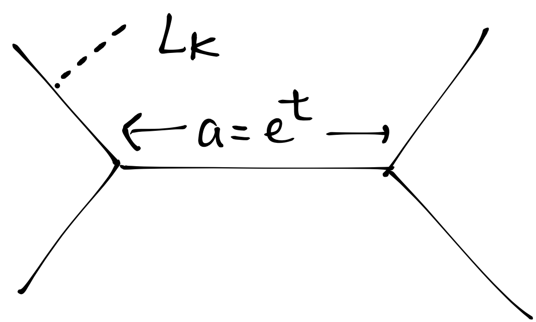
§4. Interpretation in string theory and gauge theory

$U(N)$ Chern–Simons theory on S^3 can be realized.

in A-model topological string theory. on T^*S^3.

deformed conifold $\xleftrightarrow{\text{geometric transition}}$ resolved conifold





$$N_K = \{(x.p) \in T^*S^3 \mid x = x(s), \dot{x}(s) \perp p\}$$

$$\begin{array}{c} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^1 \end{array} = X$$

	$S^1 \times \mathbb{R}^4 \times T^*S^3$	$\xrightarrow{\text{Large } N}$	$\boxed{S^1 \times \mathbb{R}^4 \times X}$
N M5	$S^1 \times \mathbb{R}^2 \times S^3$		
M5	$S^1 \times \mathbb{R}^2 \times N_K$		$\boxed{\text{M5} \quad S^1 \times \mathbb{R}^2 \times L_K}$

$\downarrow$ geom. engineering.

$U(1)$ gauge theory w/ a surface operator S_K

associated to knot K

superpoly of $(m.n)$-torus knot.

$$\overline{\mathcal{P}}_{\square}(K(m.n) : a.q.t) = Z^{\text{Nek}}_{U(1), S_{K(m.n)}}(a.q.t)$$

Gorsky–Negut.

resolved conifold $\longrightarrow$ 4d $U(1)$ gauge thy with a surf. op S_K associated to a knot K

Klemm - Katz - Vafa.

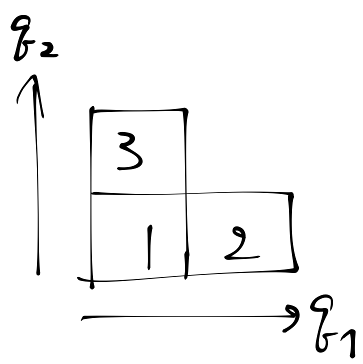
A : Coulomb branch parameter

(q_1, q_2) : equivariant parameter $\quad \begin{array}{cc} U(1)_{q_1} \times & U(1)_{q_2} \\ \circlearrowleft & \circlearrowleft \\ \mathbb{R}^2 & \mathbb{R}^2 \end{array}$

Gorsky - Negut

$$Z^{\mathrm{Nek}}_{U(1),\,S_{K(m,n)}}(A, q_1, q_2) = \sum_{SYT} \frac{\prod \chi_i^{S(i)} (1 - A\chi_i)(q_1\chi_i - t)}{\left(1 - \frac{q_1\chi_2}{q_2\chi_1}\right)\cdots\left(1 - \frac{q_1\chi_n}{q_2\chi_{n-1}}\right)} \prod_{i<j} \omega\left(\frac{\chi_i}{\chi_j}\right)$$

where sum is over SYT of size n



χ_i - (q_1, q_2) - content of box labelled i

$\chi_1 = 1 \quad \chi_2 = q_1$

$\chi_3 = q_2$

$$\omega(x) = \frac{(1 - q_1 x)(1 - q_2 x)}{(1 - x)(1 - q_1 q_2 x)}$$

$$S(i) = \left\lfloor \frac{im}{n} \right\rfloor - \left\lfloor \frac{(i-1)m}{n} \right\rfloor$$

After suitable change of variables, it is equal to Poincare poly'l of HOMFLY homology of $T(m,n)$

$$\overline{P}(T(m,n), a, q, t) = Z^{\mathrm{Nek}}_{U(1),\,S_{K(m,n)}}(A = -at,\ q_1 = q,\ q_2 = qt^2)$$

proof of geometric transition!

What is $Z^{Nek}_{U(1),\,S_{K(m,n)}}$ mathematically?

$\mathrm{Hilb}^n \mathbb{C}^2$ = moduli sp. of codim n ideals in $\mathbb{C}[x,y]$

ideal

$F\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ = moduli sp of $\mathcal{O}_{\mathbb{C}^2,0} \supset I_1 \supset I_2 \supset \cdots \supset I_n$

dim $I_k = k$, all supported at 0.

$\mathcal{L}_i \rightsquigarrow I_i / I_{i+1}$ line bundle on $F\mathrm{Hilb}^n(\mathbb{C}^2, 0)$

Th

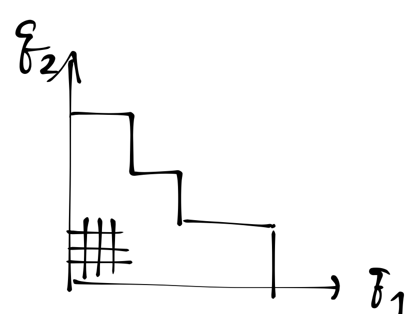
$$P(T(m,n)) = \sum_i \chi_{(\mathbb{C}^\times)^2}\left(F\mathrm{Hilb}^n, \mathcal{L}_1^{S(1)} \cdots \mathcal{L}_n^{S(n)} \otimes \Lambda^i T^*\right) a^i$$

tautological bundle

$$T = \mathbb{C}[x,y] / I_n$$

Fixed pts = flags of monomial ideals.

χ_i (q_1, q_2)-character of $\mathcal{L}_i$



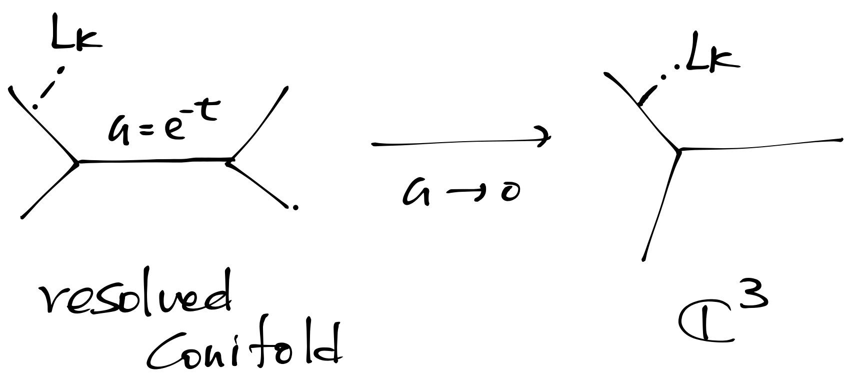
SYT.

issue! Complete intersection resolution

$F\mathrm{Hilb}^2 = \mathbb{CP}^1$

$F\mathrm{Hilb}^3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-3) \to \mathbb{P}^1)$ Hirzebruch Surface.

$a \to 0$. limit : turn off 4d gauge dynamics
dynamics on the surface op. S_K remains
$\Rightarrow$ vortex partition fn.



Gorsky - Gukov - Stosic.

Vortex Partition function

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} F = r \qquad \text{vortex number.}$$

moduli sp. $\mathcal{M}_r^{U(1)}$ of vortex partition fn with vortex # r

$$\mathcal{M}_r^{U(1)} = \left\{ (A, \phi) \;\middle|\; \begin{array}{c} *F_A = i\phi - it \\ \bar{\partial}_A \phi = 0 \end{array} \right\}$$

$$\Rightarrow \mathcal{M}_r^{U(1)} = \mathrm{Sym}^r \mathbb{C} = \underset{\substack{\| \\ (x_1, \cdots, x_r)}}{\mathbb{C}^r / S_r} \cong \underset{\substack{\| \\ (a_1, \cdots, a_r)}}{\mathbb{C}^r}$$

$$f(x) = \prod_{j=1}^{r} (x - x_j) = x^r + a_1 x^{r-1} + \cdots + a_r$$

$U(1)_q$ equivariant character $\mathrm{Ch}_q(\mathcal{M}_r^{U(1)})$ of $\mathcal{M}_r$

(counting monomials w.r.t. $(a_1 \cdots a_r)$)

$$\mathrm{Ch}_q(\mathcal{M}_r^{U(1)}) = \frac{1}{(1-q)\cdots(1-q^r)}$$

$$= \lim_{a\to 0} \overline{P}_{[r]}(\bigcirc; a, q).$$

non-abelian generalization

$U(p+1)$ gauge theory with $(p+1)$ fund. field ϕ_j

$$\mathcal{M}_r^{U(p+1)} = \left\{ (A, \Phi) \,\middle|\, \begin{array}{l} *F_A = i \sum_{j=1}^{p+1} \phi_j \phi_j^* - it \\ \bar{\partial}_A \Phi = 0 \end{array} \right\}$$

$$\dim \mathcal{M}_r^{U(p+1)} = 2r(p+1)$$

For general r, explicit construction of $\mathcal{M}_r^{U(p+1)}$ is not known

$r=1$

$$\mathcal{M}_{r=1}^{U(p+1)} = \underset{\substack{\uparrow \\ \text{center-of-mass} \\ \text{motion}}}{\overset{U(1)_q \circlearrowleft}{\mathbb{C}}} \times \mathbb{CP}^p \nwarrow \text{internal deg. of freedom}$$

$U(1)_t$ is related to homological deg.

Betti number $\displaystyle\sum_{i=0}^{\infty} b_i(\mathbb{CP}^p)\, t^{2i} = \sum_{i=0}^{p} t^{2i}$

$U(1)_q \times U(1)_t$ equiv. charater

$$\mathrm{Ch}_{q,t}(\mathcal{M}^{U(p+1)}_{r=1}) = \frac{1}{1-q} \underbrace{\sum_{i=1}^{p} t^{2i}}$$

$\parallel$ Poincare poly of $\mathbb{C}P^p$
(t: homological degree)

$$= \lim_{a\to 0} \overline{\mathcal{P}}_{\square}(T(2,2p+1); a,q,t)$$

Conjecture

$$\mathrm{Ch}_{q,t}(\mathcal{M}^{U(p+1)}_{r}) = \lim_{a\to 0} \overline{\mathcal{P}}_{[r]}(T(2,2p+1): a,q,t)$$

$$= \frac{q^{-pr}}{(q;q)_r} \sum_{k_1\cdots k_p} \begin{bmatrix} r \\ k_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}$$

$$\times\, q^{(2r+1)(k_1+\cdots+k_p) - \sum k_i k_{i+1}}\, t^{2(k_1+\cdots+k_p)}$$

3d/3d correspondence.

Dimofte-Gukov-Hollands
Yamazaki Terashima
Dimofte Gukov Gaiotto.

deformed conifold side.

$S^1 \times \mathbb{R}^4 \times T^*S^3$

$\boxed{N\ \mathrm{M5}\quad S^1 \times \mathbb{R}^2 \times S^3}$

$\mathrm{M5}\quad S^1 \times \mathbb{R}^2 \times N_K$

on $S^1 \times \mathbb{R}^2$ → $SL(N,\mathbb{C})$ CS on $S^3 \setminus K$

on $S^3 \setminus K$ → 3d $\mathcal{N}=2$ gauge theory $T_{sl(N)}[K]$ on $S^1 \times \mathbb{R}^2$

3d/3d duality

Partition fn of $T_{sl(2)}[K]$ on $S^1 \times \mathbb{R}^2$

$$I_K(a.q.t.\{x\}) = \mathrm{Tr}\,(-1)^F a^{m_1} t^{m_2} q^{\frac{R}{2}-J} x_i^{e_i}$$

$$= \int_\Gamma \frac{ds}{s} \left[\theta(z;q)\ \cdots\right] B_\Delta(z_1;q) \cdots B_\Delta(z_\ell;q)$$

CS coupling
FI Coupling

chiral field.

$$z_i = z_i(s.a.t.\{x\})$$

$$B_\Delta(z:q) = \sum_{n=0}^{\infty} \frac{z^{-n}}{(q^{-1};q^{-1})_n}$$ holomorphic block.

Beem-Dimofte-Pasquetti.

$$\mathcal{P}(K;a.q.t) \text{ “$=$” } I_{T_{sl(2)}[K]}(a.q.t)\ ??$$

The Thursday Colloquium
"The Algebra & Geometry of Modern Physics"

Lecture Notes

CHERN–SIMONS THEORY, QUANTUM KNOT INVARIANTS, AND VOLUME CONJECTURES

MATHEMATICAL SUPPLEMENT

SATOSHI NAWATA
(NIKHEF)

Cahier 5 (Suppl.) *December 5th & 6th, 2013*

M a compact 3-mfd, $G = SU(N)$

P principal G-bundle over M.

$$\pi_1(G) = 1 \implies P = M \times G$$

$\mathcal{A}_M$ space of connections on P. $\qquad \mathcal{A}_M \cong \Omega^1(M, \mathfrak{g})$

$\mathcal{G}$ gauge transf. $\qquad \mathcal{G} \cong \mathrm{Map}(M, G)$

Chern-Simons action

$$CS(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

The bilinear form $B: \Omega^1(M, \mathfrak{g}) \times \Omega^2(M, \mathfrak{g}) \to \mathbb{R}$ defined by

$$B(\alpha, \beta) = \int_M \mathrm{Tr}(\alpha \wedge \beta) \qquad \text{non-deg.}$$

Therefore, $\Omega^2(M, \mathfrak{g}) = (\Omega^1(M, \mathfrak{g}))^*$. The tangent space of $\mathcal{A}_M$ can be identified with $\Omega^1(M, \mathfrak{g})$

$$CS(A + ta) - CS(A) = \frac{t}{4\pi^2} \int \mathrm{Tr}(F_A \wedge a) + O(t^2)$$

Hence,

$$dCS(A) = \frac{1}{4\pi^2} F_A$$

Suppose M has a boundary $\partial M = \Sigma$

$$g^* A = g^{-1} A g + g^{-1} dg \qquad \text{gauge transf'n}$$

$$CS(g^* A) - CS(A) = \frac{1}{4\pi} \int_{\partial M} \mathrm{Tr}(A \wedge g^{-1} dg) - \frac{1}{12\pi} \int_M \mathrm{Tr}\left[(g^{-1} dg)^3\right]$$

Especially, $\partial M = \phi$.

$$CS(g^*A) = CS(A) - \int_M g^*\sigma \qquad \sigma : \text{Maurer-Cartan form}$$

Since $\int_M g^*\sigma = \deg(M \to 3\text{-cycle of } G) \in \mathbb{Z}$,

$$CS : \mathcal{A}_M/\mathcal{G} \longrightarrow \mathbb{R}/\mathbb{Z}$$

$\partial M = \Sigma$.

principal G-bundle over Σ is $\Sigma \times G$

$$\Rightarrow \mathcal{A}_\Sigma = \Omega^1(\Sigma . \mathfrak{g}) \qquad \infty\text{-dim'l symp. mfd}$$

$$\omega(\alpha . \beta) = -\frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(\alpha \wedge \beta) \qquad \alpha . \beta \in \Omega^1(\Sigma . \beta)$$

symplectic form

For $a \in \mathcal{A}_\Sigma$ and $g \in \mathcal{G}_\Sigma$, we denote A is an extension of a on M and $\tilde{g}$ is an extension of g on M

$$c(a, g) = e^{2\pi i (CS(\tilde{g}^*A) - CS(A))}$$

$$= \exp\left(2\pi i \left\{ \int_\Sigma \frac{1}{8\pi^2} \mathrm{Tr}(g^{-1} a g \wedge g^{-1} dg) - \int_M g^*\sigma \right\}\right)$$

This is indep. of an extension $\tilde{g}$, A. M.

In other words,

$$e^{2\pi i\, CS(g^* A)} = C(a.g|_\Sigma)\, e^{2\pi i\, CS(A)}$$

Therefore, we consider an action of $\mathcal{G}_\Sigma$ on $\mathbb{C} \times \mathcal{A}_\Sigma$

$$\mathcal{G}_\Sigma : \mathbb{C} \times \mathcal{A}_\Sigma \to \mathbb{C} \times \mathcal{A}_\Sigma \ ; (z, a) \mapsto (c(a.g)\, z,\ g^* a)$$

Chern-Simons functional integral can be regarded as $\mathcal{G}_\Sigma$-invariant holomorphic section of $\mathbb{C}^{\otimes k} \times \mathcal{A}_\Sigma$.

$$Z_k(M)(a) = \int_{\mathcal{A}_a / \mathrm{Kernel}(\mathcal{G} \to \mathcal{G}_\Sigma)} e^{2\pi i k\, CS(A)}\, \mathcal{D}A.$$

$\mathcal{A}_a$ the space of connexions on M w $A|_\Sigma = a$

$\mathcal{G}_\Sigma$-action on $\mathbb{C}^{\otimes k} \times \mathcal{A}_\Sigma$ is given by

$$Z_k(M)(g^* a) = C(a.g)^k\, Z_k(M)(a)$$

This can be identified with holomorphic section of line bundle over symplectic quotient space.

$$\mathcal{A}_\Sigma // \mathcal{G}_\Sigma = \mu^{-1}(0) / \mathcal{G}_\Sigma$$

where moment map

$$\mu : \mathcal{A}_\Sigma \to \mathrm{Lie}(\mathcal{G}_\Sigma)^* ;\ A \to F_A$$

$\mu^{-1}(0)/\mathcal{G}_\Sigma = \mathcal{M}_{flat}(\Sigma, G)$ moduli space of G flat connexions over Σ.

$$\boxed{\begin{array}{l}\underline{\text{Conclusion}}: \text{Let } M \text{ be an oriented 3-manifold with boundary } \Sigma \\ k \in \mathbb{Z} \text{ CS level, then CS functional integral } Z_k(M) \\ \text{is considered to be a section of the complex line bundle } \mathcal{L}^{\otimes k} \\ \qquad Z_k(M) \in H^0(\mathcal{M}_{flat}(\Sigma, G), \mathcal{L}^{\otimes k})\end{array}}$$

$H^0(\mathcal{M}_{flat}(\Sigma, G), \mathcal{L}^{\otimes k})$ is called quantum Hilbert space $\mathcal{H}_\Sigma$.

<u>Remark</u>: Strictly speaking, the moduli space $\mathcal{M}_{flat}(\Sigma^J, \mathcal{L}^{\otimes k})$ depends on the complex structure J of Σ. However, if you consider conformal block bundle over Teichmuller space, the bundle has a natural flat connection. In this sense, the quantum Hilbert space is indep. of cpx str. J.

If $M = M_1 \cup M_2$ with $\partial M_1 = \Sigma$ and $\partial M_2 = -\Sigma$

$$\exp(2\pi i k\, CS(A)) = \langle \exp(2\pi i k\, CS(A_1)), \exp(2\pi i k\, CS(A_2)) \rangle$$

where A is a connexion of $M \times G$, A_1 and A_2 are restriction of A to M_1 and M_2 $\Rightarrow$ axiom of TQFT.

Let us consider the case of a link L in a 3-mfd M.

We assign reps R_i to components C_i $(i=1, \cdots, |L|)$ of L.

Wilson loop operator

$$W_{C_j, R_j}(A) = \mathrm{Tr}_{R_j}\left(\mathrm{Hol}_{C_j}(A)\right)$$

The invariant of the link by Witten's formulation.

$$Z_k(M : (C_j, R_j)) = \int \mathcal{D}A \; e^{2\pi i k CS(A)} \prod_{j=1}^{|L|} W_{C_j R_j}(A)$$

The quantum Hilbert space associated with the above integral is given by conformal blocks of $\hat{\mathfrak{g}}_k$ WZW model.

$\mathcal{M}_{p_1 \cdots p_n}$ the vector sp of meromorphic fns on $\mathbb{C}P^1$ with poles of any order at most at $p_1 \cdots p_n$

$$\mathfrak{g}(p_1 \cdots p_n) = \mathfrak{g} \otimes \mathcal{M}_{p_1 \cdots p_n}$$

To each pt p_i $(1 \leq i \leq n)$, we associate the integrable higest weight H_{λ_i}. We define diagonal action Δ of $\mathfrak{g}(p_1 \cdots p_n)$ on the tensor product $H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n}$

$$\Delta(\varphi)(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_j (\xi_1 \otimes \cdots \otimes i_j(\varphi)\xi_j \otimes \cdots \otimes \xi_n)$$

where $i_j : \mathfrak{g}(p_1 \cdots p_n) \longrightarrow \hat{\mathfrak{g}}_j$

The space of conformal blocks

$$\mathrm{Hom}_{\mathfrak{g}(p_1 \cdots p_n)}(\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}, \mathbb{C})$$

is defined to be the space of linear forms

$$\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \longrightarrow \mathbb{C}$$

which are invariant under the diagonal action Δ of $\mathfrak{g}(p_1 \cdots p_n)$

Definition of colored quantum invariants

Braid group B_n

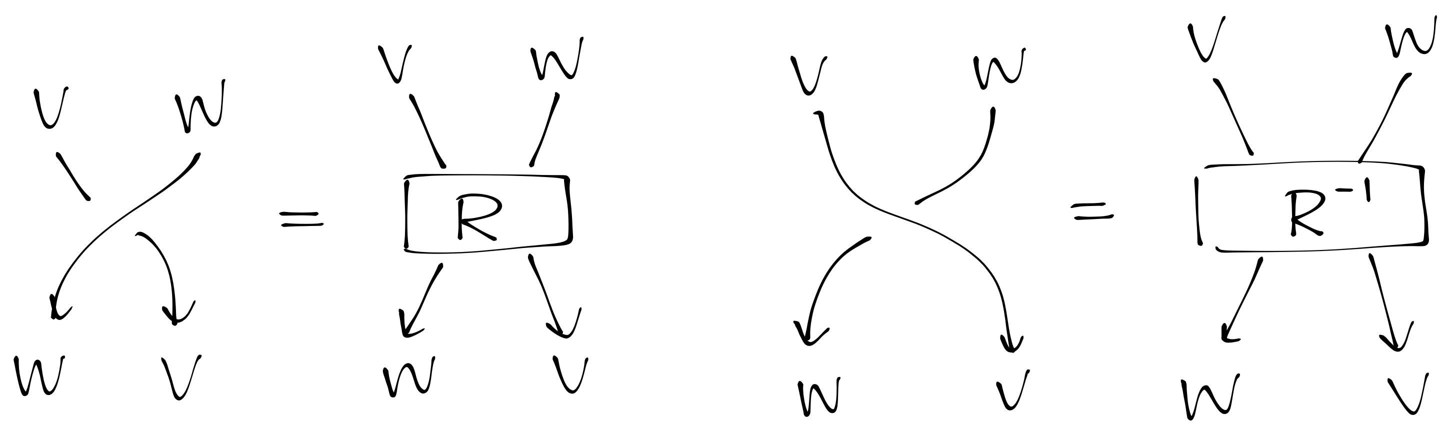
$$B_n = \langle \sigma_1 \cdots \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j|>1), \ \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \rangle$$

Th (Alexander) Every link can be realized as a closure of a certain braid.

Def

1. $U_q(\mathfrak{g})$-module V, W

$$\check{R}_{V,W} : V \otimes W \longrightarrow W \otimes V$$



which satisfy

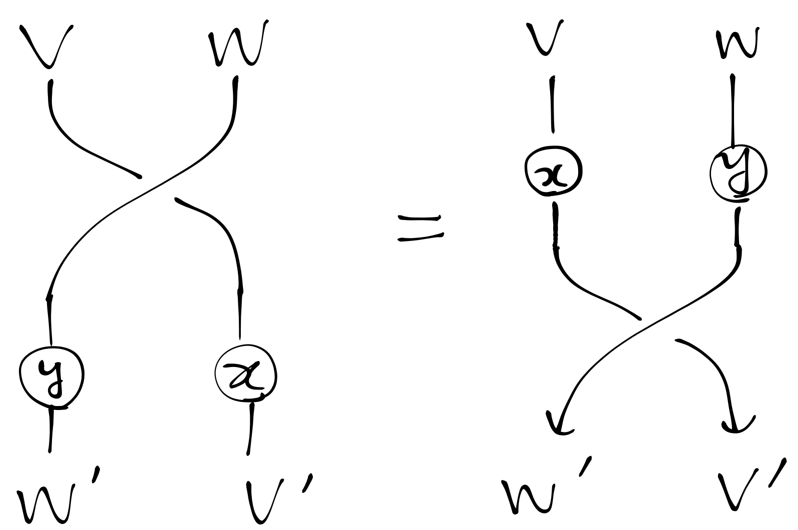
$$\check{R}_{U\otimes V,W} = (\check{R}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \check{R}_{V,W})$$

$$\check{R}_{U,V\otimes W} = (\mathrm{id}_V \otimes \check{R}_{U,W})(\check{R}_{U,V} \otimes \mathrm{id}_W)$$

It is natural in the sense that

$$(y \otimes x)\, \check{R}_{V,W} = \check{R}_{V',W'}\,(x \otimes y)$$

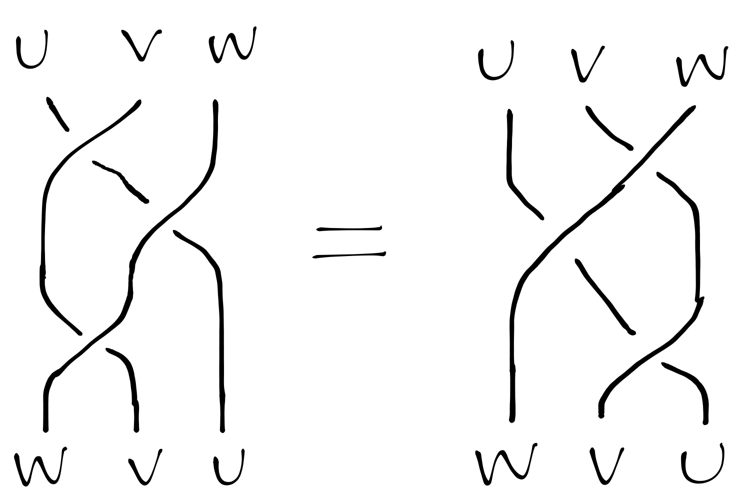
for $x \in \mathrm{Hom}_{U_q(\mathfrak{g})}(V, V')$ & $y \in \mathrm{Hom}_{U_q(\mathfrak{g})}(W, W')$



These equs imply Yang–Baxter equs

$$(\check{R}_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes \check{R}_{U,W})(\check{R}_{U,V} \otimes \mathrm{id}_W)$$

$$= (\mathrm{id}_W \otimes \check{R}_{U,V})(\check{R}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \check{R}_{V,W})$$



2. There exists an element $K_{2\rho} \in U_q(\mathfrak{g})$

$$K_{2\rho}(v \otimes w) = K_{2\rho}(v) \otimes K_{2\rho}(w)$$

for $v \in V$, $w \in W$

for every $z \in \mathrm{End}_{U_q(\mathfrak{g})}(V \otimes W)$ with $z = \sum x_i \otimes y_i$

$x_i \in \mathrm{End}(V)$ & $y_i \in \mathrm{End}(W)$, the quantum trace is defined

$$\mathrm{Tr}_W(z) = \sum_i \mathrm{Tr}(y_i K_{2\rho})\, x_i \in \mathrm{End}_{U_q(\mathfrak{g})} V$$

Given a link L with the component $L_1, \cdots, L_\ell$ labeled by $U_q(\mathfrak{g})$-module $V_1, \cdots, V_\ell$. The braid diagram β of the link L define

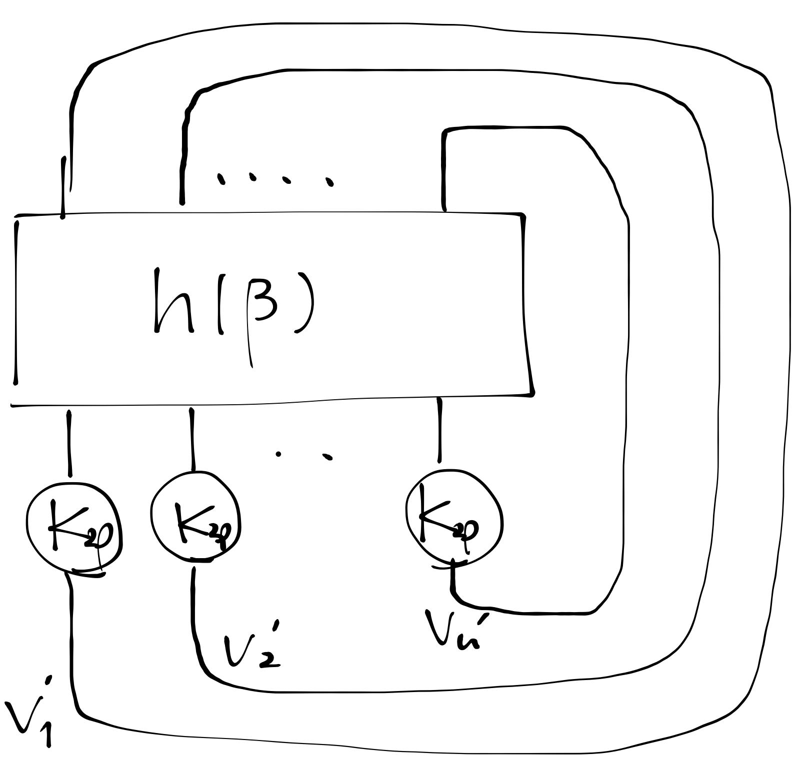
$$\begin{array}{ccc} h : B_n & \longrightarrow & \mathrm{End}_{U_q(\mathfrak{g})}(V_1' \otimes \cdots \otimes V_n') \\ \Psi & & \Psi \\ \beta & \longmapsto & h(\beta) \end{array}$$

Then, the quantum invariant of $(L_1, \cdots, L_\ell; V_1, \cdots, V_\ell)$

$$J_{(\mathfrak{g}; V_1 \cdots V_\ell)}(L) = \mathrm{Tr}_{V_1' \otimes \cdots \otimes V_n'}(h(\beta))$$

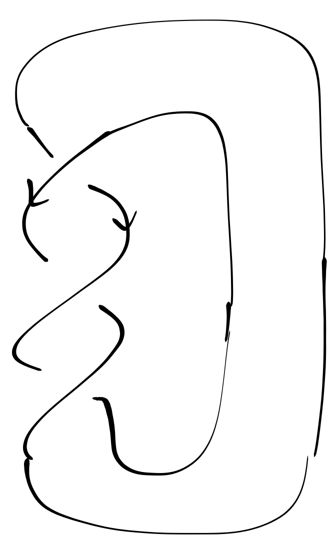
$\uparrow$

fraiming dependent i.e. invariant under Reidemeister move 2 and 3.



Ex $(\mathfrak{sl}(2), \square)$

$$R = \begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & q^{1/4} - q^{-3/4} & q^{-1/4} & 0 \\ 0 & q^{-1/4} & 0 & \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix} \qquad K_{2\rho} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$



$(\mathfrak{sl}(2), \square)$

$$\Rightarrow J_{(\mathfrak{sl}(2), \square)}(3_1) = \mathrm{Tr}_{V_{1/2} \otimes V_{1/2}} \left(R^3 (K_{2\rho} \otimes K_{2\rho}) \right)$$
$$\propto (q^{1/2} + q^{-1/2})(q^{-1} + q^{-3} - q^{-4})$$

Drinfeld constructed universal R-matrix associated to $U_q(\mathfrak{g})$

So, in principle, one can write $\check{R}_{V,W}$ for any rep. V, W.