QUANTUM DEFORMATION OF THE LORENTZIAN EPRL MODEL

based on [arXiv: 1012.4784] with C. Meusburger

From the Ponzano-Regge model (3d QG, $\Lambda = 0$)
to the Turaev-Viro (3d QG, $\Lambda > 0$)

M triangulated 3-manifold

$\pi_I : SU(2) \to \mathrm{Aut}\, V_I$, $I \in \mathbb{N}/2$,

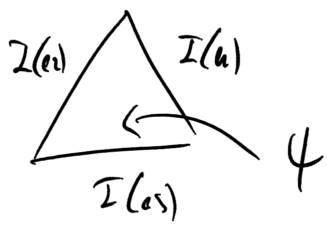
spin I UIR of $SU(2)$

Ingredients:

- Assignments $I : \text{edges} \to \mathrm{Irrep}(SU(2))$
  $e \mapsto I(e)$

  of spins to edges $e \in M$

- State spaces $H_\Delta = \mathrm{Inv}_{SU(2)}\left(\bigotimes_{e \in \partial\Delta} I(e)\right)$

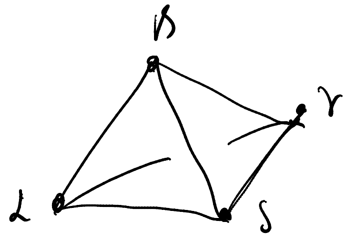


$\cong \mathrm{Hom}_{SU(2)}\left(\bigotimes_{e \in \partial\Delta} I(e), \mathbb{C}\right)$

associated to the triangles of M

Amplitudes: $A_t : \bigotimes_{\Delta \in \partial t} H_\Delta \longrightarrow \mathbb{C}$

$$A_t(\alpha \otimes \beta \otimes \gamma \otimes \delta) = \{6j\}_t$$



The model $PR(M) = \sum_I \prod_e \underbrace{\dim I(e)}_{\dim I = 2I+1} \prod_t \{6j\}_t$

Sum over all possible assignments

[Ponzano-Regge '68]

$PR(M)$ regularization of $Z(M) = \int \mathcal{D}g \, e^{i\int_M d^3x \sqrt{g} R}$

Problem: $PR(M)$ diverges for a large class of M

→ need a regularisation

<u>Idea</u>: quantum group $U_q(su(2))$ with $q^r = 1$ admits only a finite number of irreducible representation

$$I \in \left\{0, \tfrac{1}{2}, 1, \dots, \tfrac{r-2}{2}\right\}$$

↳ Replace $SU(2)$ in PR model by $U_q(su(2))$

$$TV_r(M) = K \sum_{I} \prod_{e} \underbrace{[\dim I(e)]_q}_{q\text{-dimension}} \prod_{t} \underbrace{\{6J\}_q}_{\text{quantum 6J-symbol}}$$

finite sum

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

↳ $TV_r(M)$ converges for all manifolds M

$TV_r(M)$ is a regularisation of

$$Z_\Lambda(M) = \int \mathcal{D}g \, e^{i\int d^3x \sqrt{g}\,(R - 2\Lambda)} , \quad \Lambda > 0$$

$$r = k + 2 , \quad k = \pi / \ell_P \sqrt{\Lambda}$$

Idea: apply the same procedure to regularise the Lorentzian EPRL model

Outline: I The quantum Lorentz group

II The quantum EPRL intertwiner

III The 4-simplex amplitude

$\underline{\text{III}}$ The '4-simplex amplitude'

$\underline{\text{IV}}$ The model

Hopf algebras

G Lie group, $\Pi_V : G \to \mathrm{Aut}\, V$

representation on V (field k)

Consider:

i) Tensor product of representations

$g \in G$, $\quad \Pi_{V\otimes W}(g) = \Pi_V(g) \otimes \Pi_W(g)$

ii) Dual representation

$\alpha \in V^*$ $\quad \Pi_V^*(g)\,\alpha = \alpha \circ \Pi_V(g^{-1})$

iii) Trivial representation

$\Pi_k(g) = \mathrm{id}_k$

Question: Suppose I replace G with an (associative, unital) H. What structure do I need to put on H for representations to mimic those of G?

i) Co-product $\Delta : H \to H \otimes H$

$a \in H$ $\quad \Pi_{V\otimes W}(a) = \Pi_V \otimes \Pi_W(\Delta(a))$

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$(\Delta g = g \otimes g)$

ii) Antipode : $S : H \to H$

$\Pi_V^+(a)\, \alpha = \alpha \circ \Pi(S(a))$

$(S(g) = g^{-1})$

iii) Co-unit $\varepsilon : H \to k$

$\Pi_k(a) = \varepsilon(a)\, id_k$

↳ These structures must satisfy compatibility and consistency conditions

↳ H becomes a Hopf algebra

I The quantum Lorentz group (QLG)

I.1 Hopf algebra structures

QLG : quantum double of $U_q(su(2))$
with $q = e^{-\kappa} \in]0,1[$ real

Ingredients : i) Hopf algebra $U_q(su(2))$

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⊛ Algebra structure : generators $q^{\pm J_z}$, $J_\pm$

+ relations : $q^{\pm J_z} q^{\mp J_z} = \mathbb{1}$

$$q^{J_z} J_\pm q^{-J_z} = q^{\pm 1} J_\pm$$

$$[J_+, J_-] = [2 J_z]_q$$

⊛ Co-algebra structure :

$$\Delta(q^{\pm J_z}) = q^{\pm J_z} \otimes q^{\pm J_z},$$

$$\Delta(J_\pm) = q^{-J_z} \otimes J_\pm + J_\pm \otimes q^{J_z}$$

$$\varepsilon(q^{\pm J_z}) = 1 \quad , \quad \varepsilon(J_\pm) = 0$$

⊛ Antipode $S(q^{\pm J_z}) = q^{\mp J_z}$

$$S(J_\pm) = -q^{\pm 1} J_\pm$$

⊛ UIR behave as in the classical case

→ $\pi_I : U_q(su(2)) \to \mathrm{End}\, V_I$, $I \in \mathbb{N}/2$,

spin I UIR of $U_q(su(2))$, $V_I = \mathbb{C}\{e_a\}_{a=-I \dots I}$

→ Fusion rules

$$V_I \otimes V_J \cong \bigoplus_{K=|I-J|}^{I+J} V_K$$

Clebsch-Gordon maps (CG)

$$C^K{}_{IJ} : V_I \otimes V_J \to V_K \quad , \quad C^{IJ}{}_K : V_K \to V_I \otimes V_J ,$$

defined as :

$$C^K{}_{IJ}\left(\overset{I}{e}_b \otimes \overset{J}{e}_c\right) = \sum_a{}' \begin{pmatrix} a & I & J \\ K & b & c \end{pmatrix} \overset{K}{e}_a \quad ,$$

$$C^{IJ}{}_K\left(\overset{K}{e}_a\right) = \sum_{b,c}{}' \begin{pmatrix} b & c & K \\ I & J & a \end{pmatrix} \overset{I}{e}_b \otimes \overset{J}{e}_c$$

CG coefficients ($\in \mathbb{R}$)

ii) Dual Hopf algebra

$$U_q(su(2))^* = F_q(SU(2))$$

↳ q-analogue of the algebra of functions on $SU(2)$

QLG : Hopf algebra

$$D\,U_q(su(2)) = U_q(su(2)) \hat{\otimes} F_q(SU(2))^{op}$$

2 copies do not commute

Rem : This is a quantum deformation of

$$sl(2,\mathbb{C})_{\mathbb{R}} \cong so(3,1)$$

Iwasawa decomposition : $sl(2,\mathbb{C})_{\mathbb{R}} \cong su(2) \oplus an(2)$

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$an(2)$ Lie algebra $AN(2) = \left\{ \begin{pmatrix} \lambda & n \\ 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda \in \mathbb{R}^{+*}, n \in \mathbb{C} \right\}$

($SL(2,\mathbb{C})_{\mathbb{R}} \cong SU(2) \ltimes AN(2)$)

$$D\, U_q(su(2)) = U_q(su(2)) \hat{\otimes} U_q(an(2))$$

$$\mathcal{L}_s \left| \; D\, U_q(su(2)) = U_q(sl(2,\mathbb{C})_{\mathbb{R}}) \right.$$

II.2 Irreducible representations

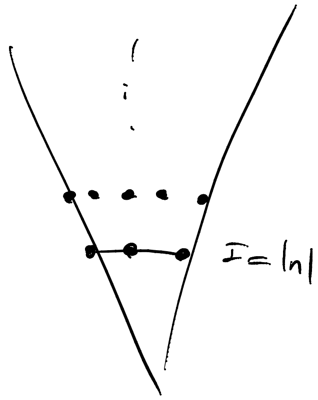
Unitary irreducible representations (of the principal series) of the QLG are labelled

by $\alpha = (n, p)$, $n \in \mathbb{Z}/2$, $p \in [0, \frac{4\pi}{\kappa}[$

($q = e^{-\kappa}$)

Decomposition in terms of representations of $U_q(su(2))$

$$V_\lambda = \bigoplus_{I=|n|}^{\infty} V_I$$



II The quantum EPRL intertwiner

$$\mathrm{Irrep}\left(U_q(sl(2,\mathbb{C})_{\mathbb{R}})\right) = \left\{ (n,\rho) \mid n \in \mathbb{Z}/2,\ \rho \in \left[0, \tfrac{4\pi}{\kappa}\right[\right\}$$

$$\begin{aligned} \phi_r : \mathrm{Irrep}(U_q(su(2))) &\longrightarrow \mathrm{Irrep}(U_q(sl(2,\mathbb{C})_{\mathbb{R}})) \\ K &\longmapsto (n(K), \rho(K)) := \underbrace{(K, rK)}_{q\text{-EPRL representation}} \end{aligned}$$

$r \in \mathbb{R}^+$ fixed parameter

Rem: pre-image of ϕ_r is restricted

$$\mathcal{L} = \left\{ K \in \mathbb{N}/2 \;\middle|\; K \leq \frac{4\pi}{r\kappa} \right\}$$

II.2 Quantum EPRL intertwiner

Need to generalise expressions of the form

$$\int_{SL(2,\mathbb{C})_{\mathbb{R}}} dX \bigotimes_{i=1}^{n} \Pi_{\lambda_i}(X) : \bigotimes_{i=1}^{n} V_{\lambda_i} \to \bigotimes_{i=1}^{n} V_{\lambda_i} \quad (\mathrm{I})$$

Introduce a Haar measure on the Hopf algebra $F_q(SL(2,\mathbb{C})_{\mathbb{R}})$ dual to $U_q(sl(2,\mathbb{C})_{\mathbb{R}})$

$$h : F_q(SL(2,\mathbb{C})_{\mathbb{R}}) \longrightarrow \mathbb{C}$$

[Buffenoir, Roche '99]

<u>Satisfies</u>: $(h \otimes id)\,\Delta(x) = (id \otimes h)\,\Delta(x) = h(x)\,1$

<u>Ex</u>: $C(G)$, $h(f) = \int_G dg\, f(g)$

(I) is generalised as

$$T_{\alpha_1,\dots,\alpha_n} = \sum_A \hat{\bigotimes_{i=1}} \Pi_{\alpha_i}\left(\Delta^{(n)}(x^A)\right)\, h(x_A)$$

dual basis of $U_q(sl(2,\mathbb{C})_{\mathbb{R}})$

basis of $F_q(SL(2,\mathbb{C})_{\mathbb{R}})$

[Noui, Roche '02]

The EPRL representation $\alpha(K) = (K, \gamma K)$ factorises as $V_{\alpha(K)} = \bigoplus_{I \geq K} V_I$

Let $f_\alpha^K : V_\alpha \to V_K$ be the projection on lowest weight factor on lowest weight factor

The dual map induces an embedding

$$f^*: \mathrm{Hom}_{U_q(su(2))}\left(\bigotimes_{i=1}^{n} V_{K_i}, \mathbb{C}\right) \to \mathrm{Hom}_{U_q(sl(2,\mathbb{C})_{\mathbb{R}})}\left(\bigotimes_{i=1}^{n} V_{\lambda_i}, \mathbb{C}\right)$$

The EPRL intertwiner is defined as

$$\Lambda_{\lambda(K)} = f^*(\Lambda_K) = \Lambda_K \circ \bigotimes_{i=1}^{n} f_{\lambda_i}^{K_i} \circ T_{\lambda_1,\dots,\lambda_n}$$

where $\Lambda_K : \bigotimes_{i=1}^{n} V_{K_i} \to \mathbb{C}$ is an $U_q(su(2))$ – intertwiner.

infinite sum

<u>Theorem</u>: The evaluation of the q-EPRL intertwiner $\Lambda_{\lambda(K)} = f^*(\Lambda_K)$ on the basis element $\bigotimes_{i=1}^{4} e^{L_i}_{c_i}$ of $\bigotimes_{i=1}^{4} V_{\lambda_i}$ is a multiple series that converges absolutely.

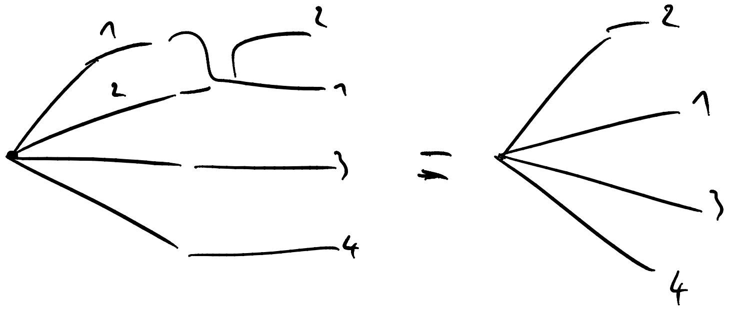
$\{ e^{L}_{c} \mid L \geq |n|,\ c = -L, \dots, L \}$ basis of V_λ

<u>Property</u>: The q-EPRL intertwiner is <u>not</u> invariant under the braiding

$$C_{\alpha_1,\alpha_2} = \tilde{\tau}_{\alpha_1,\alpha_2} \circ \Pi_{\alpha_1}\otimes\Pi_{\alpha_2}(R) : V_{\alpha_1}\otimes V_{\alpha_2} \to V_{\alpha_2}\otimes V_{\alpha_1}$$

permutation R-matrix of QLG

<u>Rem</u>: The q-BC intertwiner [Noui, Roche '02] <u>is</u> invariant

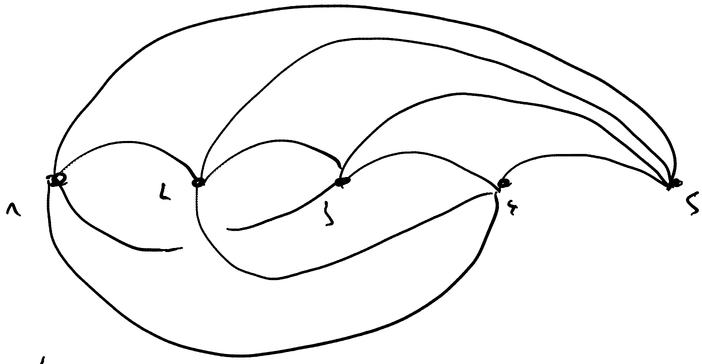


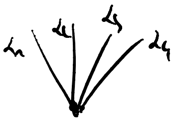
<u>III The 4-simplex amplitude</u>

We define a procedure to associated a number (possibly infinite) $ev(\Gamma_n)$ to all closed diagrams Γ_n with n vertices

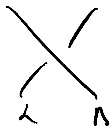
III.2 Amplitude for the 4-simplexes

$A_\sigma = ev(\Gamma_5)$ 4-simplex amplitude

$T_5 =$ 


q-EPRL intertwiner


bilinear form $V_\alpha \otimes V_\alpha \to \mathbb{C}$


$c_{\alpha,\beta}$

Theorem: $ev(T_5)$ is a multiple series that converges absolutely

IV The model

M triangulated 4-manifold

$$Z_\kappa(M) = \sum_{\kappa,\mathcal{J}}{}' \prod_\Delta [2\kappa(\Delta)+1]_q \prod_\sigma A_\sigma(\kappa(\Delta), \psi_\sigma)$$

sum ranges over all $\kappa \in \mathcal{L}$

Property: $Z_\kappa(M)$ is finite

Physics: Setting $q = e^{-\ell_P^2/\ell_c^2}$, $\ell_P = \sqrt{G\hbar}$, $\ell_c = \frac{1}{\sqrt{\Lambda}}$

we obtain a bound:

$$\mathrm{Area}(\Delta) \leq 32\pi^2 \ell_c^2 \quad (\ell_p \ll \ell_c)$$

for the area of Δ.