Bose-Einstein condensates and nonlinear waves

Michał Matuszewski
Institute of Physics, Polish Academy of Sciences, Warsaw

Exciton-polariton BEC Research Group

Institute of Physics
Polish Academy of Sciences

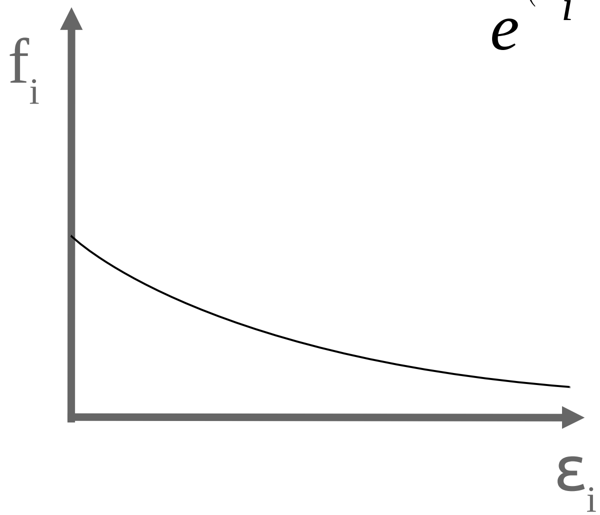
$$\frac{\partial n_R}{\partial t} =$$

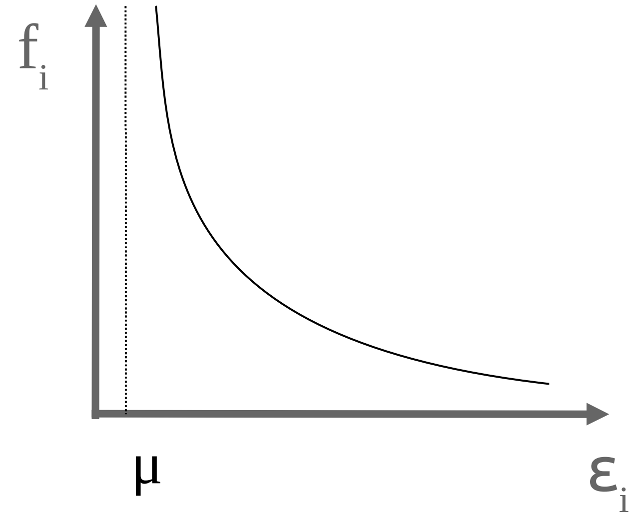
$$i\frac{\partial \psi}{\partial t} = \left[-\frac{\partial^2}{\partial x^2} + \frac{i}{2}\left(Rn_R - \gamma\right) + g|\psi|^2 + \right.$$

Bose-Einstein condensation

At some energy, divergence may occur
This means that the occupation of the lowest state becomes macroscopic

$$f_i = \frac{1}{e^{(\epsilon_i - \mu)/k_B T} - 1}$$





This can be achieved in two ways: decreasing T or increasing μ (number of particles)

(Berezinskii-)Kosterlitz-Thouless phase transition

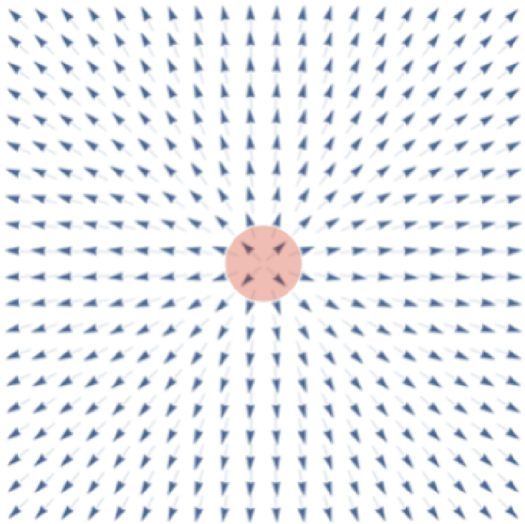
- Nobel prize in Physics 2016

- Awarded to D. J. Kosterlitz, J.M. Thouless and F. D. M.Haldane “for theoretical discoveries of topological phase transitions and topological phases of matter" (in part for the prediction of the KT transition)

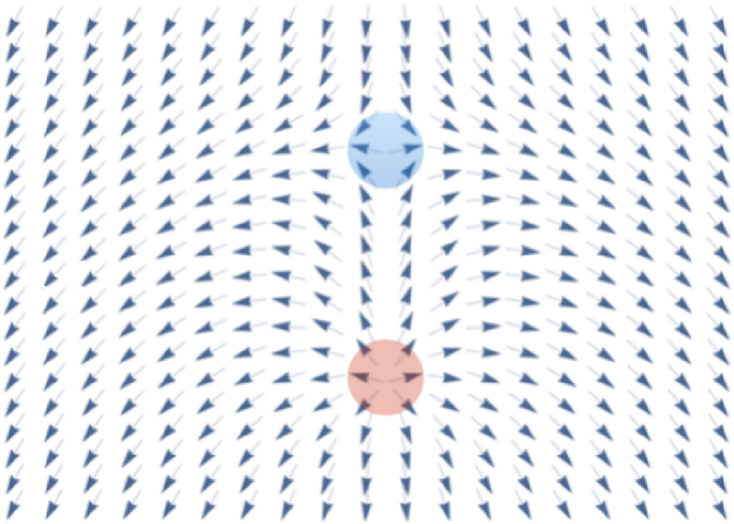
2D XY model and vortices

- A classical model of an array of spins in 2D

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$



Vortex



Vortex-Antivortex pair

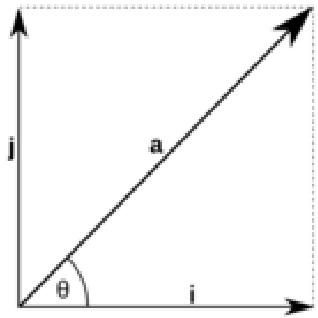
2D XY model and vortices

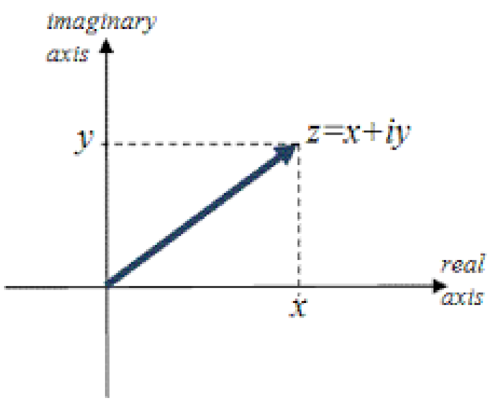
- Correspondence with complex fields

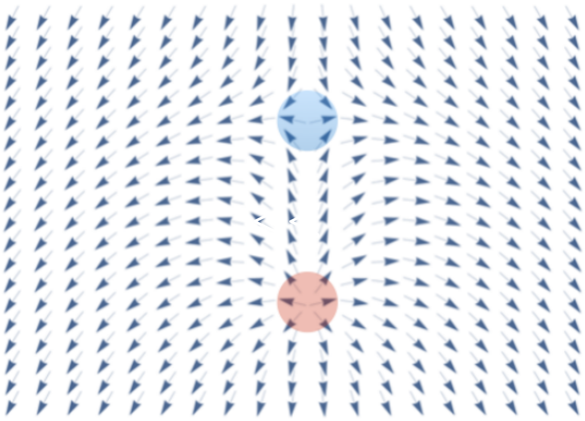
$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

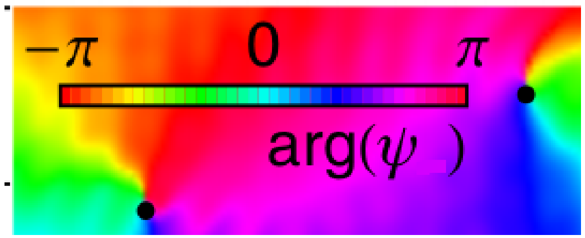
$$H_{XY} = \frac{J}{2} \int d^2r \, (\vec{\nabla}\theta(\vec{r}))^2 \, .$$

$$\psi = A\, e^{i\theta}$$



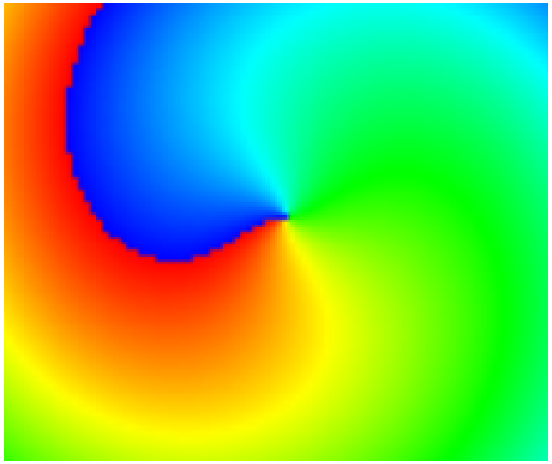


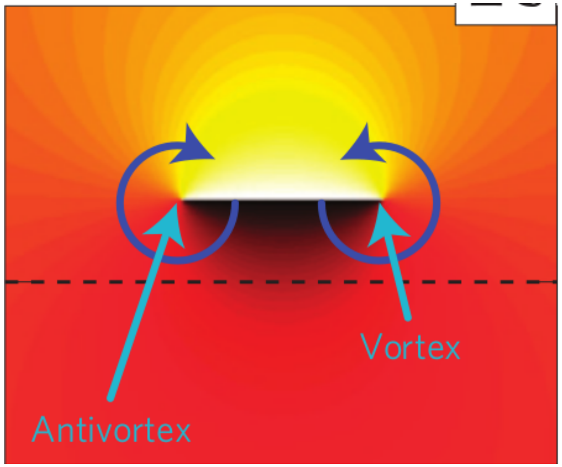




KT transition

- Phase circulation far from vortex core: Energy of a vortex diverges logarithmically with system size!





- Vortex-antivortex pairs have finite energy

- Considering free energy, even the state with free vortices can exist

$$F = E - TS = J\pi \ln\left(\frac{L}{a}\right) - Tk_B \ln\left(\frac{L^2}{a^2}\right) \qquad T_{KT} = J\pi/2k_B$$

KT transition

- The phase with bound vortex-antivortex pairs has a nonzero superfluid fraction and power-law decay of coherence

$$\langle e^{i(\theta(\vec{r})-\theta(\vec{0}))}\rangle \sim \left(\frac{a}{r}\right)^{\frac{k_B T}{2\pi J}}$$

in contrast to a BEC, where coherence does not decay to zero at r → ∞

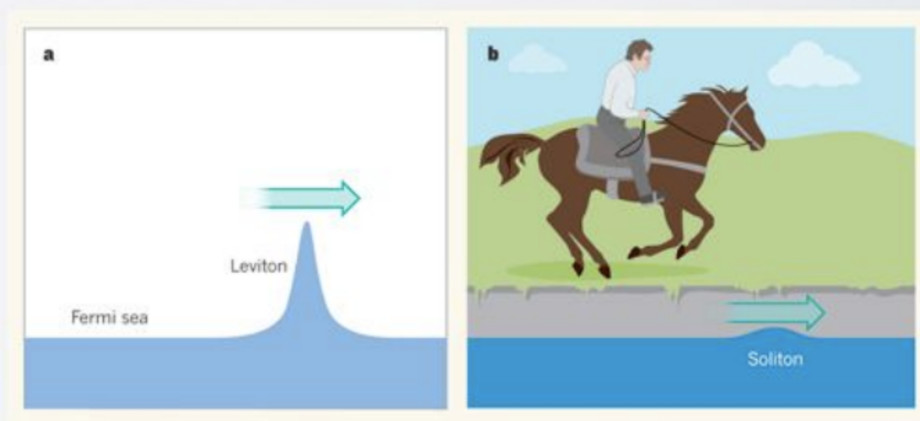
- The phase with free vortices has a standard exponential decay of coherence

Solitons
history

Scott Russell in 1834 observed a heap of water in a canal that propagated undistorted over several kilometer

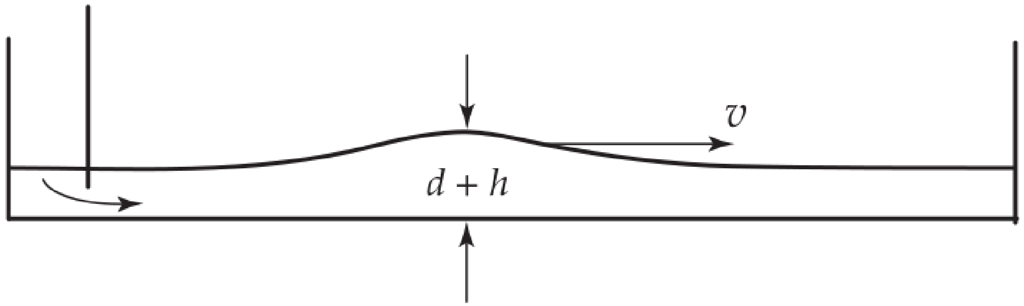
„a rounded, smooth and well-defined heap of water, which continued its curse along the channel apparently without change of form or diminution of speed. I followed it on a horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height."

after „*Nonlinear Fiber Optics*" G. P. Agraval



Quantum physics: Single electrons pop out of the Fermi sea
Ch. Flindt, Nature 502, 630–632 (31 October 2013)

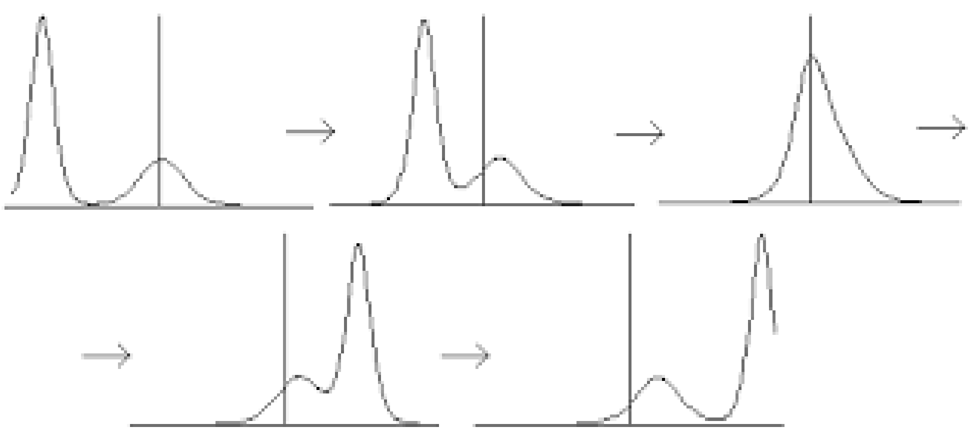
Solitons



$$v = \sqrt{g(d+h)}$$

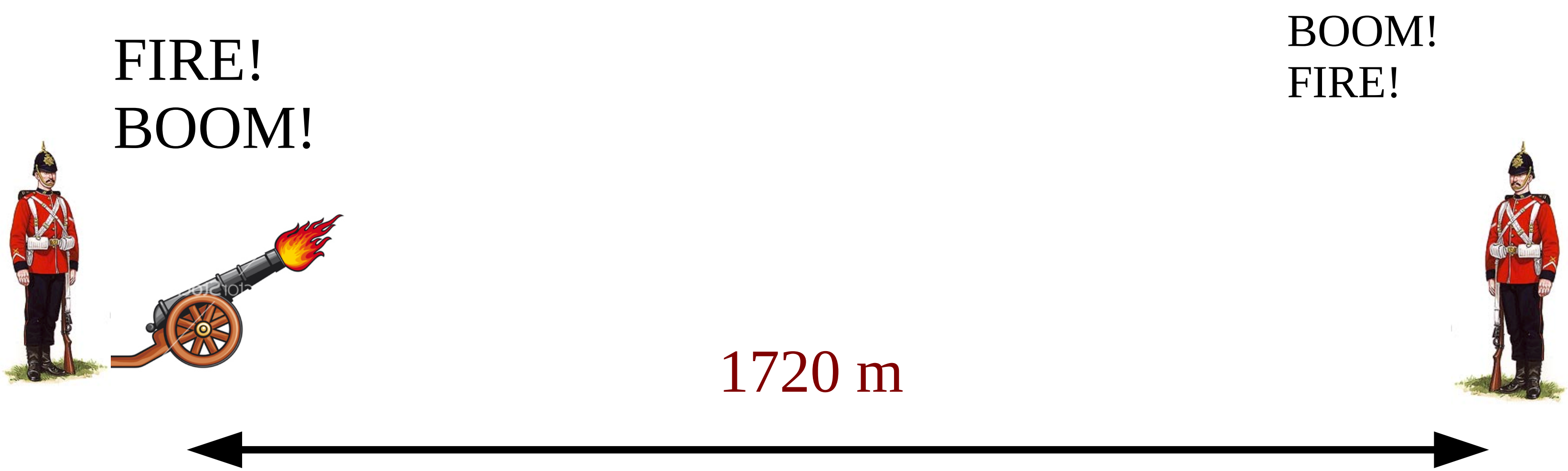
Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$



Captain Parry's observation

- Second (unsuccesful) expedition to the North Pole, 1822
- The sound of the cannon arrived ½ second before the "fire" command



Fermi-Pasta-Ulam-Tsingou paradox

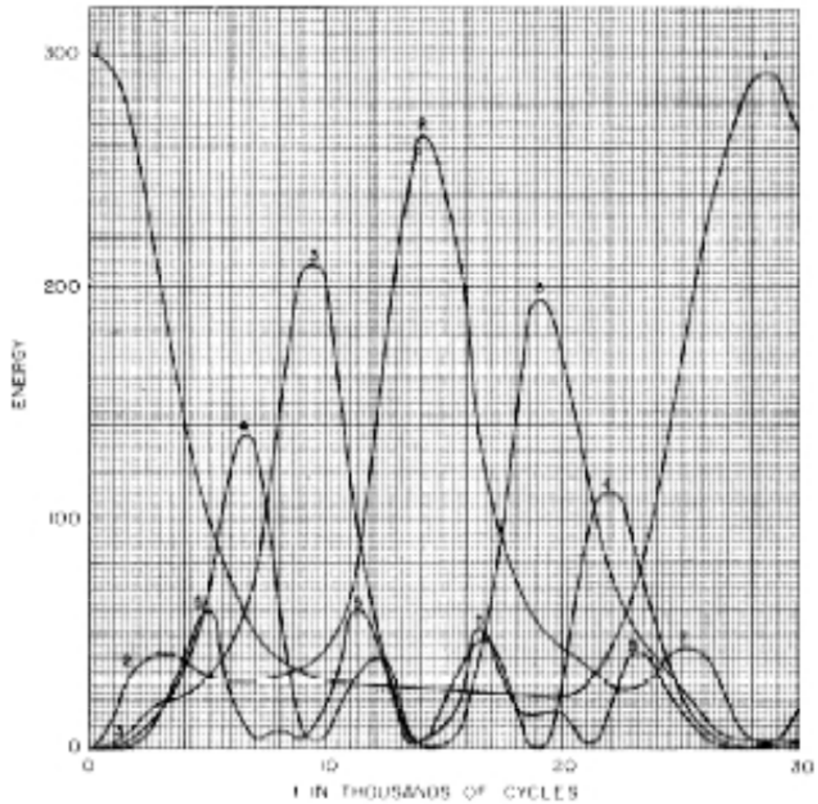
- Array of particles connected by anharmonic springs

$$H(P,Q) = \frac{1}{2}\sum_{i=1}^{N-1} P_i^2 + \frac{1}{2}\sum_{i=0}^{N-1}(Q_{i+1}-Q_i)^2 + \frac{\alpha}{3}\sum_{i=0}^{N-1}(Q_{i+1}-Q_i)^3$$

- Equivalent to the Korteweg–de Vries equation in the continuous (many small oscillators) limit

- Inevstigated numerically on MANIAC computer in search for ergodicity and thermalization

Fermi-Pasta-Ulam-Tsingou paradox

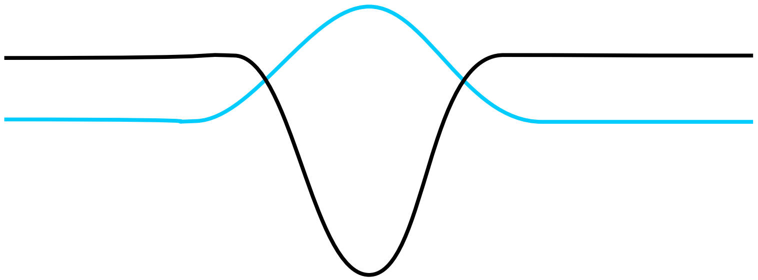


- Instead of thermalization, periodic revivals occurred
- The behaviour has been explained by Zabusky and Kruskal in terms of solitonic interference

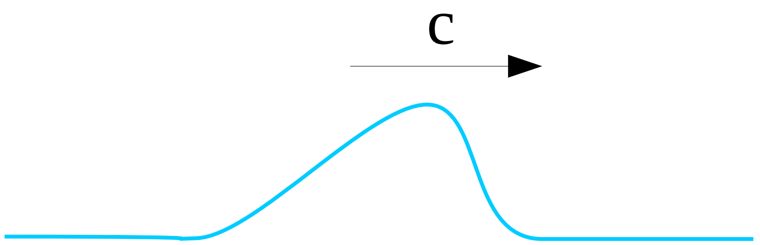
What is NOT a soliton?

- Defect or bound state

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$



- Dispersionless linear wave



$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Solitons are self-localized, i.e. their existence is the reason why they do not decay. In particular, they cannot exist in the low-amplitude (linear) limit.

Integrability

Aristocratic solitons
(in "mathematical" sense)

Constant shape

Self-Localized

Immune to collisions

Integrable models

Minority

Plebeian solitons
(in "physical" sense)

Constant shape

Self-Localized

Not immune to collisions

Non-integrable models

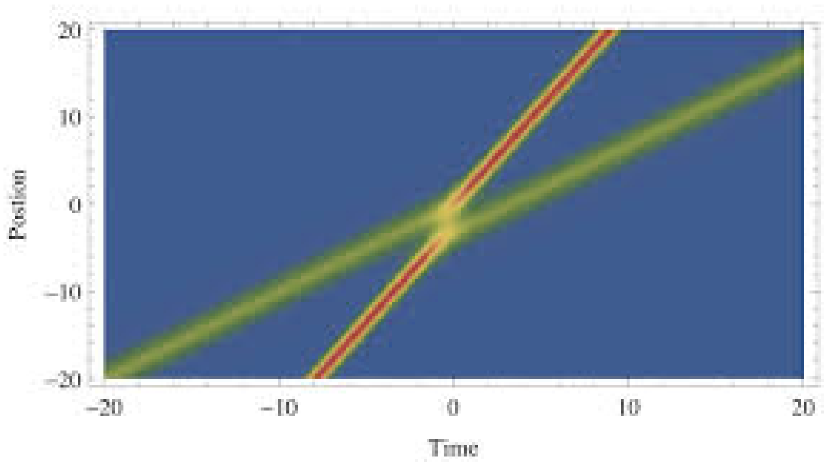
Majority

Integrability

Some equations (eg. 1D NLS, KdV) can be solved exactly!

Inverse scattering method (IST) is a "Nonlinear Fourier Transform"

- Solutions are decomposed into nonlinear waves (solitons) and linear "radiation", all of which evolve (almost) independently
- In result, solitons are immune to collisions, except for a phase and trajectory shifts



Integrability

Inverse scattering method (Kruskal, Gardner, Greene, Miura 1967)

- Let u(x,t) be the solution of a nonlinear equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

- Then, let's write time-independent (!) linear scattering equation:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + [\lambda + u(x,t)]\psi = 0$$

- The general solution to this problem gives discrete eigenvalues λ which correspond to solitons, and are *independent of time t*

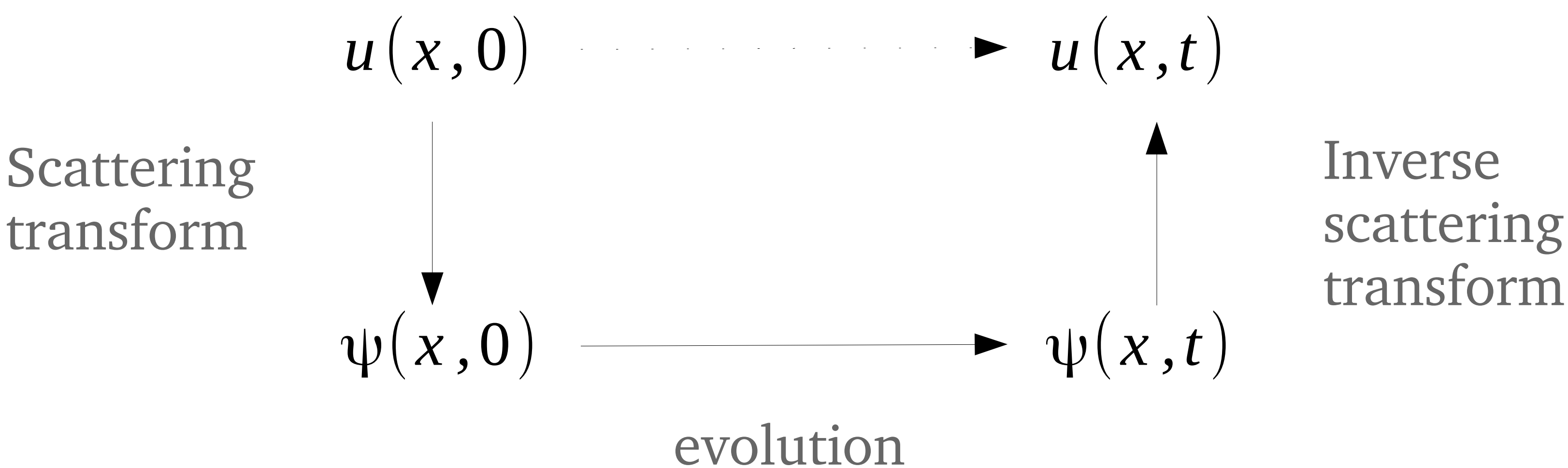
- Knowing how ψ changes with time, we can recover u(x,t) – this is the inverse scattering problem, solved Gel'fand, Levitan, and Marchenko in the 1950's

Integrability

Inverse scattering method (Kruskal, Gardner, Greene, Miura 1967)

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + [\lambda + u(x,t)]\psi = 0$$

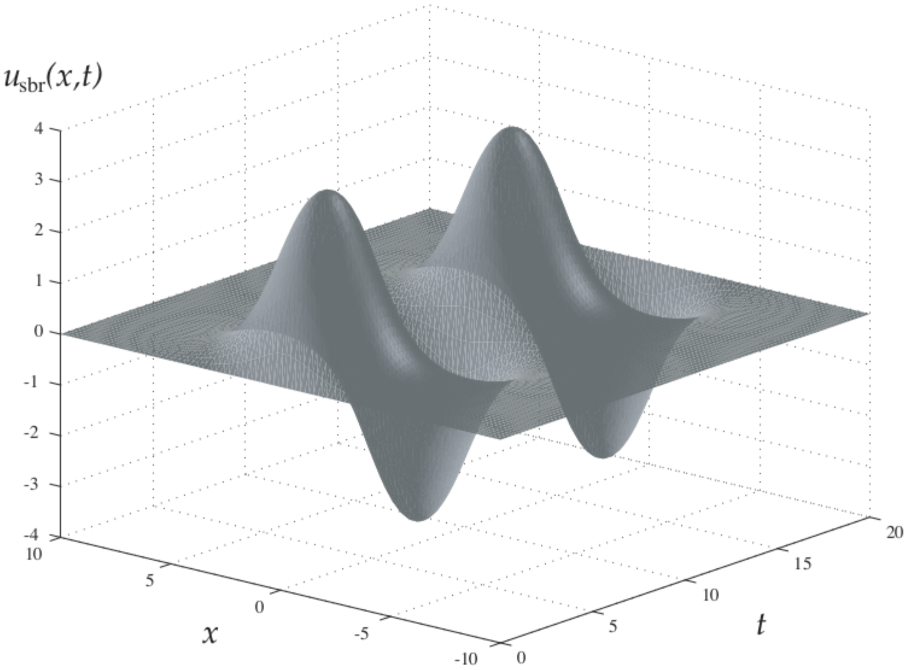


Breathers

- Solitons are associated with a particular frequency (ω or μ), which depends on the amplitude and velocity

$$u(x,t) = \frac{\sqrt{2}}{W}\operatorname{sech}\frac{x - vt}{W}\exp\left[\frac{ivx}{2} + i\left(\frac{1}{W^2} - \frac{v^2}{4}\right)t\right]$$

- If two solitons with different frequencies but the same velocity overlap, the “breather” solution appears

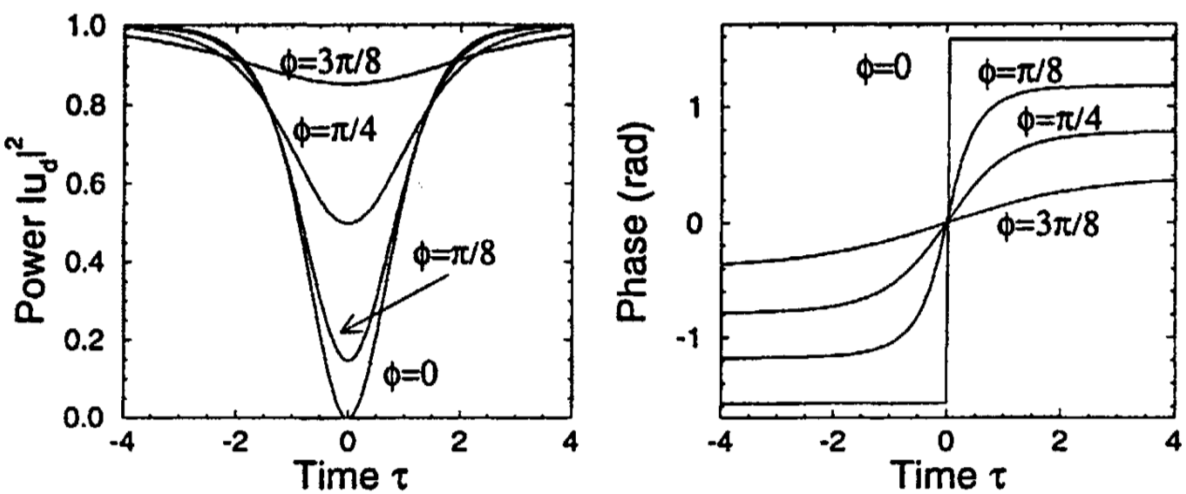


Dark solitons

- If we invert the sign of nonlinearity to repulsive, we can find dark soliton solutions

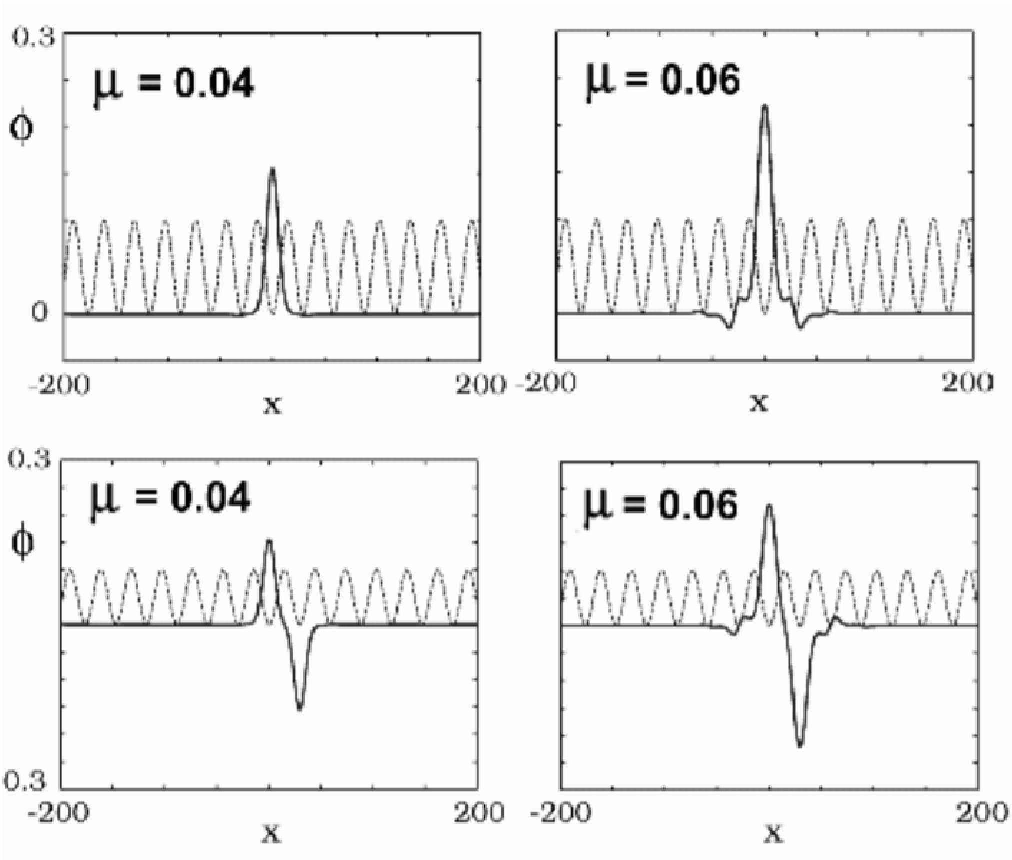
$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi$$

$$\psi=a\{B\tanh[aB(x-Aat)]+iA\}e^{-ia^2t} \qquad A^2+B^2=1$$



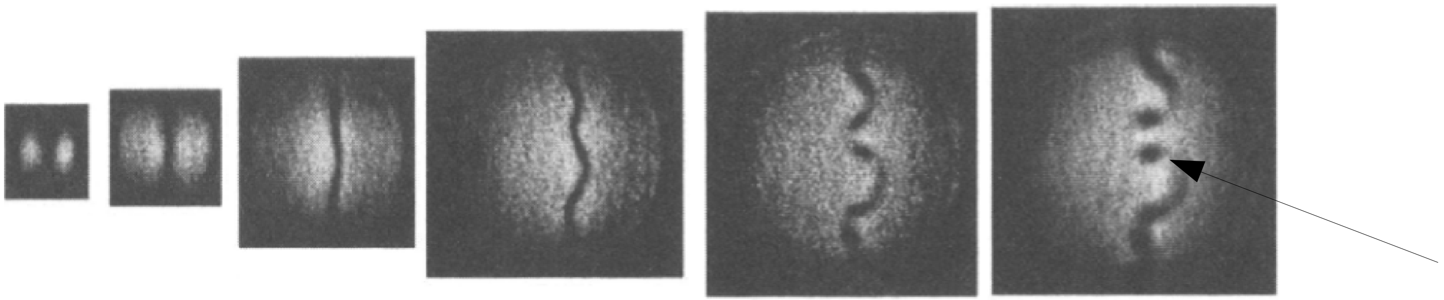
$$\phi=\arctan A/B$$

Gap solitons



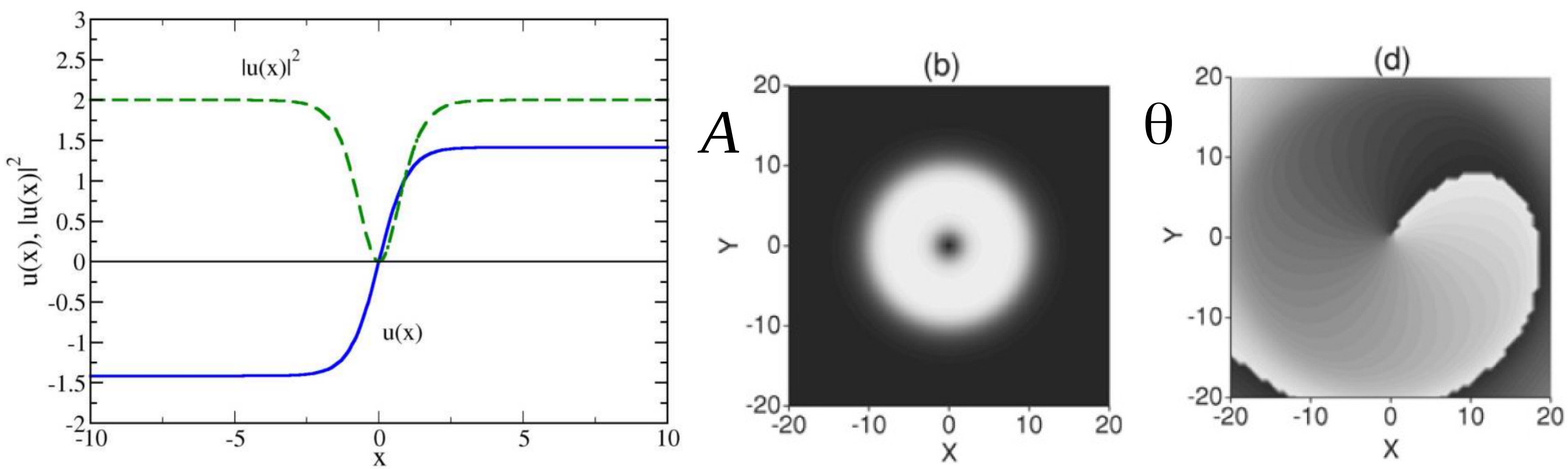
Dark solitons in 2D – snaking instability

- Dark soliton stripes in 2D are unstable, vortex solitons appear



- Vortex solitons are 2D analogs of dark solitons

$$\psi = A\,e^{i\theta}$$

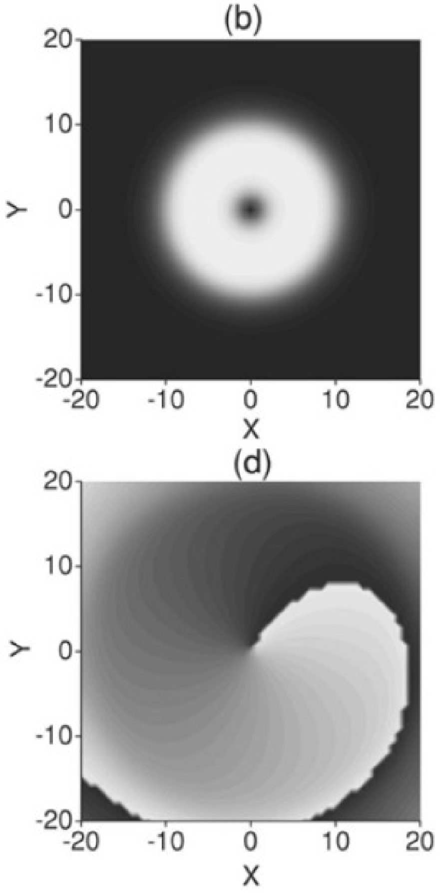


Skyrmions

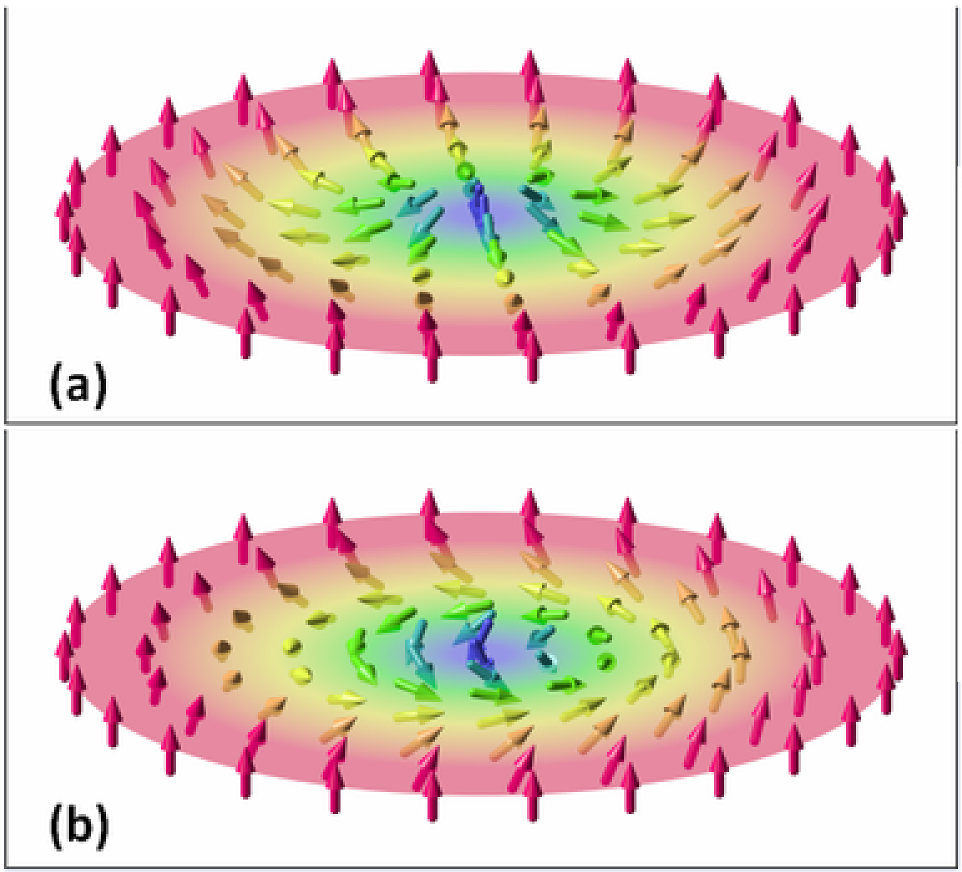
- The phase winding of a vortex is encoded in its asymptotic (topological) behaviour – there is no way to "unwind" it

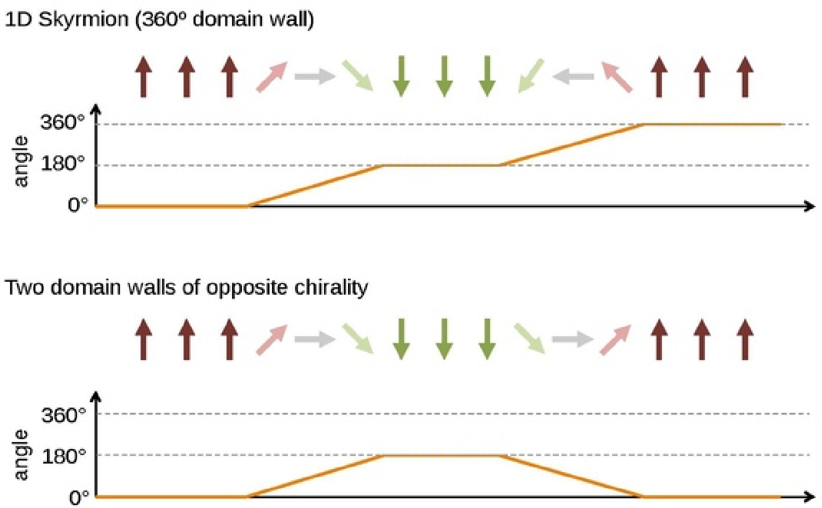
- Singularity exists in the center of a vortex

Skyrmion: a topological defect with no singularity, nontrivial subject to a given boundary condition



Skyrmions


(a)
(b)


1D Skyrmion (360º domain wall)
angle
360°
180°
0°
Two domain walls of opposite chirality
angle
360°
180°
0°

Skyrmions