# Phase estimation without *'a priori'* phase knowledge in the presence of loss

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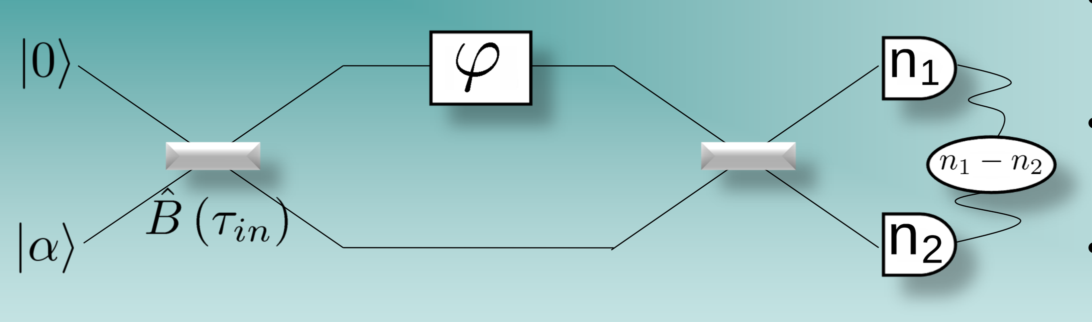
## Abstract

We consider the problem of **phase estimation** when **no *'a priori'* knowledge** about its initial value is present. We use the **covariant** positive operator valued measurement **(POVM)** scheme, in order to find the optimal states that yield the highest **estimation fidelities**. We investigate the **effect of losses** in the system, by introducing a fictitious beamsplitter and derive the **optimal usage of coherent and N-photon input states**. We **prove analytically** that in the asymptotic limit of infinite photons the **quantum precision enhancement** amounts at most to a **constant factor improvement over classical strategies**.

## 1) Typical approach – Mach-Zehnder Interferometer

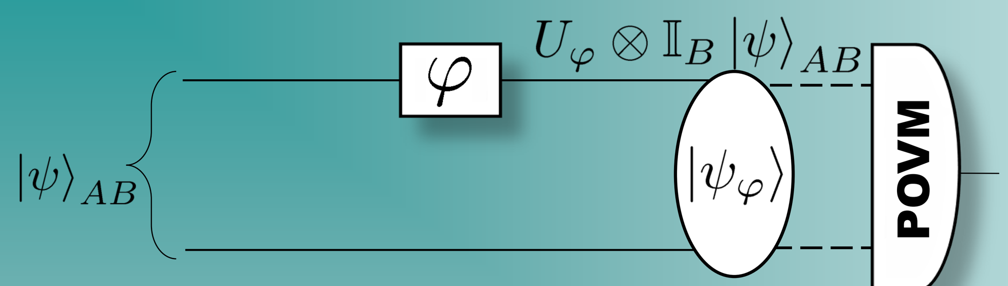



- **Measurement** $\langle \hat{n}_1 - \hat{n}_2 \rangle = |\alpha|^2 \cos\varphi$
- **Estimator** $\tilde{\varphi}(n_1, n_2) = \arccos\left(\frac{n_1 - n_2}{|\alpha|^2}\right)$
- **Variance** $\Delta^2\tilde{\varphi} = \frac{1}{|\alpha|^2 \sin^2\varphi} = \frac{1}{\bar{n}\sin^2\varphi}$

$\Longrightarrow$ $\Delta^2\tilde{\varphi} \propto \frac{1}{\bar{n}}$ **Standard Quantum Limit**

*"classical" states of light*

## 2) General Quantum Interferometer




- Looking for schemes that beat SQL.
- A mixed state at the input, will always be worse than a pure state.

- By the method of maximising the **Quantum Fisher Information**, [1], it has been proved that **non-classical**, **entangled** states can greatly improve the precision, ideally leading to the (for *N* photon input states)

$\Longrightarrow$ $\Delta^2\tilde{\varphi} \propto \frac{1}{N^2}$ **Heisenberg Limit** *non -"classical" states of light*

- The most celebrated example that saturates the bound is the ***NOON*** state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + |0,N\rangle\right)$$

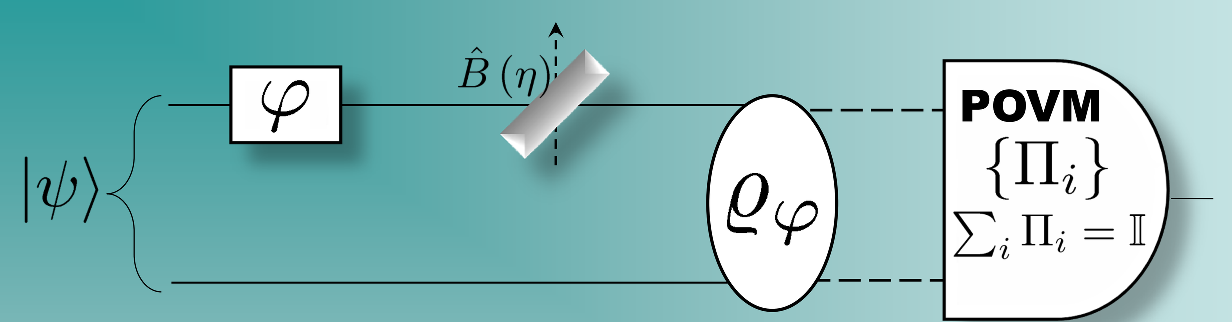
highly entangled ⇒ extremely fragile: loss of one photon makes it useless ⇒ **only optimal for lossless systems !**

- In order to achieve $\frac{1}{N^2}$, we need to be estimating within small variations from the **"a priori" known initial phase**, $\varphi_0$.

e.g. for NOON we can effectively estimate only within $\varphi_0 \pm \frac{\pi}{N}$, ⇒ **local** as $U_{\frac{2\pi}{N}} \otimes \mathbb{I}_B |NOON\rangle = |NOON\rangle$

⇒ **NOT optimal when no 'a priori' knowledge** is present !

## 3) Interferometric setup considered




- $\varrho_\varphi$ is possibly mixed due to losses.
- The set of POVMs serves to form an estimator of $\tilde{\varphi}$.

- We consider again a pure input state. However, if $|\psi\rangle$ is a superposition over total photon numbers, then, as we have no extra reference beam, we have to average over the external phase [2].

$$|\psi\rangle = \sum_N \beta_N |\psi^{(N)}\rangle \xrightarrow{\langle\dots\rangle_{ref}} \varrho_{in} = \sum_N |\beta_N|^2 |\psi^{(N)}\rangle\langle\psi^{(N)}|$$

- State evolution in the system

$$|\psi\rangle \xrightarrow{\hat{U}_\varphi} |\psi_\varphi\rangle \xrightarrow{\hat{B}(\eta)} \varrho_\varphi \xrightarrow{POVM} p(i|\varphi) = Tr\{\Pi_i \varrho_\varphi\} \Longrightarrow Estimator\ \tilde{\varphi}(i)$$

- **Covariant measurement** ↔ optimally to parameterise the POVM by the estimated parameter's group G, [3]. Here: G = U(1) and

$$discrete\ \{\Pi_i\} \longrightarrow continuous\ \{\Pi_\varphi = U_\varphi \Pi_0 U_\varphi^\dagger\}$$

completeness constraint $\sum_i \Pi_i = \mathbb{I} \longrightarrow \int d\varphi \Pi_\varphi = \mathbb{I}$

Estimator: $\tilde{\varphi}(\varphi) = \varphi$

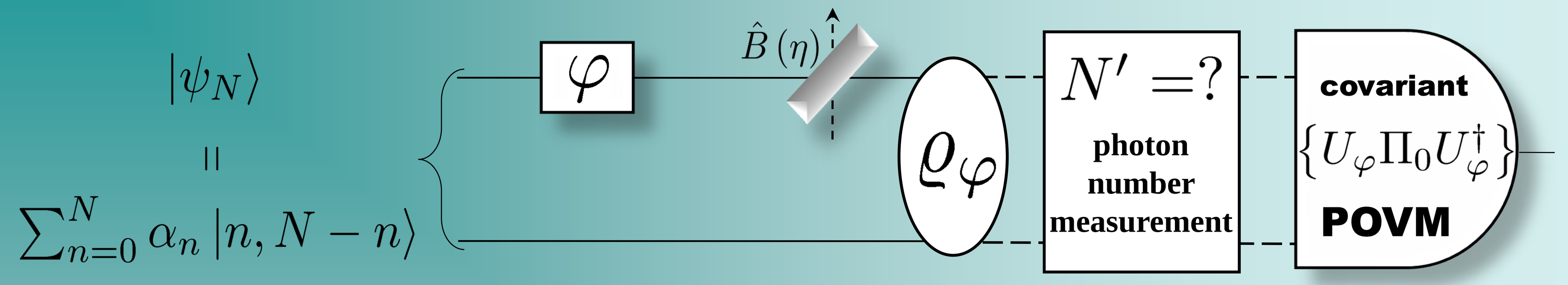
## 4) Average fidelity of estimation

The functional that quantifies the precision of estimation: **Average Fidelity**

$$F = \int_{U(1)} d\varphi\, p(\varphi) \int_{U(1)} d\tilde{\varphi}\, A(\varphi, \tilde{\varphi})\, p(\tilde{\varphi}|\varphi) \Rightarrow \Delta^2\tilde{\varphi} = 4(1-F)$$

| 'a priori' φ distribution | gain (figure of merit) function | probability of $\tilde{\varphi}$ estimation |
|---|---|---|
| $p(\varphi) = \frac{1}{2\pi}$ | $A(\varphi,\tilde{\varphi}) = \sum_{m=-\infty}^{\infty} \mathcal{A}_n e^{in(\varphi-\tilde{\varphi})}$ | $p(\tilde{\varphi}\|\varphi) = Tr\left\{U_{\tilde{\varphi}} \Pi_0 U_{\tilde{\varphi}}^\dagger \varrho_\varphi\right\}$ |

## 5) N-photon input state




- The output state – mixture over the number of photons lost, *l* :

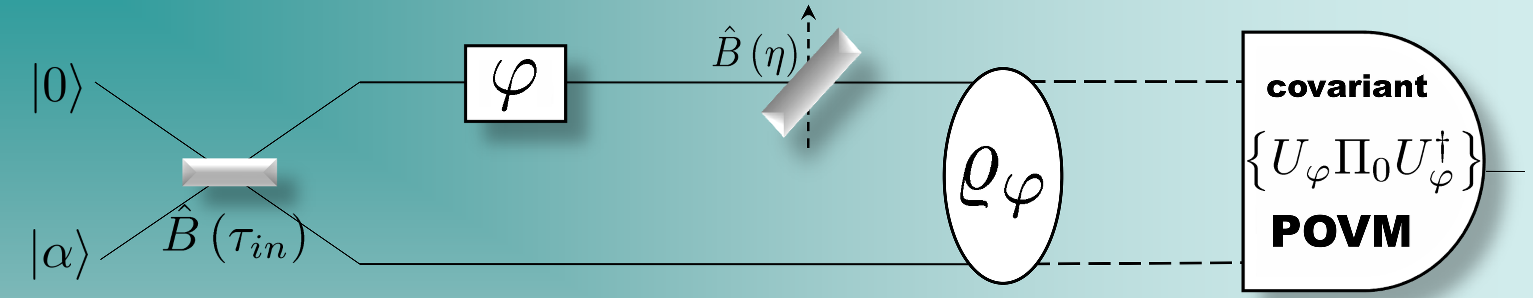
$$\varrho_\varphi = \sum_{l=0}^{N} p_l |\psi_l\rangle\langle\psi_l| \quad \begin{cases} p_l = \sum_{m=l}^{N} |\alpha_m|^2 B(\eta)_k^n \\ |\psi_l\rangle = \frac{1}{\sqrt{p_l}} \sum_{n=l}^{N} \alpha_n e^{in\varphi} \sqrt{B(\eta)_l^n}\, |n-l, N-n\rangle \end{cases}$$

where $B(\eta)_k^n = \binom{n}{k} (\eta)^{n-k}(1-\eta)^k$ are the binomial factors parameterised by beamsplitter's transmission coefficient, $\eta$.

- **Average fidelity**: $F_N = \sum_{l=0}^{N} \sum_{n,n'=l}^{N} \alpha_n \alpha_{n'}^* \sqrt{B(\eta)_l^n B(\eta)_l^{n'}}\, \mathcal{A}_{n-n'} \langle n'-l, N-n'| \Pi_0 |n-l, N-n\rangle$
- Maximal when $\Pi_0 = \sum_{n,n'=0}^{N} e^{i(\theta_{n-l} - \theta_{n'-l})} |n, N-n\rangle\langle n', N-n'|$, $\theta_n = arg(\alpha_n)$
- Optimal input ⇒ eigenvector corr. to maximal eigenvalue of: $F_N = \underline{\alpha}^\dagger \underline{\underline{\mathbf{M}}} \underline{\alpha}$
- Gain function: $A(\varphi,\tilde{\varphi}) = \cos^2\left(\frac{\varphi - \tilde{\varphi}}{2}\right) \Longleftrightarrow \mathcal{A}_0 = \frac{1}{2}, \mathcal{A}_{\pm 1} = \frac{1}{4}$
  ⇒ $\underline{\underline{\mathbf{M}}}$ – tridiagonal matrix
- For $\eta$ **= 1**, analytically solvable, [4], → **Heisenberg Limit** for large N ⇒ $\Delta^2\tilde{\varphi} \sim \frac{1}{N^2}$

$\alpha_n = \mathcal{N} \sin\left(\frac{(n+1)\pi}{N+2}\right)$ and for N≫1, $F_N^{max} \approx 1 - \frac{\pi^2}{4N^2}$

## 6) Coherent input state




- Average **fidelity**:

$$F_{coh} = \int_{U(1)} \frac{d\varphi}{2\pi} A_{diff}(\varphi) \langle \alpha\sqrt{\tau_{in}}\sqrt{\eta}e^{i\varphi}, \alpha\sqrt{1-\tau_{in}} | \Pi_0 | \alpha\sqrt{\tau_{in}}\sqrt{\eta}e^{i\varphi}, \alpha\sqrt{1-\tau_{in}} \rangle$$

- Maximal when $\Pi_0 = \sum_{N=0}^{\infty} \sum_{n,n'=0}^{N} |n, N-n\rangle\langle n', N-n'| = \sum_{N=0}^{\infty} |e_N\rangle\langle e_N|$, Gain: $A_{diff}(\varphi) = \cos^2\left(\frac{\varphi}{2}\right)$
- **Optimal** $\tau_{in}$ for

$$\frac{\partial}{\partial\tau_{in}} F_{coh} = 0 \Rightarrow (1-\eta)|\alpha|^2 = \frac{1}{\tau_{in}} \frac{S_{\frac{3}{2}}(|\alpha|^2\tau_{in}\eta)}{S_{\frac{1}{2}}(|\alpha|^2\tau_{in}\eta)} - \frac{1}{1-\tau_{in}} \frac{S_{\frac{3}{2}}(|\alpha|^2(1-\tau_{in}))}{S_{\frac{1}{2}}(|\alpha|^2(1-\tau_{in}))} - \frac{1-2\tau_{in}}{2\tau_{in}(1-\tau_{in})}$$

where $S_a(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} n^a$ are *Bell polynomials of fractional order* → compute numerically.

- In weak and strong beam (**Standard Quantum Limit**) regimes:

$|\alpha| \ll 1$: $\tau_{in} = \frac{1}{2} + \gamma(1-\eta)$, $\gamma = \frac{1}{8}(1+\sqrt{2})(3\sqrt{2}-4)|\alpha|^2$

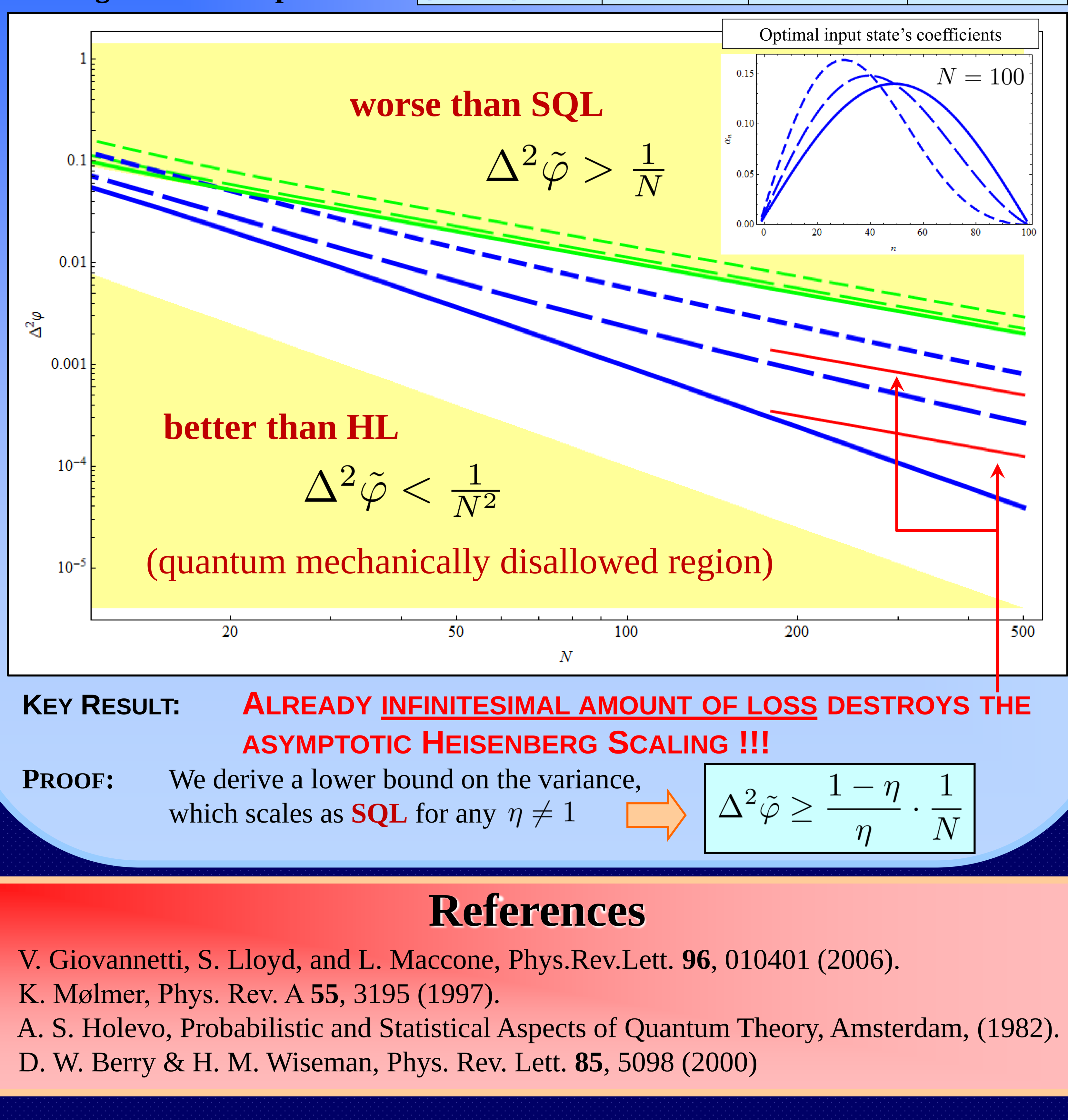
$|\alpha| \gg 1$: $S_{\frac{3}{2}}(x) \approx \frac{\sqrt{xe^x}(e^x(1+2x)-1-x)}{2\sqrt{(e^x-1)}}$, $\tau_{in} = \frac{1}{1+\sqrt{\eta}}$, $S_{\frac{1}{2}}(x) \approx \sqrt{xe^x(e^x-1)}$

⇒ $F_{str.coh}^{max} = 1 - \frac{1}{16|\alpha|^2} \frac{(1+\sqrt{\eta})^2}{\eta}$, $\Delta^2\tilde{\varphi} \sim \frac{(1+\sqrt{\eta})^2}{4\eta} \cdot \frac{1}{\bar{n}}$

## 7) Results

PLOT : The **variance** against the **average number of photons**

| input | no loss: $\eta = 1$ | $\eta = 0.8$ | $\eta = 0.5$ |
|---|---|---|---|
| coherent | solid green line | dashed green line | dotted-dashed green line |
| pure *N* photon | solid blue line | dashed blue line | dotted-dashed blue line |




**KEY RESULT:** **ALREADY INFINITESIMAL AMOUNT OF LOSS DESTROYS THE ASYMPTOTIC HEISENBERG SCALING !!!**

**PROOF:** We derive a lower bound on the variance, which scales as **SQL** for any $\eta \neq 1$ ⇒ $\Delta^2\tilde{\varphi} \geq \frac{1-\eta}{\eta} \cdot \frac{1}{N}$

## References

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