IBM Quantum – Bell inequality

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2026-03-24/25

Single qubit gate - reminder

Single qubit basis $|0\rangle$, $|1\rangle$

$$G|q\rangle = \sum_{q'} G_{q'q}|q'\rangle, \ \langle q'|G|q\rangle = G_{q'q}$$

$$G = \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{10} \end{pmatrix}$$

Identity $I|q\rangle = |q\rangle$.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

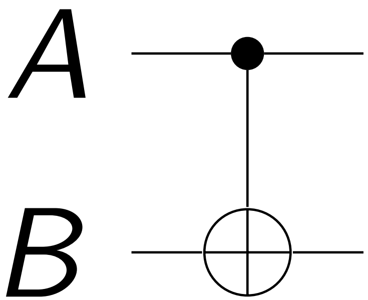
Two qubits – tensor product, single qubit gate

Two qubit basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. States $|ab\rangle$ with a - first qubit (A) states, b - second qubit (B) state.
$G_A = G \otimes I, \;\; G_B = I \otimes G$
$G_A|ab\rangle = \sum_{a'} G_{a'a}|a'b\rangle, \;\; G_B|ab\rangle = \sum_{b'} G_{b'b}|ab'\rangle$

$$G_A = \begin{pmatrix} G_{00}I & G_{01}I \\ G_{10}I & G_{11}I \end{pmatrix} = \begin{pmatrix} G_{00} & 0 & G_{01} & 0 \\ 0 & G_{00} & 0 & G_{01} \\ G_{10} & 0 & G_{11} & 0 \\ 0 & G_{10} & 0 & G_{11} \end{pmatrix}$$

$$G_B = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} G_{00} & G_{01} & 0 & 0 \\ G_{10} & G_{11} & 0 & 0 \\ 0 & 0 & G_{00} & G_{01} \\ 0 & 0 & G_{10} & G_{11} \end{pmatrix}$$

Two qubits, CX gate aka $CNOT$



Notation $CX|AB\rangle$, A – control $\bullet$, B – target $\oplus$.
$CX|00\rangle = |00\rangle$, $CX|01\rangle = |01\rangle$, $CX|10\rangle = |11\rangle$, $CX|11\rangle = |10\rangle$.
Basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$$

Entangled state

Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(self-reverse $H^2 = HH = I$)

$$H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$$

add second qubit, $H_A|00\rangle = (|00\rangle + |10\rangle)/\sqrt{2}$, and apply CX:

$$CX \cdot H_A|00\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$

Bell inequality

- A and B: two chosen measurements, indexed by 0 and 1.
- outcome assignment: $0 \to +1$, $1 \to -1$
- outcome notation A_0, A_1, B_0, B_1, for each choice and party.

Clauser, Horne, Shimony, Holt (CHSH)

$$A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 \leq 2$$

Statistical test

We apply RZ_a to A and RZ_b to B on the entangled state, to get

$$(|00\rangle + e^{i(a+b)}|11\rangle)/\sqrt{2}$$

and H to each qubit ($H_A H_B$)

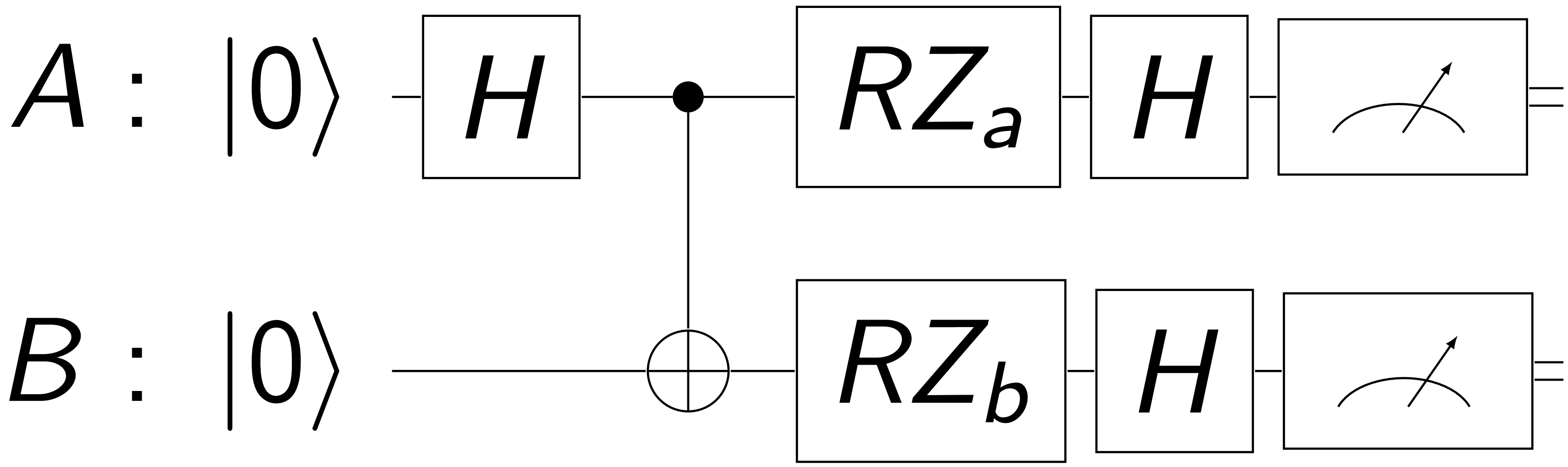
$$[(|00\rangle + |11\rangle)(1 + e^{i(a+b)}) + (|10\rangle + |01\rangle)(1 - e^{i(a+b)})]/\sqrt{8}$$

Final measurement gives probabilities

$$P(00) = P(11) = (1 + \cos(a+b))/4,$$

$$P(01) = P(10) = (1 - \cos(a+b))/4$$

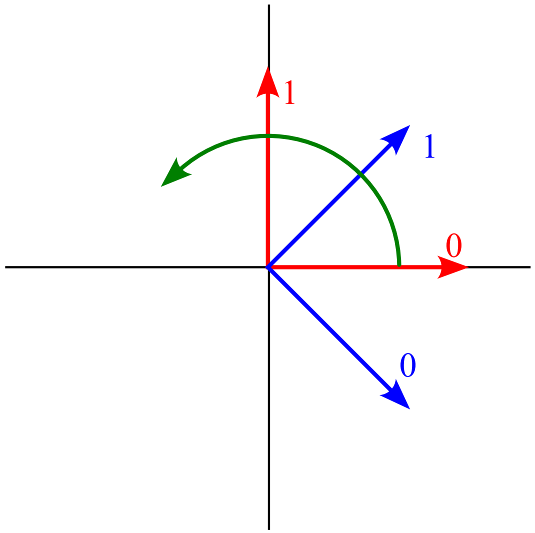
Circuit

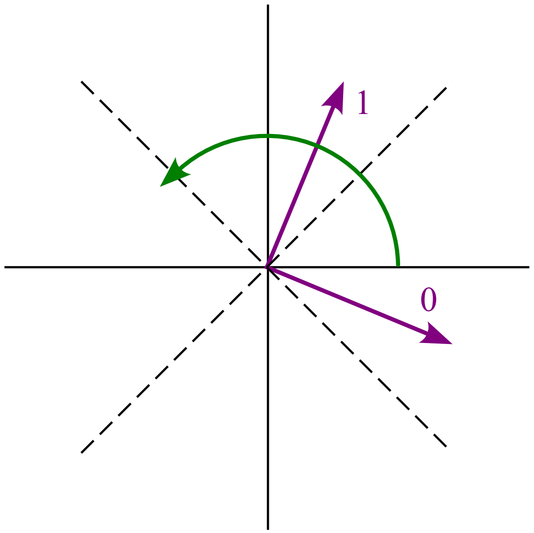


Angles

	0	1
a	0	$\pi/2$
b	$-\pi/4$	$\pi/4$

or $0 \to -\pi/8$, $1 \to 3\pi/8$





Violation!

$$\langle AB\rangle = P(00) + P(11) - P(01) - P(10) = \cos(a+b)$$

	0	1	a
0	$-\pi/4$	$\pi/4$	
1	$\pi/4$	$3\pi/4$	
b			

$$\langle A_0 B_0\rangle = \langle A_1 B_0\rangle = \langle A_0 B_1\rangle = -\langle A_1 B_1\rangle = 1/\sqrt{2}$$

and

$$\langle A_0 B_0\rangle + \langle A_1 B_0\rangle + \langle A_0 B_1\rangle - \langle A_1 B_1\rangle = 2\sqrt{2} \simeq 2.828 > 2$$

Statistical error

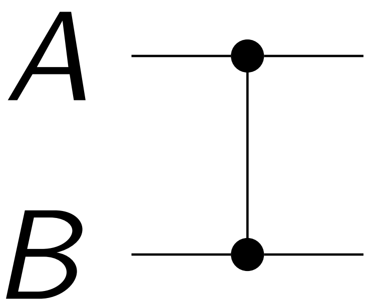
A single shot

$$\langle [\Delta(AB)]^2 \rangle = 1 - \langle AB \rangle^2$$

For N shots of each of 4 circuits

$$N\sigma^2 = 4 - \langle A_0 B_0 \rangle^2 - \langle A_0 B_1 \rangle^2 - \langle A_1 B_0 \rangle^2 - \langle A_1 B_1 \rangle^2$$

CZ – standard native gate



$CZ|00\rangle = |00\rangle$, $CZ|01\rangle = |01\rangle$, $CZ|10\rangle = |10\rangle$, $CZ|11\rangle = -|11\rangle$,
$CZ|ab\rangle = (-1)^{ab}|ab\rangle$.

$$\begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Gate transpilation

$$CX = H_B CZ H_B$$