Renormalization of NC Quantum Fields: The GW Model Zakopane 11.3.2011

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Introduction

- Motivation:
  - Two pillars: QT and relativity: Do they match?
  - Improve QFT in 4 dimensions, add "gravity" effects
- Renormalizable local Quantum Fields: triviality, not summmable?
- Space-Time
- Renormalizable Noncommutative Quantum Fields formulation, IR/UV mixing
- GW Model H G + R Wulkenhaar ...
  - Renormalizable ncQFTs: Scalar Higgs model
  - Taming the Landau Ghost
  - Borel summable almost solvable, nontrivial !
- Fermions
- Gauge models + M Wohlgenannt
- properties: Wedge locality + G Lechner
- Outlook

Introduction

- 4 Fermi interaction is not renormalisable, needs cutoff around 300 GeV, unitarity! Solved by W^+, Z^0, W^-,...
- Quantum field theory for standard model (electroweak+strong) is renormalisable
- Gravity is not renormalisable !?
- Classical field theories for fundamental interactions (electroweak, strong, gravitational) are of geometrical origin

Renormalisation group interpretation

- space-time being smooth manifold $\Rightarrow$ gravity scaled away
- weakness of gravity determines Planck scale where geometry is something different

promising approach: noncommutative geometry
unifies standard model with gravity as classical field theories

Program

- Wightman QFT — Euclidean QFT
  - UV, IR, convergence problem, Landau ghost, triviality
  - NC QFT ideas, formulation, regularization
  - Euclidean NC QFT IR/UV mixing
- Euclidean NC QFT
  - Renormalization
  - taming Landau ghost, β function vanishes
- Construction
  - Ward identity
  - Schwinger-Dyson equation
  - integral equation for renormalized N pt functions
- Fermions - Spectral triple
- gauge Higgs model-matrix model: Spectral action principle
- Wedge locality
- Outlook

Requirements

Quantum mechanical properties

- states are vectors of a separable Hilbert space H
- $\Phi(f)$ on D dense, $\Phi(f) = \int d^4x \Phi(x) f^*(x)$,
  Ω is cyclic
- Space-time translations are symmetries:
  spectrum $\sigma(P_\mu)$ in closed forward light cone
  Ground state $\Omega \in H$ invariant under $e^{ia_\mu P^\mu}$

Relativistic properties

- $U_{(a,\Lambda)}$ unitary rep. of Poincaré group on H, Covariance
- Locality
  $[\phi(f), \phi(g)]_\pm \Psi = 0 \qquad \text{for} \qquad suppf \subset (suppg)'$

Define Wighman functions $W_N(f_1 \otimes ... \otimes f_N) :=< \Omega\phi(f_1) \cdots \phi(f_N)\Omega >$

Euclidean

Schwinger distributions: $S_N(x_1, ..., x_N) = \int \phi(x_1)...\phi(x_N) d\nu(\phi)$

$$S_N(x_1 \dots x_N) = \frac{1}{Z} \int [d\phi]\, e^{-\int dx \mathcal{L}(\phi)} \prod_i^N \phi(x_i)$$

extract free part $d\mu(\phi) \propto [d\phi]\, e^{-\frac{m^2}{2}\int \phi^2 - \frac{a}{2}\int (\partial_\mu \phi)(\partial^\mu \phi)}$

Two point correlation: $\langle \phi(x_1)\phi(x_2)\rangle = C(x_1, x_2)$

$\tilde{C}(p_1, p_2) = \delta(p_1 - p_2)\frac{1}{p_1^2 + m^2}$

$$\int d\mu(\phi)\phi(x_1)\dots\phi(x_N) = \sum_{\text{pairings}} \prod_{l\in\gamma} C(x_{i_l} - x_{j_l})$$

add $\frac{\lambda}{4!}$ Φ^4 interaction, expand

Φ^4 Interaction

$$S_N(x_1 \dots x_N) = \sum_n \frac{(-\lambda)^n}{n!} \int d\mu(\phi) \prod_j^N \phi(x_j) \left(\int dx \, \frac{\phi^4(x)}{4!} \right)^n$$

$$= \sum_{\text{graph } \Gamma_N} \frac{(-\lambda)^n}{Sym_{\Gamma_N}(G)} \int_V \prod_{l \in \Gamma_N} C_\kappa(x_l - y_l) \sim \Lambda^{\omega_D(G)}$$

put cutoffs, e.g.: $\tilde{C}_\kappa(p) = \int_{\kappa=1/\Lambda^2}^{\infty} d\alpha e^{-\alpha(p^2+m^2)}$

degree of divergence given by $\omega_D(G) = (D-4)n + D - \frac{D-2}{2}N$

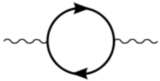
$\omega_2(G) = 2 - 2n,$ $\omega_4(G) = 4 - N$

n order # of vertices, N # of external lines, $(4n+N)!!$ is nr of Feynman graphs

use Stirling and $\frac{1}{n!}$ from exponential large order behavior $K^n n!$ ... no Taylor (Borel) convergence

Renormalization

For QED: polarization + higher order terms....



Leads to Vacuum fluctuations, Casimir effect, Lambshift, pair production, virtual particles...

$$\tilde{G}_2(p) = \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Sigma\frac{1}{p^2+m^2} + ... = \frac{1}{p^2+m^2-\Sigma}$$

Impose renormalization conditions need 3

$$G_2(p^2=0) = \frac{1}{m^2_{phys}},\ \frac{d}{dp^2}G_2(p^2=0) = -\frac{a^2}{m^4_{phys}},\ G_4(p^2=0) = \lambda_{phys}$$

$a = Z, m, \lambda$ depend on scale ! Renormalize

if NO NEW interactions are generated.... model is renormalizable

RG FLOW

Wilson RG-Flow

divide covariance for free Euclidean scalar field into slices

$$\Phi_m = \sum_{j=0}^{m} \phi_j, \quad C_j = \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \frac{e^{-m^2\alpha - x^2/4\alpha}}{\alpha^{D/2}}$$

integrate out degrees of freedom

$$Z_{m-1}(\Phi_{m-1}) = \int d\mu_m(\phi_m) e^{-S_m(\phi_m + \Phi_{m-1})}$$

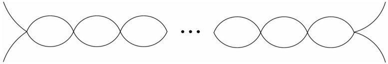
$$Z_{m-1}(\Phi_{m-1}) = e^{-S_{m-1}(\Phi_{m-1})}$$

to evaluate: use loop expansion

Landau Ghost

QFT on undeformed R^4

- superficial degree of divergence for Graph Γ $\omega(\Gamma) = 4 - N$
- BPHZ Theorem: renormalizability generate no new terms
- but: certain chain of finite subgraphs with m bubbles grows like



$$\int \frac{d^4 q}{(q^2+m^2)^3} (\log|q|)^m \simeq C^m m!$$

- not Borel summable

$$\lambda_j \simeq \frac{\lambda_0}{1 - \beta\lambda_0 j}$$

- sign of β positive: Landau ghost, triviality QED, Higgs model,...
- sign of β negative: Asymptotic freedom, QCD

Space-Time, History

- Interacting models in $D = 2, 3$ are constructed
- $D = 4$ use renormalized pertubation theory RG FLOW
- add "Gravity" or quantize Space-Time

Project

merge general relativity with quantum physics through noncommutative geometry

Limited localisation in space-time $D \geq R_{ss} = G/c^4 hc/\lambda \geq G/c^4 hc/D$

$D \geq l_p \quad 10^{-35} m$

- Riemann, Schrödinger, Heisenberg,...
- 1947 Snyder,...
- 1986 Connes NCG,...
- 1992 H G and J Madore,...
- 1995 Filk Feynman rules,...
- 1999 Schomerus: obtained nc models from strings,...

Algebra, fields, diff. calculus,...

- Space-Time, Gelfand - Naimark theorem,
- Deform Algebra, gives associative nonlocal star product

Moyal space

algebra of decaying funct over D-dim Eucl. space, $\star$-product

$$(a \star b)(x) = \int d^D y d^D k a(x + \tfrac{1}{2}\Theta \cdot k) b(x+y)\, e^{iky}$$

where $\Theta = -\Theta^T \in M_D(\mathbb{R})$

- Fields, sections of bundles,...proj. modules over A
- Differential Calculus

Can we make sense of renormalisation in NCG?

First step: construct QFT on simple nc geometries, e.g. the Moyal space

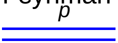
- construction of field theories with non-local interaction

Model - Feynman rules

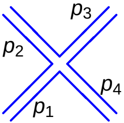
- naïve ϕ^4-action (ϕ-real, Euclidean space) on Moyal plane

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$$

- Feynman rules:



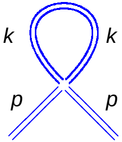
$$= \frac{1}{p^2 + m^2}$$



$$= \frac{\lambda}{4} \exp \Big(-\frac{\mathrm{i}}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu} \Big)$$

- cyclic order of vertex momenta is essential
  $\Rightarrow$ ribbon graphs

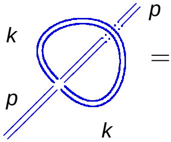
- one-loop two-point function, *planar contribution*:



$$= \frac{\lambda}{6} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}$$

to be treated by usual regularisation methods, can be put to 0

- planar nonregular contribution:



$$= \frac{\lambda}{12} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot\Theta\cdot p}}{k^2+m^2} \sim (\Theta p)^{-2}$$

- non-planar graphs finite (noncommutativity as a regulator) but $\sim p^{-2}$ for small momenta (renormalisation not possible)

$\Rightarrow$ leads to non-integrable integrals when inserted as subgraph into bigger graphs: UV/IR-mixing

The UV/IR-mixing problem and its solution

- *observation*: euclidean quantum field theories on Moyal space suffer from UV/IR mixing problem which destroys renormalisability if quadratic divergences are present

Theorem

The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \Big)(x)$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$, ϕ – real, Euclidean metric
is perturbatively renormalisable to all orders in λ.

The additional oscillator potential $\Omega^2 \tilde{x}^2$

- implements mixing between large and small distance scales
- results from the renormalisation proof
- add. term connected to curvature M Buric + M Wohlgenannt

The matrix base of the Moyal plane

- central observation (in 2D):
  $f_{00} := 2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)} \quad \Rightarrow \quad f_{00} \star f_{00} = f_{00}$

- left and right creation operators:
  $$f_{mn}(x_1,x_2) = \frac{(x_1+\mathrm{i}x_2)^{\star m}}{\sqrt{m!(2\theta)^m}} \star \left(2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)}\right) \star \frac{(x_1-\mathrm{i}x_2)^{\star n}}{\sqrt{n!(2\theta)^n}}$$
  $$f_{mn}(\rho,\varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}}\mathrm{e}^{\mathrm{i}\varphi(n-m)} \left(\sqrt{\frac{2}{\theta}}\rho\right)^{n-m} \mathrm{e}^{-\frac{\rho^2}{\theta}} L_m^{n-m}\left(\frac{2}{\theta}\rho^2\right)$$

- satisfies: $(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$
  $$\int d^2x\, f_{mn}(x) = \delta_{mn}$$

- Fourier transformation has the same structure

Extension to four dimensions

non-vanishing components: $\theta = \Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43}$
double indices

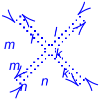
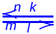
non-local $\star$-product becomes simple *matrix product*

$$S[\phi] = \sum_{m,n,k,l \in \mathbb{N}^2} \left(\frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right)$$

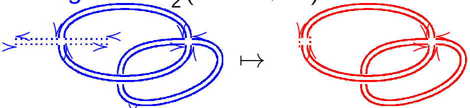
important: $\Delta_{mn;kl} = 0$ unless $m - l = n - k$
$SO(2) \times SO(2)$ angular momentum conservation

- diagonalisation of Δ yields recursion relation for Meixner polynomials
- closed formula for propagator $G = (\Delta)^{-1}$
- $G_{\substack{m\ m\\0\ 0};\substack{m\ m\\0\ 0}} \sim \frac{\theta/8}{\sqrt{\frac{4}{\pi}(m+1) + \Omega^2 (m+1)^2}}$
- $G_{\substack{m_1\ m_1\\m_2\ m_2};\substack{0\ 0\\0\ 0}} = \frac{\theta}{2(1+\Omega)^2 (m_1+m_2+1)} \left(\frac{1-\Omega}{1+\Omega} \right)^{m_1+m_2}$

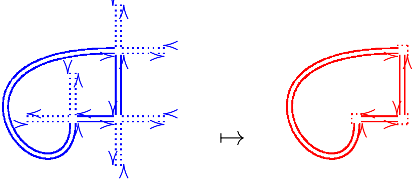
Ribbon graphs

Feynman graphs are ribbon graphs with V vertices  and I edges  $= G_{mn;kl}$ and N external legs

- leads to F faces, B of them with external legs
- ribbon graph can be drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(F - I + V)$ with B holes



$F = 1$	$g = 1$
$I = 3$	$B = 1$
$V = 2$	$N = 2$

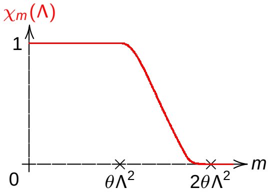


$L = 2$	$g = 0$
$I = 3$	$B = 2$
$V = 3$	$N = 6$

Exact renormalisation group equations

QFT defined via partition function $Z[J] = \int \mathcal{D}[\phi]\, e^{-S[\phi]-\mathrm{tr}(\phi J)}$

- Wilson's strategy: integration of field modes ϕ_{mn} with indices $\geq \theta\Lambda^2$ yields effective action $L[\phi, \Lambda]$
- variation of cut-off function $\chi(\Lambda)$ with Λ modifies effective action:



exact renormalisation group equation [Polchinski equation]

$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} Q_{mn;kl}(\Lambda) \left(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\, \partial \phi_{kl}} \right)$$

with $Q_{mn;kl}(\Lambda) = \Lambda \frac{\partial (G_{mn;kl}\, \chi_{mn;kl}(\Lambda))}{\partial \Lambda}$

- renormalisation = proof that there exists a regular solution which depends on only a finite number of initial data

Renormalisation

power-counting degree of divergence of graphs Γ
$\omega(\Gamma) = 4 - N - 4(2g + B - 1)$

Conclusion

All non-planar graphs and all planar graphs with ≥ 6 external legs are convergent

Problem: infinitely many planar 2- and 4-leg graphs diverge
Solution: discrete Taylor expansion about reference graphs
difference expressed in terms of
$|G_{np;pn} - G_{0p;p0}| \leq K_3 M^{-i} \frac{\|n\|}{M^i} e^{-c_3\|p\|}$

- similar for all $A^{planar}_{mn;nk;kl;lm}$ $A^{planar}_{mn;nm}$ and $A^{planar}_{\substack{m^1+1 \\ m^2}\substack{n^1+1 \\ n^2};\substack{n^1 \\ n^2}\substack{m^1 \\ m^2}}$

Renormalisation of noncommutative ϕ^4_4-model to all orders

by normalisation conditions for mass, field amplitude, coupling constant and oscillator frequency

History of the renormalisation proof

- exact renormalisation group equation in matrix base

  H. G., R.Wulkenhaar

  - simple interaction, complicated propagator
  - power-counting from decay rate and ribbon graph topology

- multi-scale analysis in matrix base

  V. Rivasseau, F. Vignes-Tourneret, R.Wulkenhaar

  - rigorous bounds for the propagator (requires large Ω)

- multi-scale analysis in position space

  R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret

  - simple propagator (Mehler kernel), oscillating vertex
  - distinction between sum and difference of propagator ends

- Schwinger parametric representation

  R. Gurau, V. Rivasseau, T. Krajewski,...

- New topological Graph Polynomials and ncQFT

  T. Krajewski, V. Rivasseau, A. Tanasa, Zhituo Wang

Summary

- Renormalisation is compatible with noncommutative geometry
- We can renormalise models with new types of degrees of freedom, such as dynamical matrix models
- Equivalence of renormalisation schemes is confirmed
- Tools (multi-scale analysis) are worked out
- Construction of NCQF theories is promising
- Other models
  - Gross-Neveu model $D = 2$ F. Vignes-Tourneret
  - Degenerate Θ matrix model H. G. F. Vignes-Tourneret needs five relevant/marginal operators !
  - $1/p^2$ model R. Gurau, J. Magnen, V. Rivasseau, A. Tanasa
  - Fermions
  - induced Yang-Mills theory ? A. de Goursac, J.-C. Wallet, R.Wulkenhaar; H. G, M. Wohlgenannt

Taming Landau ghost, calculate β function

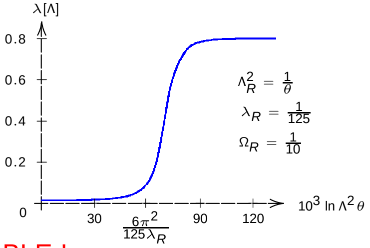
evaluate β function, H. G. and R. Wulkenhaar,

$$\Lambda \frac{d\lambda}{d\Lambda} = \beta_\lambda = \lambda^2 \frac{(1-\Omega^2)}{(1+\Omega^2)^3} + \mathcal{O}(\lambda^3)$$

flow bounded, L. ghost killed!
Due to wave fct. renormalization
$\Omega = 1$ betafunction vanishes
to all orders M. Disertori, R. Gurau, J. Magnen, V. Rivasseau

$$\Omega^2[\Lambda] \leq 1$$

($\lambda[\Lambda]$ diverges in comm. case)



construction possible! Almost SOLVABLE !

Matrix model

- Action in matrix base at $\Omega = 1$
- Action functionals for *bare* mass μ_{bare}
- Wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$.

Fix $\theta = 4$, $\phi_{mn} = \overline{\phi_{nm}}$ real:

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\,,$$

$$H_{mn} = Z\left(\mu^2_{bare} + |m| + |n|\right)\,, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\,,$$

- Λ is cut-off. μ_{bare}, Z divergent
- No infinite renormalisation of coupling constant

$m, n, \ldots$ belong to $\mathbb{N}^2$, $|m| := m_1 + m_2$.

Ward identity

- inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ, infinitesimally
  $\phi_{mn} \mapsto \phi_{mn} + \mathrm{i}\sum_{k\in\mathbb{N}^2_\Lambda}(B_{mk}\phi_{kn} - \phi_{mk}B_{kn})$
- not a symmetry of the action, but invariance of measure $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$ gives

$$0 = \frac{\delta W}{\mathrm{i}\delta B_{ab}} = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\,\Big(-\frac{\delta S}{\mathrm{i}\delta B_{ab}} + \frac{\delta}{\mathrm{i}\delta B_{ab}}(\mathrm{tr}(\phi J))\Big)e^{-S+\mathrm{tr}(\phi J)}$$
$$= \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\,\sum_n \Big((H_{nb} - H_{an})\phi_{bn}\phi_{na} + (\phi_{bn}J_{na} - J_{bn}\phi_{na})\Big)e^{-S+\mathrm{tr}(\phi J)}$$

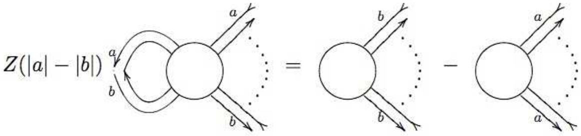
where $W[J] = \ln \mathcal{Z}[J]$ generates connected functions

trick $\phi_{mn} \mapsto \frac{\partial}{\partial J_{nm}}$

$$0 = \Big\{\sum_n\Big((H_{nb} - H_{an})\frac{\delta^2}{\delta J_{nb}\,\delta J_{an}} + \Big(J_{na}\frac{\delta}{\delta J_{nb}} - J_{bn}\frac{\delta}{\delta J_{an}}\Big)\Big)$$
$$\times \exp\Big(-V\Big(\frac{\delta}{\delta J}\Big)\Big)e^{\frac{1}{2}\sum_{p,q} J_{pq}H^{-1}_{pq}J_{qp}}\Big\}_c$$

Interpretation

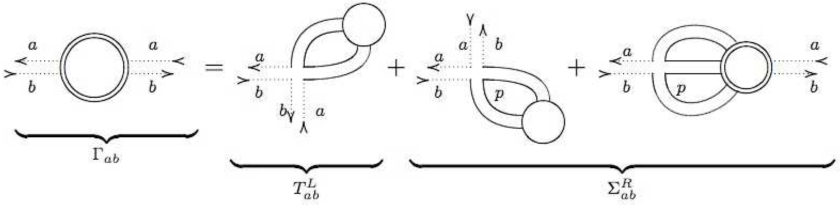
The insertion of a special vertex $V_{ab}^{ins} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$ into an external face of a ribbon graph is the same as the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$



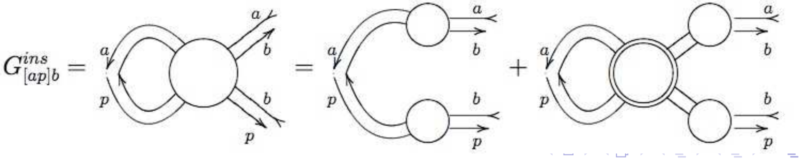
The dots stand for the remaining face indices.

$$Z(|a| - |b|)G^{ins}_{[ab]\dots} = \quad G_{b\dots} \qquad - \quad G_{a\dots}$$

SD equation 2



- vertex is $Z^2\lambda$, connected two-point function is G_{ab}:
  first graph equals $Z^2\lambda \sum_q G_{aq}$
- open p-face in Σ^R and compare with insertion into connected two-point function; insert either into 1P reducible line or into 1PI function:



Amputate upper G_{ab} two-point function, sum over p, multiply by vertex $Z^2\lambda$, obtain: Σ^R_{ab}:

$$\Sigma^R_{ab} = Z^2\lambda \sum_p (G_{ab})^{-1} G^{ins}_{[ap]b} = -Z\lambda \sum_p (G_{ab})^{-1} \frac{G_{bp} - G_{ba}}{|p| - |a|} .$$

case $a = b = 0$ and $Z = 1$ treated.
Use $G^{-1}_{ab} = H_{ab} - \Gamma_{ab}$ and $T^L_{ab} = Z^2\lambda \sum_q G_{aq}$
gives for 2 point function:

$$Z^2\lambda \sum_q G_{aq} - Z\lambda \sum_p (G_{ab})^{-1} \frac{G_{bp} - G_{ba}}{|p| - |a|} = H_{ab} - G^{-1}_{ab} .$$

Symmetry $\Gamma_{ab} = \Gamma_{ba}$ is not manifest!
express SD equation in terms of the 1PI function Γ_{ab}

Renormalize

perform renormalisation for the 1PI part

$$\Gamma_{ab} = Z^2\lambda \sum_p \Big(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}}\frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)}\Big) .$$

Taylor expand: $\Gamma_{ab} = Z\mu_{bare}^2 - \mu_{ren}^2 + (Z-1)(|a|+|b|) + \Gamma_{ab}^{ren}$,
$\Gamma_{00}^{ren} = 0 \qquad (\partial\Gamma^{ren})_{00} = 0$, $\partial\Gamma^{ren}$ is derivative in a_1, a_2, b_1, b_2. Implies

$$G_{ab}^{-1} = |a| + |b| + \mu_{ren}^2 - \Gamma_{ab}^{ren} .$$

... change variables

$$|a| =: \mu^2\frac{\alpha}{1-\alpha} , \quad |b| =: \mu^2\frac{\beta}{1-\beta} , \quad |p| =: \mu^2\frac{\rho}{1-\rho} , \quad |p|\,d|p| = \mu^4\frac{\rho\,d\rho}{(1-\rho)^3}$$

$$\Gamma_{ab} =: \mu^2\frac{\Gamma_{\alpha\beta}}{(1-\alpha)(1-\beta)} , \quad \Lambda =: \mu^2\frac{\xi}{1-\xi}$$

...get expression for ren.constant,

$$Z^{-1} = 1 + \lambda\int_0^\xi d\rho\frac{G_{0\rho}}{1-\rho} - \lambda\int_0^\xi d\rho\Big(G_{0\rho} - \frac{G'_{0\rho}}{1-\rho}\Big)$$

Results

....Since the model is renormalisable, the limit $\xi \longrightarrow 1$ can be taken...:

Theorem

The renormalised planar two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory (with continuous indices) satisfies the integral equation

$$\begin{aligned} G_{\alpha\beta} = 1 + \lambda \Big(& \frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) \\ & + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}) - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) \\ & + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta} - 1)\mathcal{Y}\Big) , \end{aligned}$$

$$\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} , \qquad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho} , \qquad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha}$$

$\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$ *at the self-duality point.*

expansion

- Integral equation for Γ_{ab} is non-perturbatively defined. Resisted an exact treatment.
- We look for an iterative solution $G_{\alpha\beta} = \sum_{n=0}^{\infty} \lambda^n G^{(n)}_{\alpha\beta}$.
- This involves iterated integrals labelled by rooted trees.

Up to $\mathcal{O}(\lambda^3)$ we need

$$I_{\alpha} := \int_0^1 dx \, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha) \,,$$

$$I_{\underset{\bullet}{\alpha}} := \int_0^1 dx \, \frac{\alpha \, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2$$

$$I_{\underset{\bullet\;\bullet}{\alpha}} = \int_0^1 dx \, \frac{\alpha \, I_x \cdot I_x}{1-\alpha x} = -2\,\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big)$$

$$I_{\underset{\underset{\bullet}{\bullet}}{\alpha}} = \int_0^1 dx \, \frac{\alpha \, I_{\underset{\bullet}{x}}}{1-\alpha x} = -2\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big) - 2\mathrm{Li}_3(\alpha) - \ln(1-\alpha)\zeta(2)$$

$$+ \ln(1-\alpha)\mathrm{Li}_2(\alpha) + \frac{1}{6}\big(\ln(1-\alpha)\big)^3$$

Observations

Polylogarithms and multiple zeta values appear in singular part of individual graphs of e.g. ϕ^4-theory (Broadhurst-Kreimer)
We encounter them for regular part of all graphs together

Conjecture

- $G_{\alpha\beta}$ takes values in a polynom ring with generators $A, B, \alpha, \beta, \{I_t\}$, where t is a rooted tree with root label α or β
- at order n the degree of A, B is $\leq n$,
  the degree of α, β is $\leq n$,
  the number of vertices in the forest is $\leq n$.

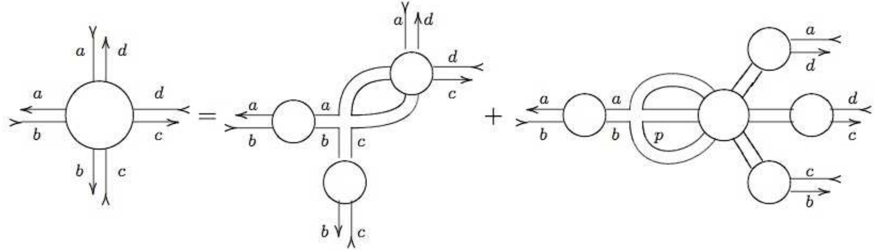
If true:

- There are $n!$ forests of rooted trees with n vertices at order n
- estimate: $|G^{(n)}_{\alpha\beta}| \leq n!(C_{\alpha\beta})^n$

may lead to Borel summability?.

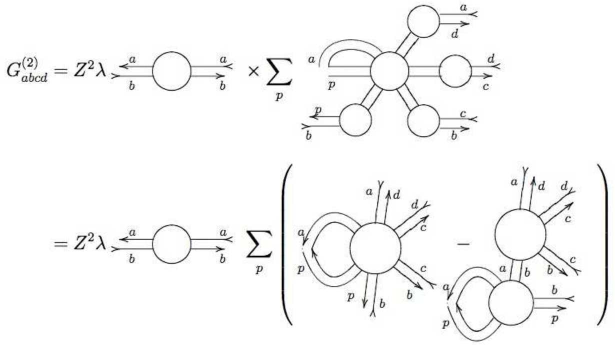
Schwinger-Dyson equ 4 pt fct

Follow *a*-face, there is a vertex at which *ab*-line starts:



1. First graph: Index c and a are opposite
   It equals $Z^2 \lambda G_{ab} G_{bc} G^{ins}_{[ac]d}$
2. Second graph: Summation index p and a are opposite. We open the p-face to get an insertion.

   This is not into full connected four-point function, which would contain an *ab*-line not present in the graph.



second graph equals

$$= Z^2\lambda\Big(\sum_p G_{ab}G^{ins}_{[ap]bcd} - G^{ins}_{[ap]b}G_{abcd}\Big)$$

1PI four-point function

$$\Gamma^{ren}_{abcd} = Z\lambda\Big\{\frac{G^{-1}_{ad}-G^{-1}_{cd}}{|a|-|c|} + \sum_p \frac{G_{pb}}{|a|-|p|}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}\Big)\Big\}$$

Theorem

The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ^4_4-theory satisfies

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\Big(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)}{G_{\alpha\delta}+\lambda\Big((\mathcal{M}_\beta-\mathcal{L}_\beta-\mathcal{Y})G_{\alpha\delta}+\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)}$$

Corollary

$\Gamma_{\alpha\beta\gamma\delta} = 0$ is not a solution!
We have a non-trivial (interacting) QFT in four dimensions!

Conclusions

- Studied model at $\Omega = 1$
- RG flows save
- Used Ward identity and Schwinger-Dyson equation
- ren. 2 point fct fulfills nonlinear integral equ
- ren. 4 pt fct linear inhom. integral equ
  perturbative solution:

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2 \Big(\frac{(1-\gamma)(I_\alpha - \alpha) - (1-\alpha)(I_\gamma - \gamma)}{\alpha - \gamma} + \frac{(1-\delta)(I_\beta - \beta) - (1-\beta)(I_\delta - \delta)}{\beta - \delta} \Big) + \mathcal{O}(\lambda^3)$$

- is nontrivial and cyclic in the four indices
- nontrivial Φ^4 model ?

Spectral triples

Connes "On the spectral characterization of manifolds"

Definition (commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension $p \in \mathbb{N}$)

...given by a Hilbert space $\mathcal{H}$, a commutative involutive unital algebra $\mathcal{A}$ represented in $\mathcal{H}$, and a selfadjoint operator $\mathcal{D}$ in $\mathcal{H}$ with compact resolvent, with

1. *Dimension:* k^{th} characteristic value of resolvent of $\mathcal{D}$ is $\mathcal{O}(k^{-\frac{1}{p}})$
2. *Order one:* $[[\mathcal{D}, f], g] = 0 \quad \forall f, g \in \mathcal{A}$
3. *Regularity:* f and $[\mathcal{D}, f]$ belong to domain of δ^k, where $\delta T := [|\mathcal{D}|, T]$
4. *Orientability:* $\exists$ Hochschild p-cycle $\boldsymbol{c}$ s.t.$\pi_{\mathcal{D}}(\boldsymbol{c}) = 1$ for p odd, $\pi_{\mathcal{D}}(\boldsymbol{c}) = \gamma$ for p even with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma\mathcal{D} = -\mathcal{D}\gamma$
5. *Finiteness and absolute continuity:* $\mathcal{H}_\infty := \cap_k \mathrm{dom}(\mathcal{D}^k) \subset \mathcal{H}$ is finitely generated projective $\mathcal{A}$-module, $\mathcal{H}_\infty = e\mathcal{A}^n$.

Spectral triples are interesting for physics!

- equivalence classes of spectral triples describe Yang-Mills theory (inner automorphisms; exist always in nc case) and possibly gravity (outer automorphisms)
- inner fluctuations: $\mathcal{D} \mapsto \mathcal{D}_A = \mathcal{D} + A\ , \quad A = \sum f[\mathcal{D}, g]$
  for almost-commutative manifolds: A=Yang-Mills+Higgs

Spectral action principle [Chamseddine+Connes, 1996]

As an automorphism-invariant object, the (bosonic) action functional of physics has to be a function of the spectrum of $\mathcal{D}_A$, i.e. $S(\mathcal{D}_A) = \mathrm{Tr}(\chi(\mathcal{D}_A))$.

for almost-commutative 4-dim compact manifolds:

- $S(\mathcal{D}_A) = \int_X d\,vol\ (\mathcal{L}_\Lambda + \mathcal{L}_{\text{EH+W}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Higgs-kin}} + \mathcal{L}_{\text{Higgs-pot}})$

  for any function of the spectrum (universality of RG)
- structure of the standard model more or less unique

Harmonic oscillator spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_i)$

Morse function $h = \frac{1}{2}\|x\|^2$

implies constant $[a_\mu, a_\nu^\dagger] = 2\omega\delta_{\mu\nu}$

Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \bigwedge(\mathbb{C}^d)$: declare ONB

$$\Big\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d} |0\rangle \;:\; n_\mu \in \mathbb{N}\,, s_\mu \in \{0,1\}\Big\}$$

TWO Dirac operators $\mathcal{D}_1 = \mathfrak{Q} + \mathfrak{Q}^\dagger$, $\mathcal{D}_2 = \mathrm{i}\mathfrak{Q} - \mathrm{i}\mathfrak{Q}^\dagger$

$$\mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} = \sum_{\mu=1}^d \big(a_\mu^\dagger a_\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b_\mu\big)$$
$$= 2\omega(N_b + N_f) = H \otimes 1 + \omega \otimes \Sigma$$

where

$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu$ – harmonic oscillator hamiltonian

$\Sigma = \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu]$ – spin matrix

algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness

All axioms of spectral triples satisfied, with minor adaptation

U(1)-Higgs model for commutative algebra

tensor $(\mathcal{A}, \mathcal{H}, \mathcal{D}_1)$ with $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1, \sigma_3)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_1 \otimes \sigma_3 + 1 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_1 & M \\ M & -\mathcal{D}_1 \end{pmatrix}$ $\qquad \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathcal{A}_{tot}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A := \mathcal{D}_1 + \sum_i a_i [\mathcal{D}_1, b_i]$, $a_i, b_i \in \mathcal{A}_{tot} = \mathcal{A} \oplus \mathcal{A}$, are of the form
$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_1 + \mathrm{i} A^\mu \otimes (b^\dagger_\mu - b_\mu) & \phi \otimes 1 \\ \overline{\phi} \otimes 1 & -(\mathcal{D}_1 + \mathrm{i} B^\mu \otimes (b^\dagger_\mu - b_\mu)) \end{pmatrix}$$
for $A_\mu = \overline{A_\mu}$, $B_\mu = \overline{B_\mu}$, $\phi \in \mathcal{A}$
- $F_A = (-\{\partial^\mu, A_\mu\} - \mathrm{i} A^\mu A_\mu) \otimes 1 + \frac{1}{4} F_A^{\mu\nu} \otimes [b^\dagger_\mu - b_\mu, b^\dagger_\nu - b_\nu]$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = Tr\left(\chi(\mathcal{D}_A^2)\right)$

- Laplace transf. + asympt. exp'n $\mathrm{Tr}(e^{-t\mathcal{D}_A^2}) \sim \sum_{n=-dim/2}^{\infty} a_n(\mathcal{D}_A^2) t^n$

  leads to $S(\mathcal{D}_A) = \sum_{n=-dim/2}^{\infty} \chi_n \,\mathrm{Tr}(a_n(\mathcal{D}_A^2))$

  with $\chi_z = \frac{1}{\Gamma(-z)} \int_0^\infty ds\, s^{-z-1} \chi(s)$ for $z \notin \mathbb{N}$

  $\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$
- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_{\mathcal{A}}^2 = \mathrm{H}_0 - V$

$$e^{-t(\mathrm{H}_0 - V)} = e^{-t\mathrm{H}_0} + \int_0^t dt_1 \left(e^{-(t-t_1)(\mathrm{H}_0 - V)} V e^{-t_1 \mathrm{H}_0}\right)$$

. . . iteration

Vacuum trace

Mehler kernel (in 4D)

$$e^{-t(H+\omega\Sigma)}(x,y) = \frac{\omega^2(1-\tanh^2(\omega t))^2}{16\pi^2\tanh^2(\omega t)} e^{-t\omega\Sigma 1_2 - \frac{\omega}{4}\frac{\|x-y\|^2}{\tanh(\omega t)} - \frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}$$

$$\mathrm{tr}(e^{-t\omega\Sigma}) = (2\cosh(\omega t))^d$$

$$\begin{aligned}\mathrm{Tr}(e^{-t(H+\omega\Sigma)\otimes 1_2}) &= 2\mathrm{tr}\Big(\int d^4x \, (e^{-t(\mathrm{H}+\omega\Sigma)})(x,x)\Big) \\ &= \frac{2}{\tanh^4(\omega t)} = 2(\omega t)^{-4} + \frac{8}{3}(\omega t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2)\end{aligned}$$

- Spectral action is finite, in contrast to standard $\mathbb{R}^4$! (This is meant by “finite volume”)
- expansion starts with t^{-4} $\Rightarrow$ corresponds to 8-dim. space

The spectral action

$$
\begin{aligned}
S(\mathcal{D}_A) = & \, const \\
& + \frac{\chi_0}{\pi^2} \int d^4x \, \Big\{ D^\mu \phi \overline{D_\mu \phi} + \frac{5}{12} (F^A_{\mu\nu} F^{\mu\nu}_A + F^B_{\mu\nu} F^{\mu\nu}_B) \\
& \quad + \Big((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0} |\phi|^2 + 2\omega^2 \|x\|^2 |\phi|^2 \Big) \Big\} + \mathcal{O}(\chi_1)
\end{aligned}
$$

- spectral action is finite
- only difference in field equations to infinite volume is additional harmonic oscillator potential for the Higgs
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) $\phi \in \mathbb{R}$, rotationally invariant

The spectral action: noncommutative case

$$S(\mathcal{D}_A) = const + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4x \Big\{ 2D_\mu\phi \star \overline{D_\mu\phi}$$
$$+\Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu} + F^B_{\mu\nu}\star F_B^{\mu\nu}\big)$$
$$+\Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}X_A^\mu \star X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$+\Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X_B^\mu \star X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$-2\Big(\frac{4\Omega^2}{1+\Omega^2}X_0^\mu \star X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(x) + \mathcal{O}(\chi_1)$$

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}$$

$$\omega = \frac{2\Omega}{\theta}$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $X_{A\mu}(x) = (\Theta^{-1})_{\mu\nu}x^\nu + A_\mu(x)$ appear with Higgs field ϕ in unified potential; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

Gauge model

H G + M. Wohlgenannt + Blaschke + Schweda,...Steinacker,...
Couple scalar field to "external" gauge field

$$S[A] = Tr\left(\phi[X^\mu, [X_\mu, \phi]] + \Omega^2 \phi\{X^\mu, \{X_\mu, \phi\}\} + \mu^2\phi^2 + \frac{\lambda}{4}\phi^4\right)$$

where $X^\mu = \tilde{x}^\mu + A^\mu$ are covariant coordinates.
Gauge transformation

$$X^\mu \to U^\dagger X^\mu U,\ A^\mu \to -iU^\dagger[\tilde{x}^\mu, U] + U^\dagger A^\mu U\,.$$

One-loop regularized effective action

$$\Gamma_\epsilon[A] = -\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t} Tr\left(e^{-tH} - e^{-tH_0}\right)\,.$$

Use Duhamel expansion

$$e^{-tH} = e^{-tH_0} - \int_0^t dt_1 e^{-t_1 H_0} V_A e^{-(t-t_1)H_0} + \ldots$$

Identify: $\frac{\delta^2 S}{\delta\phi\delta\phi} = H_0 + V_A$ expansion up to order V^4 gives

$$S[A] = -\frac{1}{4}\int F^2 + \alpha\int((\tilde{X}_\nu \star \tilde{X}^\nu)^{\star 2} - (\tilde{x}^2)^2) + \beta\int((\tilde{X}_\nu \star \tilde{X}^\nu) - (\tilde{x}^2))$$

$F^{\mu,\nu} = [X^\mu, X^\nu] - i(\Theta^{-1})^{\mu,\nu}$ yields matrix models IKKT
Generalize BRST complex to nc gauge models with oscillator:

BRST invariant renormalization? tadpole ?

QFT on noncommutative Minkowski space

Definition of quantum fields on NC Minkowski space

$$\phi_\otimes(x) := \int dp\, e^{ip\cdot\hat{x}} \otimes e^{ip\cdot x}\, \tilde{\phi}(p)$$

- $\phi_\otimes(f)$ acts on $\mathcal{V} \otimes \mathcal{H}$,
- Vacuum $\omega_\theta := \nu \otimes \langle \Omega, .\Omega \rangle$ independent of ν
- We relate the antisymmetric matrices to Wedges:
  $W_1 = \left\{ x \in \mathbb{R}^D | x_1 > |x_0| \right\}$ act by Lorentz transformations.
- Stabilizer group is $SO(1,1)xSO(2)$
- Get isomorphism $(\mathcal{W}, i_\Lambda) \cong (\mathcal{A}, \gamma_\Lambda = \Lambda\Theta\Lambda^\dagger)$

Wedge locality

With this isomorphism define $\Phi_W(x) := \Phi_{\Theta(W)}(x)$. Transformation properties

$$U_{y,\Lambda}\Phi_W(x)U^\dagger_{y,\Lambda} = \Phi_{\gamma_\Lambda(\Theta(W))}(\Lambda x + y)$$

Theorem

Let $\kappa_e \geq 0$ the family $\Phi_W(x)$ is a wedge local quantum field on Fockspace:

$$[\phi_{W_1}(f), \phi_{-W_1}(g)](\psi) = 0,$$

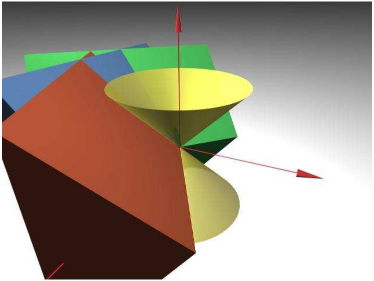
for $supp(f) \subset W_1$, $supp(g) \subset -W_1$.

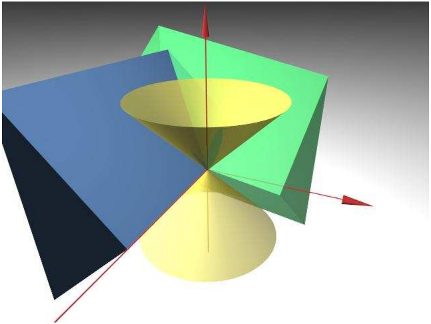
Show that

$$[a_{W_1}(f^-), a^\dagger_{-W_1}(g^+)] + [a^\dagger_{W_1}(f^+), a_{-W_1}(g^-)] = 0$$

$\vartheta = \sinh^{-1}\left(p_1/(m^2 + p_2^2 + p_3^2)^{1/2}\right)$
Use analytic continuation from R to $R + i\pi$ in ϑ .

Wedges





- Need well localized states for asymptotics $t \to \pm\infty$ ("in/out")
- Wedge-locality is good enough for two particle scattering
- Find for two-particle scattering with $p_1^1 > p_2^1$, $q_1^1 > q_2^1$:
  ${}^{\theta_1}_{\text{out}}\langle p_1, p_2 \,|q_1, q_2\rangle^{\theta_1}_{\text{in}} = e^{\frac{i}{2}p_1\theta p_2} e^{\frac{i}{2}q_1\theta q_2}\, {}_{\text{out}}\langle p_1, p_2 \,|q_1, q_2\rangle_{\text{in}}$ non-trivial S-matrix, **deformation induces interaction!**
- measurable effects of the noncommutativity (time delay)

Outlook

Obtained renormalized II summable nc QFT. III locality weakened to Wedge locality

Learn Principles
Learn to obtain
ren. sum. nc Standard Model?
ren. sum. Quantum Gravity?

- implications for cosmology?
  cosmological constant problem ?
  dark matter, dark energy?
  information paradox,....? Inflation?
- implications for experiments???