4 marca 2011
19:08

$$A_\sigma(k,n) = \int_{SL(2,\mathbb{C})^{n}} \prod_a dX_a \; \underbrace{\delta(X_s)}_{\text{gauge fixing}} \; \prod_{a<b} \rho_{ab}$$

<u>Kernel</u>: $$\rho_{ab} = \int_{\mathbb{CP}^1} \Omega \, \langle X_a^\dagger z, X_a^\dagger z \rangle^{-1-i\rho_{ab}-n_{ab}} \langle X_a^\dagger z, \xi_{ab} \rangle^{2n_{ab}} \times$$

$$\times \langle X_b^\dagger z, X_b^\dagger z \rangle^{-1+i\rho_{ab}-n_{ab}} \langle J\xi_{ba}, X_b^\dagger z \rangle^{2n_{ab}}$$

where: $\langle , \rangle$ Hermitian inner product on $\mathbb{C}^2$

$\zeta \in \mathbb{C}^2$ are CS in the $k=\frac{1}{2}$ representation

$(n,\rho) = (k, \gamma k)$

II Asymptotic analysis

Semi-classical limit:

$$\text{Area}(ab) \sim 8\pi G \hbar \gamma k_{ab}$$

$\hbar \to 0$ keeping $\text{Area}(ab)$ fixed

$\hookrightarrow$ All spins rescalled $k_{ab} \to \lambda k_{ab}$ with $\lambda \to \infty$

II.1 Asymptotic problem and critical points

Notation: $Z_{ab} = X_a^{+} z_{ab}$, $Z_{ba} = X_b^{+} z_{ab}$.
$a<b$

Amplitude rewritten as:

$$A_\Gamma(k,n) = \int_{SL(2,\mathbb{C})^n} \prod_a dX_a \, \delta(X_5) \int_{\mathbb{C}P^n} \prod_{a<b} \Omega_{ab} \, e^{\lambda S_{kn}[X,z]}$$

/

$$\Omega_{ab} = \frac{\Omega}{\langle Z_{ab}, Z_{ab}\rangle \langle Z_{ba}, Z_{ba}\rangle}$$

with the action S given by :

$$S_{k,n}[X, z] = \sum_{a<b} k_{ab} \ln \frac{\langle Z_{ab}, \zeta_{ab}\rangle^2 \langle J\zeta_{ba}, Z_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle \langle Z_{ba}, Z_{ba}\rangle}$$

$$+ i\gamma k_{ab} \ln \frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}$$

$\hookrightarrow$ Asymptotics of A_σ can be computed using stationary phase

Asymptotic formula dominated by critical points of S , i.e, stationary points ($\delta S = 0$) for which Re S is a maximum

Critical points :

i) Condition on Re S

$\mathrm{Re}\, S \leq 0$

$\mathrm{Re}\, S = 0$

$$\Longleftrightarrow (1)\ (X_a^{+})^{-1} \zeta_{ab} = \frac{\|Z_{ba}\|}{\|Z_{ab}\|} e^{i\theta_{ab}} (X_b^{+})^{-1} J \zeta_{ba}$$

$\forall\, a < b$

θ_{ab} is a phase , $\|z\|^2 = \langle x, x\rangle$

θ_{ab} is a phase , $\|z\|^2 = \langle X, X \rangle$

(ii) Evaluation δS on (1)

$\circledast$ $\delta_{z_{ab}} S = 0$

on shell $\Leftrightarrow$ $\quad X_a \zeta_{ab} = \frac{\|z_{ab}\|}{\|z_{ba}\|} e^{i\theta_{ab}} X_b J \zeta_{ba} \quad (2)$

$\forall\, a < b$

$\circledast$ $\delta X = X \circ L \;,\; \delta X^\dagger = L^\dagger \circ X^\dagger$

$L \in sl(2,\mathbb{C})_{\mathbb{R}}$

$$\delta_{X_a} S = 0 \overset{\text{on shell}}{\Leftrightarrow} \forall a,\ \sum_{b:\, b\neq a} k_{ab}\, n_{ab} = 0 \quad (3)$$

Use: $L = \alpha^i J_i + \beta^i K_i \;,\; \alpha^i, \beta^i \in \mathbb{R}$

$J_i = \frac{i}{2}\sigma_i \;,\; K_i = \frac{1}{2}\sigma_i$

$\left(sl(2,\mathbb{C})_{\mathbb{R}} = (su(2)^{\mathbb{C}})_{\mathbb{R}} = su(2) \oplus i\, su(2) \right)$

+ properties of σs

II.2 Geometry of critical points

Idea : bivectorology

Geometry of critical points : spinors $\longleftrightarrow$ null vectors

Identification :

i) $\gamma : \mathbb{R}^{3,1} \longrightarrow \mathbb{H}$

$x \longmapsto \gamma(x) = x^0 \mathbb{1} + x^i \sigma_i$

$\mathbb{H}$: 2x2 hermitian matrices

$\det \gamma(x) = -\eta(x,x) \quad , \quad \eta = -+++$

ii) $\mathbb{H}_0^+ = \{ h \in \mathbb{H} \mid \det h = 0 \; , \; \mathrm{tr}\, h > 0 \}$

↳ γ identifies future null cone $\mathcal{C}^+ \subset \mathbb{R}^{3,1}$ with $\mathbb{H}_0^+ \subset \mathbb{H}$

iii) $\varsigma : \mathbb{C}^2 \longrightarrow \mathbb{H}_0^+$

$$z \longmapsto \varsigma(z) = z \otimes z^\dagger = \begin{pmatrix} |z_0|^2 & z_0 \bar{z}_1 \\ z_1 \bar{z}_0 & |z_1|^2 \end{pmatrix}$$

↳ Define a map

$n : \mathbb{C}^2 \longrightarrow \mathcal{C}^+ \subset \mathbb{R}^{3,1}$

The map n associates 2 null vectors to the coherent state ς :

$$n(\varsigma) = \frac{1}{2}(1, \vec{n})$$

time component spatial part

$$n(J\zeta) = \frac{1}{2}(1, -\vec{n})$$

$$\zeta(\zeta) = \zeta \otimes \zeta^{+} = \frac{1}{2}(\mathbb{1} + \sigma \cdot n) \xrightarrow{\gamma^{-1}} \frac{1}{2}(1, \vec{n})$$

From these 2 vectors, construct the bivector

$$b = 2 * n(J\zeta) \wedge n(\zeta) = * \mathcal{T} \wedge \mathcal{N}$$

$$\mathcal{T} = (1, \vec{0}) \qquad \mathcal{N} = (0, \vec{n})$$

$$* : \Lambda^2(\mathbb{R}^{3,1}) \to \Lambda^2(\mathbb{R}^{3,1}) \quad \text{Hodge}$$

$$b = * \begin{bmatrix} 0 & \vec{n} \\ \vec{n} & \bigcirc \end{bmatrix}$$

$\bigcirc$: 3x3 matrix

To each coherent state ζ_{ab}, associate the bivector

$$B_{ab} = k_{ab} \, \hat{X}_a \otimes \hat{X}_a \, b_{ab}$$

$$\hat{X}_a \in SO(3,1)$$

Critical point equations relate to the Barrett-Crane constraints for bivectors

$$B_{ab} = \mu \, B_{ab}(\sigma) \qquad \mu = \pm 1$$

bivectors of a geometric 4-simplex σ in $\mathbb{R}^{1,n}$

III Asymptotic formula

On the critical points,

$$S_{k,n} = i\gamma \sum_{a<b} k_{ab} \ln \frac{\|Z_{ba}\|^2}{\|Z_{ab}\|^2} + 2k_{ab}\theta_{ab}$$

Couple eqns (1) and (2) :

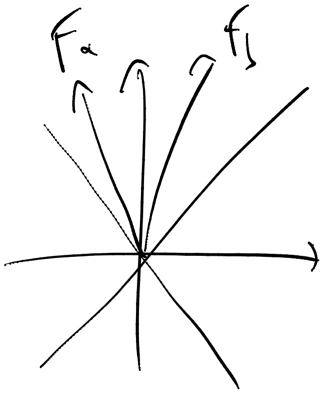
$$X_a^{-1} X_b X_b^{+} (X_a^{+})^{-1} \zeta_{ab} = \frac{\|Z_{ba}\|^2}{\|Z_{ab}\|^2} \zeta_{ab}$$

Using $(J \cdot n)\zeta = \frac{i}{2}\zeta$, $(K \cdot n)\zeta = \frac{1}{2}\zeta$

$$X_a^{-1} X_b X_b^{+} (X_a^{+})^{-1} = e^{2r_{ab} K \cdot n_{ab}}$$

$$= g(n_{ab}) \begin{pmatrix} e^{r_{ab}} & 0 \\ 0 & e^{-r_{ab}} \end{pmatrix} g(n_{ab})^{-1}$$

$$e^{r_{ab}} = \frac{\|Z_{ba}\|^2}{\|Z_{ab}\|^2}$$



Thick wedges
[Barrett, Foxon '94]

$$\cosh \theta_{ab} = - F_a \cdot F_b$$

$$= \frac{1}{2} \operatorname{tr}\left(\gamma(F_a)^{-1} \gamma(F_b)\right)$$

$$\Rightarrow \frac{1}{2} \operatorname{tr} (X_a^\dagger)^{-1} X_a^{-1} X_b X_b^\dagger = \cosh r_{ab}$$

$$= \cosh \Theta_{ab}$$

$$Y(F_a) = X_a X_a^\dagger$$

$$\Rightarrow \quad |\Theta_{ab}| = |r_{ab}|$$

Canonical choice of phase : $\theta_{ab} = 0$

Action at critical points :

$$S_{k,n} = i\gamma \sum_{a<b} k_{ab} \Theta_{ab} \qquad \text{Regge action}$$

III.2 Asymptotic formula

$$A_v(\lambda k, n) \sim \left(\frac{1}{\lambda}\right)^{12} \left[N_+ e^{i\lambda\gamma \sum_{a<b} k_{ab} \Theta_{ab}} + N_- e^{-i\lambda\gamma \sum_{a<b} k_{ab} \Theta_{ab}} \right]$$

Parity related solutions