Phase Estimation with Interfering Bose-Condensed Atomic Clouds

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Outline

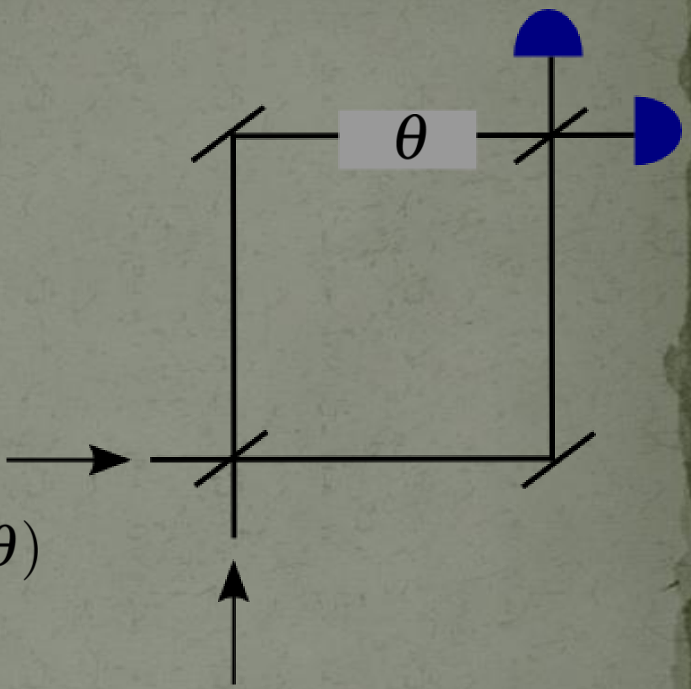
1. Main goals of interferometry
2. Formalism of the Fisher information
3. Interferometry with cold atoms
4. Phase estimation with interfering atomic clouds
5. Conclusions

Main goals of interferomtery

Estimate the phase θ with minimal possible error $\Delta\theta$

Optimize the input state $|\psi_{in}\rangle$

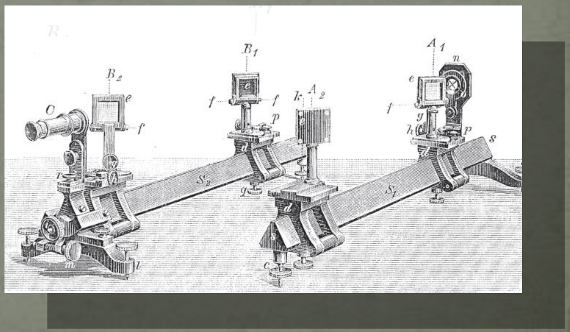
Optimize the measurement $p(\xi|\theta)$



Reference point – shot-noise limit $$\Delta\theta_{sn} = \frac{1}{\sqrt{N}}$$

The main goal of interferometry $$\Delta\theta < \Delta\theta_{sn}$$

The Holy Grail – Heisenberg limit $$\Delta\theta = \frac{1}{N}$$



Luca Pezzé and Augusto Smerzi, Phys. Rev. Lett. **102**, 100401 (2009)

Phase estimation in experiment

How does one deduce the value of θ in a real experiment?

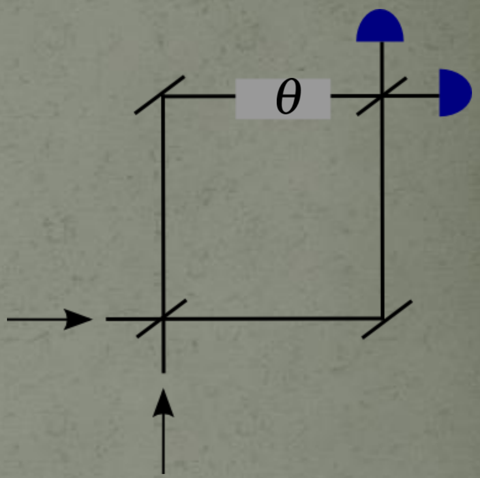
1. Choose the physical quantity ξ

2. Determine the conditional probablity $p(\xi|\theta)$

3. Measure ξ_i in the i-th experiment

4. Invert the probability and obtain $p(\theta|\xi_i)$

5. Estimate the θ_i as the maximum of this probability



$$\underbrace{\theta_1 \quad \ldots \quad \theta_m}$$

$$\langle\theta\rangle = \frac{1}{m}\sum_{i=1}^{m}\theta_i$$

$$\Delta^2\theta = \frac{1}{m}\sum_{i=1}^{m}(\theta_i - \langle\theta\rangle)^2$$

„phase sensitivity”

Phase sensitivity - theory

What is the theoretical value of the phase sensitivity?

Cramer Rao Lower Bound (CRLB)

$$F = \int \frac{d\xi}{p(\xi|\theta)} \left[\partial_{\theta} p(\xi|\theta)\right]^2$$

Fisher information

- 0
- N
- N^2

Phase sensitivity is bounded by $\Delta\theta \geq \frac{1}{\sqrt{mF}}$

Phase sensitivity – theory (2)

How can one calculate the Fisher information?

Use the evolution operator (interferometer) $\hat{U}(\theta) = e^{-i\theta\hat{h}}$

$$p(\xi|\theta) = \left|\langle\xi|\, e^{-i\theta\hat{h}}\,|\psi_{in}\rangle\right|^2$$

Example – the Mach-Zehnder interferometer

$$\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})$$

$$\hat{J}_y = \frac{1}{2i}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})$$

$$\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})$$

Evolution operator $\hat{U}(\theta) = e^{-i\theta\hat{J}_y}$

Input state $|\psi_{in}\rangle = \sum_n c_n|n, N-n\rangle$

The probability $p(m|\theta) = |\langle m, N-m|e^{-i\theta\hat{J}_y}\sum_n c_n|n, N-n\rangle|^2$

Optimalization over the possible measurements – Quantum Fisher Information

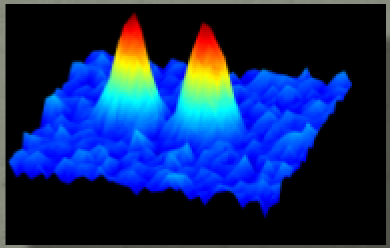
$$F_{max} = F_Q = 4\Delta^2\hat{h}$$

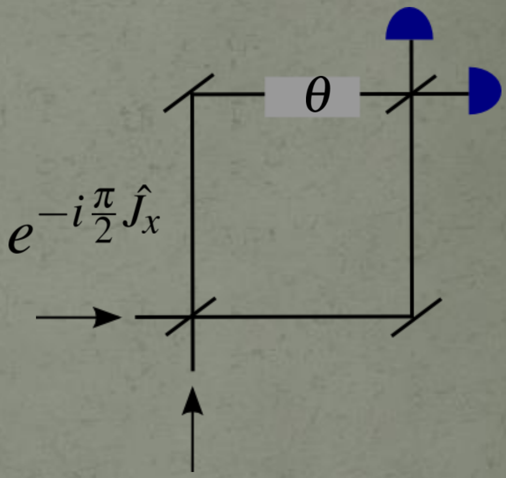
Interferometry with cold atoms

- Atoms strongly interact with external fields (gravitation, EM fields)

- Non-classical input states due to atom-atom interactions

- BEC in a double-well potential





- Beam-splitters realized by tunneling of atoms

- Limited number of atoms, $\Delta\theta < \Delta\theta_{sn}$ very important!

$$H = -E_J\hat{J}_x + E_C\hat{J}_z^2$$

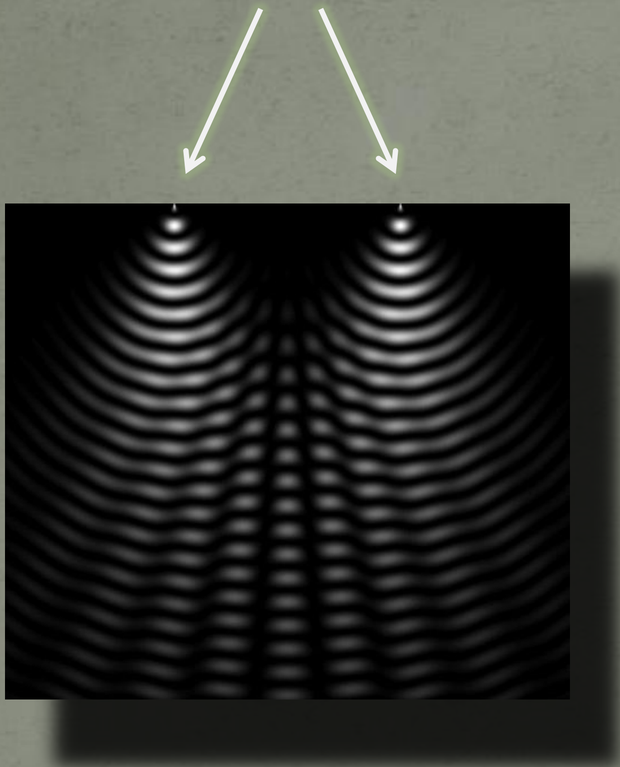
J. B. Fixler, G. T. Foster, J. M. McGuirk, M. A. Kasevich, Science **315**, 74 (2007)

J. Estève, C. Gross, A. Weller, S. Giovanazzi & M. K. Oberthaler *Nature* **455**, 1216-1219 (2008)

Phase estimation with interfering atomic clouds

A simple interferometric scheme:

Two BECs in a double-well potential



- Imprint a relative phase θ
- Let the clouds expand and form an interference pattern
- Measure positions of atoms and deduce the phase

$$\hat{\Psi}(x,\theta) = e^{i\theta\hat{J}_z}(\psi_a(x)\hat{a} + \psi_b(x)\hat{b})e^{-i\theta\hat{J}_z} = \psi_a(x)e^{-i\frac{\theta}{2}}\hat{a} + \psi_b(x)e^{i\frac{\theta}{2}}\hat{b}$$

Phase estimation with interfering atomic clouds (2)

Optimal states – identify using the QFI $\hat{U} = e^{-i\theta \hat{J}_z} \Rightarrow \hat{h} = \hat{J}_z$

$$F_Q = 4\Delta^2 \hat{J}_z$$

two-mode states $|\psi_{in}\rangle = \sum_n c_n |n, N-n\rangle$

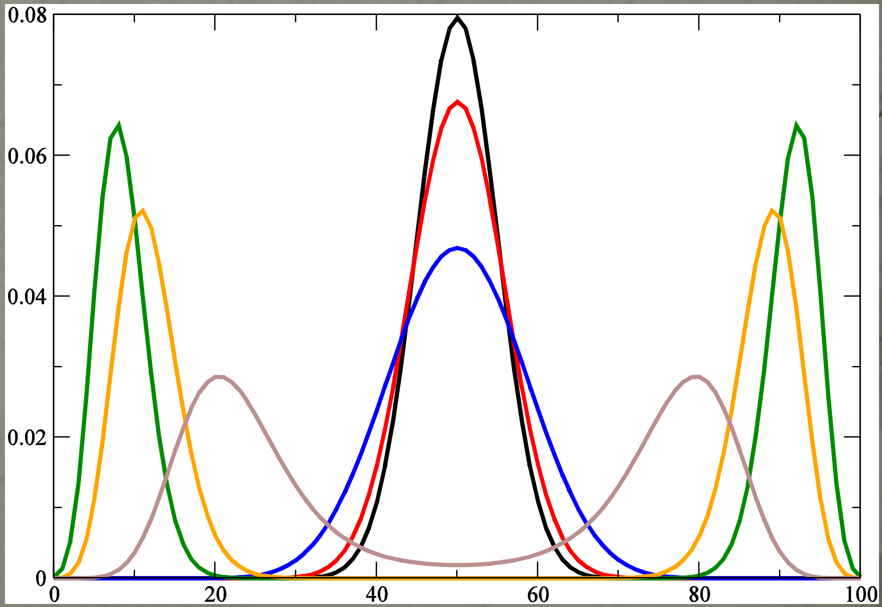
$$F_Q = 4\sum_n c_n^2 \left(n - \frac{N}{2}\right)^2$$

Good states:

Ground state of the two-mode Hamiltonian

$$H = -E_J \hat{J}_x + E_C \hat{J}_z^2$$

with attractive interactions



J. Grond, J. Schmiedmayer and U. Hohenester, New J. Phys. **12**, 065036 (2010)

What do you want to measure?

Positions of atoms forming the interference pattern.

Starting point – N-body probability

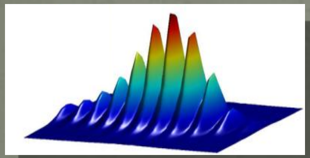
$$p_N(\vec{x}_N|\theta) = \langle \hat{\Psi}^\dagger(x_1|\theta)\ldots\hat{\Psi}^\dagger(x_N|\theta)\hat{\Psi}(x_N|\theta)\ldots\hat{\Psi}(x_1|\theta)\rangle$$

$$p_N(\vec{x}_N|\theta) = \int_0^{2\pi}\int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{d\varphi'}{2\pi}\prod_{i=1}^{N}u_\theta^*(x_i,\varphi;t)u_\theta(x_i,\varphi';t)\sum_{n,m=0}^{N}\frac{C_nC_m\cos\left[\varphi\left(\frac{N}{2}-n\right)\right]\cos\left[\varphi'\left(\frac{N}{2}-m\right)\right]}{\sqrt{\binom{N}{n}\binom{N}{m}}}$$

with $$u_\theta(x,\varphi;t) = \psi_a(x,t)e^{\frac{i}{2}(\varphi+\theta)} + \psi_b(x,t)e^{-\frac{i}{2}(\varphi+\theta)}$$

Detection schemes

Fit to the density



1. Measure the density
2. Fit the theoretical curve $p(x|\theta)$
3. Determine the phase from the least-square formula

The Fisher information

$$F = \int dx \frac{1}{p(x|\theta)} \left[\partial_\theta p(x|\theta)\right]^2 \leq N \quad \Rightarrow \quad \Delta\theta \geq \frac{1}{\sqrt{mN}}$$

No sub-shot noise sensitivity!

Idea – measure the correlations!

Detection schemes (2)

N-*th* order correlation function

$$p_N(\vec{x}_N|\theta) = \int_0^{2\pi}\int_0^{2\pi} \frac{d\varphi}{2\pi}\frac{d\varphi'}{2\pi}\prod_{i=1}^{N} u_\theta^*(x_i,\varphi;t)u_\theta(x_i,\varphi';t)\sum_{n,m=0}^{N}\frac{C_n C_m \cos\left[\varphi\left(\frac{N}{2}-n\right)\right]\cos\left[\varphi'\left(\frac{N}{2}-m\right)\right]}{\sqrt{\binom{N}{n}\binom{N}{m}}}$$

The Fisher Information

Work hard...

$$F = \int d\vec{x}_N \frac{1}{p_N(\vec{x}_N|\theta)}\left[\partial_\theta p_N(\vec{x}_N|\theta)\right]^2 = 4\sum_n c_n^2\left(n-\frac{N}{2}\right)^2 = 4\Delta^2 \hat{J}_z$$

Saturation of the Quantum Fisher Information

Can be sub shot-noise

In fact $\sqrt{N}$ is enough...

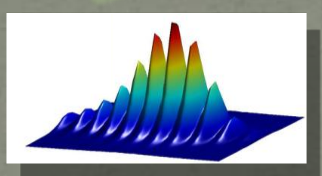
Results

- Identify the „good" states: „phase squeezing" ✓
- Detection scheme ✓

Basic tool – N-body probability

$$p(x_1 \ldots x_N|\theta) = \frac{1}{N!}\langle \hat{\Psi}^\dagger(x_1)\ldots\hat{\Psi}^\dagger(x_N)\hat{\Psi}(x_N)\ldots\hat{\Psi}(x_1)\rangle$$

Fit to the density | Correlation functions | Center of mass



$$\Delta\theta \geq \frac{1}{\sqrt{N}}$$

No sub shot-noise sensitivity

$$g_k(x_1 \ldots x_k|\theta)$$

$$\Delta\theta < \frac{1}{\sqrt{N}}$$

Only when $k > \sqrt{N}$

$$p_{cm}(x|\theta)$$

$$\Delta\theta < \frac{1}{\sqrt{N}}$$

Only when all atoms are measured

J. Ch., F. Piazza and A. Smerzi, PRA 82, 051601(R) (2010)

Conclusions

- Interference pattern „kills" the modes
- Useful correlations between the particles
- Very difficult to obtain sub shot-noise sensitivity
- Do Mach-Zehnder!