# Note Holographic Goldstone Boson

(last compiled October 16, 2006)

This is a technical note on the effective potential for $A_5$ in theories with warped extra dimensions.

We consider a 5D gauge theory with the 5th dimension being a finite interval, $x_5 \in [0, L]$. The gravitational background is warped, with the line element

$$ds^2 = a^2(x_5)\eta_{\mu\nu}dx^\mu dx^\nu - dx_5^2 \tag{0.1}$$

e.wb

For most of the discussion the warp factor is an arbitrary function.

For clarity, we first restrict to $SU(2)$ gauge group broken to $U(1)$ by boundary conditions. Then we generalize to more general 5D gauge theories with the boundary conditions breaking pattern $G \to H$. Below the Kaluza-Klein (KK) scale, apart from the massless $H$ gauge field, there are $\#G/H$ scalar excitations originating from the $A_5$ component of the 5D gauge field. These scalars are massless at tree-level. The objective is to compute one loop corrections to their masses. They are generated by interactions of the scalar with 5D nonabelian gauge field and with the 5D fermions, whose presence in the bulk we also assume.

## 1 Holographic image of Coleman and Weinberg

We prove here that a Coleman-Weinberg potential in a 5D KK theory can be rewritten in a form that resembles that of a 4D theory.

We start with the Coleman-Weinberg.jr formula

$$V = \frac{N}{2}\sum_n \int \frac{d^dk}{(2\pi)^d}\log(k^2 + m_n^2) \tag{1.2}$$

Here $N$ measures the number of degrees of freedom at each KK level. ($N = 3$ for a gauge boson, $N = -4$ for a bulk fermion). The momentum integral is harnessed by a dimensional regularization. After angular integration

$$V = \frac{N}{(4\pi)^{d/2}\Gamma(d/2)}\sum_n \int dk k^{d-3}\log\left(1 + \frac{m_n^2}{k^2}\right) \tag{1.3}$$

e.cw

where we dropped an irrelevant infinite constant. Next, we can evaluate the rest of the momentum integral.

$$V = \frac{N}{(4\pi)^{d/2}}\frac{\pi}{d\Gamma(d/2)\sin(\pi d/2)}\sum_n m_n^d \tag{1.4}$$

Using the holomorphic techniques the sum over KK modes can be traded for an integral

$$\sum_n m_n^d = \int_C \frac{dz}{2\pi i} z^d \frac{S'(z)}{S(z)} \tag{1.5}$$

where $S(z)$ is any function with simple zeros on the real axis for $z = m_n$, and the contour $C$ runs down the imaginary axis from $i\infty$ to $-i\infty$ and then via a counterclockwise big circle back to $+i\infty$. If $S(z)$ vanishes sufficiently fast at infinity, the big circle does not contribute and the contour integral reduces to a 1d integral.

$$\sum_n m_n^d = i^d d \int_{-\infty}^{\infty} \frac{dy}{2\pi i} y^{d-1} \log S(iy) \tag{1.6}$$

We can further simplify the integral assuming[1] $S(-iy) = S(iy)$. Then, we obtain

$$\sum_n m_n^d = \frac{d \sin(\pi d/2)}{\pi} \int_0^{\infty} dy y^{d-1} \log S(iy) \tag{1.7}$$

In consequence:

$$V = \frac{N}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^{\infty} dk k^{d-1} \log S(ik) \tag{1.8}$$ e.hcw

This looks like a 4D Coleman-Weinberg formula, with $1 + m^2/k^2$ replaced by some more general form factor $S(ik)$.

To go further we need to know the spectral function $S(z)$ in a theory of interest.

# 2 Toy Model

The 5D action:

$$S = \int d^4x \int_0^L dx_5 \sqrt{g} \left( -\frac{1}{2g_5^2} \mathrm{Tr}\{F_{MN}F_{MN}\} + \overline{\psi}(i\Gamma_N D_N - M)\psi \right) \tag{2.9}$$ e.tma

where

$$A_M = A_M^a \frac{\sigma^a}{2} \qquad D_N = \partial_N - iA_M \qquad F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N] \tag{2.10}$$

$$\psi = \left(\psi_L^+, \psi_L^-, \psi_R^-, \psi_R^+\right)^T \qquad \Gamma^\mu = \gamma^\mu \quad \Gamma^5 = i\gamma_5 \qquad \gamma_5 \psi_{L,R} = \mp \psi_{L,R} \tag{2.11}$$

It is invariant under local gauge transformations

$$\psi \to \omega\psi \qquad A_M \to \omega A_M \omega^{-1} - i\partial_M \omega \omega^{-1} \tag{2.12}$$

with $\omega$ being arbitrary $SU(2)$ matrix.

We shall consider the action (2.9) on the warped background (0.1). The boundary conditions are chosen as[2]

$$\partial_5 A_\mu^3|_{0,L} = A_\mu^{1,2}|_{0,L} = A_5^3|_{0,L} = 0 \tag{2.13}$$ e.tmbc

$$\psi_L^-|_{0,L} = \psi_R^-|_{0,L} = 0 \tag{2.14}$$

These boundary conditions signify that, at tree-level, $A_\mu^3$, $A_5^{1,2}$ and $\psi_{L,R}^+$ yield a massless mode.

[1] This is what happens in real life. This is a consequence of the fact that the equations defining $S(z)$ depend on $z^2$ only, which in turn is a consequence of the fact that it's $p^2$ who is a Lorentz invariant.

[2] There's no need to impose any boundary conditions on $A_5^{1,2}$ as this never appears in any boundary terms from integration by parts.

## 2.1 KK expansion

This subsection discusses KK expansion in a general warped background. For those skilled in higher dimensions there is nothing interesting here.

The quadratic action for the gauge field:

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{2}a^2(x_5)(\partial_5 A_\mu^a - \partial_\mu A_5^a)^2 \tag{2.15}$$

Integrating by parts, and using boundary conditions to get rid of the boundary terms we obtain

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{2}A_\mu^a \partial_5(a^2(x_5)\partial_5 A_\mu^a) + \frac{1}{2}a^2(x_5)(\partial_\mu A_5^a)^2 - \partial_\mu A_\mu^a \partial_5(a^2(x_5)A_5^a) \tag{2.16}$$

We rewrite the 5D gauge field in the KK basis:

$$A_\mu^a(x, x_5) = \sum_n f_n^a(x_5) A_\mu^{n,a}(x) \qquad A_5^a(x, x_5) = \sum_n \tilde{f}_n^a(x_5) A_5^{n,a}(x) \tag{2.17}$$

We wish to arrive at the quadratic action that is diagonal in $n$. To this end we choose the gauge KK profiles $f_n^a$ to be solutions of the mode equation:

$$\partial_5(a^2 \partial_5 f_n) + m_n^2 f_n = 0 \tag{2.18}$$

e.gfme

Furthermore the profiles are subject to the boundary conditions

$$\partial_5 f_n^3|_{0,L} = 0 = f_n^{1,2}|_{0,L} \tag{2.19}$$

e.gfpbc

and the normalization/orthogonality condition $g_5^{-2}\int_0^L f_n f_m = \delta_{nm}$.

Denote the two independent solutions of eq. (2.18) as $J(x_5, m_n)$ and $Y(x_5, m_n)$ (of course they depend also on the warp factor). Then the gauge KK profiles satisfying the UV brane boundary conditions can be written as

$$\begin{aligned} f_n^3 &= C_n^3 \left(J(x_5, m_n)Y'(0, m_n) - Y(x_5, m_n)J'(0, m_n)\right) \\ f_n^{1,2} &= C_n^{1,2} \left(J(x_5, \hat{m}_n)Y(0, \hat{m}_n) - Y(x_5, \hat{m}_n)J(0, \hat{m}_n)\right) \end{aligned} \tag{2.20}$$

e.gfpbc

The KK masses for the Neumann and Dirichlet modes do not have to be the same, that's why we denote them differently. The IR boundary conditions yield the quantization condition

$$\begin{aligned} J'(L, m_n)Y'(0, m_n) - Y'(L, m_n)J'(0, m_n) &= 0 \qquad a = 3\,(\mathit{Neumann}) \\ J(L, \hat{m}_n)Y(0, \hat{m}_n) - Y(L, \hat{m}_n)J(0, \hat{m}_n) &= 0 \qquad a = 1, 2\,(\mathit{Dirichlet}) \end{aligned} \tag{2.21}$$

e.gs

The solutions $m_n$ and $\hat{m}_n$ determine the KK tower for Neumann and Dirichlet modes, respectively. Note in passing that for $m_0 = 0$ we can choose the solutions as

$$J(x_5, 0) = C \qquad Y(x_5, 0) = \int_0^{x_5} dy\, a^{-2}(y) \tag{2.22}$$

Since $J'(x_5, 0) = 0$ the mode with $m_0 = 0$ satisfy the Neumann boundary condition but not the Dirichlet one.It also follows that the zero-mode profile is flat: $f_0^3 = g_5/L^{1/2} \equiv g$.

As for the Goldstone profiles the convenient choice is

$$\begin{aligned}
\tilde{f}_0^{1,2} &= C_H a^{-2}(x_5) \qquad C_H = \frac{g_5}{(\int_0^L a^{-2})^{1/2}} \\
\tilde{f}_n^a &= \partial_5 f_n^a \qquad n \neq 0
\end{aligned} \tag{2.23}$$

Choosing the Goldstone profile proportional to the 5th derivative of the respective gauge KK profile ensures that $A_5^n(x_\mu)$ and $A_\mu^{n'}(x_\mu)$ do not mix for $n = n'$. Indeed

$$\begin{aligned}
\int_0^L a^2 \partial_\mu A_5 \partial_5 A_\mu = \int_0^L \partial_5(a^2 A_5)\partial_\mu A_\mu = \sum_{n,m} \partial_\mu A_\mu^n A_5^m \int_0^L f_n \partial_5(a^2 \partial_5 f_m) \\
= -\sum_{n,m} \partial_\mu A_\mu^n A_5^m m_m^2 \int_0^L f_n f_m = -\sum_n m_n^2 \partial_\mu A_\mu^n A_5^n
\end{aligned} \tag{2.24}$$

We used the mode equation for $f_n$ and the orthonormality property. Note that also $f_0$ is a derivative of the solution for $m_0 = 0$, therefore the Goldstone zero mode will not mix with any other modes. Furthermore, if $f_n^3$ satisfies Neumann BC, $\partial_5 f_n^3$ satisfies Dirichlet BC, $\tilde{f}_n^3|_{0,L} = 0$, consistently with eq. (2.13).

After all that, the quadratic part of the 5D action can be rewritten in the KK basis

$$\begin{aligned}
S_5 = \int d^4x \left\{ -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + \tfrac{1}{2}(\partial_\mu A_5^1)^2 + \tfrac{1}{2}(\partial_\mu A_5^2)^2 \right. \\
\left. + \sum_{n\neq 0} \left( -\tfrac{1}{4} F_{\mu\nu}^{n,a} F_{\mu\nu}^{n,a} + \tfrac{1}{2} m_n^2 (A_\mu^{n,3} - \partial_\mu A_5^{n,3})^2 + \tfrac{1}{2} \hat{m}_n^2 (A_\mu^{n,i} - \partial_\mu A_5^{n,i})^2 \right) \right\}
\end{aligned} \tag{2.25}$$

where $i = 1, 2$. The first line contains the zero-mode kinetic terms. The second line describes the tower of massive gauge fields, each eating his own Goldstone.

We move to the fermions. The quadratic fermionic lagrangian reads

$$\mathcal{L} = i a^3 \overline{\psi}_{L,R}^{\pm} \gamma_\mu \partial_\mu \psi_{L,R}^{\pm} - a^4 \overline{\psi}_{L,R}^{\pm} \left( \gamma_5 \partial_5 + \gamma_5 2 \frac{a'}{a} + M \right) \psi_{R,L}^{\pm} \tag{2.26}$$

We rewrite the fermion fields in the KK basis:

$$\begin{aligned}
\psi_L^{\pm}(x, x_5) &= a^{-2}(x_5) e^{M x_5} \sum_n f_{L,n}^{\pm}(x_5) \psi_{L,n}^{\pm}(x) \\
\psi_R^{\pm}(x, x_5) &= a^{-2}(x_5) e^{-M x_5} \sum_n f_{R,n}^{\pm}(x_5) \psi_{R,n}^{\pm}(x)
\end{aligned} \tag{2.27}$$

We assume the KK profiles satisfy the mode equation

$$\begin{aligned}
\partial_5 f_R^- &= m_n a^{-1} e^{2M x_5} f_L^+ \\
-\partial_5 f_L^+ &= m_n a^{-1} e^{-2M x_5} f_R^- \\
\partial_5 f_R^+ &= m_n a^{-1} e^{2M x_5} f_L^- \\
-\partial_5 f_L^- &= m_n a^{-1} e^{-2M x_5} f_R^+
\end{aligned} \tag{2.28}$$

e.fme

the boundary conditions

$$f_L^-|_{0,L} = f_R^-|_{0,L} = 0 \tag{2.29}$$

and the normalization condition

$$\int_0^L a^{-1} e^{2M x_5} |f_{L,n}^{\pm}|^2 = \int_0^L a^{-1} e^{-2M x_5} |f_{R,n}^{\pm}|^2 = 1 \tag{2.30}$$

For $m_0 = 0$ there are simple solutions to eq. (2.28)

$$f^+_{L,0} = C_L \qquad f^-_{R,0} = 0 \qquad |C_L|^2 = \frac{1}{\int_0^L a^{-1} e^{2Mx_5}} \tag{2.31}$$

$$f^+_{R,0} = C_R \qquad f^-_{L,0} = 0 \qquad |C_R|^2 = \frac{1}{\int_0^L a^{-1} e^{-2Mx_5}} \tag{2.32}$$

The boundary conditions are trivially satisfied, and so the zero modes always exist.

For $m_n \neq 0$ from eq. (2.28) we can derive two second order equations for the Dirichlet modes

$$\left[\partial_5^2 + \left(\frac{a'}{a} + 2M\right)\partial_5 + \frac{m_n^2}{a^2}\right] f^-_{L,n} = 0 \tag{2.33} \qquad \boxed{\texttt{e.fme2}}$$

$$\left[\partial_5^2 + \left(\frac{a'}{a} - 2M\right)\partial_5 + \frac{m_n^2}{a^2}\right] f^-_{R,n} = 0 \tag{2.34} \qquad \boxed{\texttt{e.fme3}}$$

We denote the two independent solutions of the first equation by $J_M(x_5, m_n)$, $Y_M(x_5, m_n)$ (the other is solved by $J_{-M}(x_5, m_n)$, $Y_{-M}(x_5, m_n)$). Then the solution to eq. (2.28) that satisfies the boundary conditions on the UV brane can be formally written as

$$\begin{aligned}
f^-_L &= C_M \left[Y_M(0, \hat{m}_n) J_M(x_5, \hat{m}_n) - J_M(0, \hat{m}_n) Y_M(x_5, \hat{m}_n)\right] \\
f^+_R &= -C_M \frac{a}{\hat{m}_n} e^{2Mx_5} \left[Y_M(0, \hat{m}_n) J'_M(x_5, \hat{m}_n) - J_M(0, \hat{m}_n) Y'_M(x_5, \hat{m}_n)\right] \\
f^-_R &= C_{-M} \left[Y_{-M}(0, m_n) J_{-M}(x_5, m_n) - J_{-M}(0, m_n) Y_{-M}(x_5, m_n)\right] \\
f^+_L &= C_{-M} \frac{a}{\hat{m}_n} e^{-2Mx_5} \left[Y_{-M}(0, m_n) J'_{-M}(x_5, m_n) - J_{-M}(0, m_n) Y'_{-M}(x_5, m_n)\right]
\end{aligned} \tag{2.35}$$

The IR boundary condition yields the quantization condtions

$$\begin{aligned}
Y_M(0, \hat{m}_n) J_M(L, \hat{m}_n) - J_M(0, \hat{m}_n) Y_M(L, \hat{m}_n) &= 0 \\
Y_{-M}(0, m_n) J_{-M}(L, m_n) - J_{-M}(0, m_n) Y_{-M}(L, m_n) &= 0
\end{aligned} \tag{2.36}$$

The solutions $m_n$ and $\hat{m}_n$ describe the two mass towers of fermionic KK modes. Note that for $M = 0$ or $a' = 0$ we have $J_M = J_{-M}$ and, therefore, the $m_n$ and $\hat{m}_n$ towers coincide.

After all this mess the quadratic action for fermions takes the diagonal form

$$\begin{gathered}
\mathcal{L} = i\overline{\psi}_L \gamma_\mu \partial_\mu \psi_L + i\overline{\psi}_R \gamma_\mu \partial_\mu \psi_R \\
\textstyle\sum_n \Big\{ i\overline{\psi}^\pm_{L,n} \gamma_\mu \partial_\mu \psi^\pm_{L,n} + i\overline{\psi}^\pm_{R,n} \gamma_\mu \partial_\mu \psi^\pm_{R,n} \\
-m_n \overline{\psi}^+_{L,n} \psi^-_{R,n} - m_n \overline{\psi}^-_{R,n} \psi^+_{L,n} - \hat{m}_n \overline{\psi}^-_{L,n} \psi^+_{R,n} - \hat{m}_n \overline{\psi}^+_{R,n} \psi^-_{L,n} \Big\}
\end{gathered} \tag{2.37}$$

## 2.2 Tree-level effective action

We have identified the zero modes and their KK profiles. Below the scale set by the lightest massive mode we can consider an effective action that contains the zero mode only. At the tree-level this is obtained simply by replacing the 5D fields in the original 5D action (2.9) with the zero mode profile

$$
\begin{aligned}
A^3_\mu(x,x_5) &\to gA_\mu(x) \\
A^{1,2}_5(x,x_5) &\to C_H a^{-2}(x_5) A^{1,2}_5(x) \\
\psi^+_L(x,x_5) &\to C_L e^{Mx_5} a^{-2}(x_5)\psi_L(x) \\
\psi^+_R(x,x_5) &\to C_R e^{-Mx_5} a^{-2}(x_5)\psi_R(x)
\end{aligned}
\tag{2.38}
$$

where the normalization constants are

$$
g = \frac{g_5}{\sqrt{L}} \quad C_H = g\frac{L^{1/2}}{(\int_0^L a^{-2})^{1/2}} \quad C_L = \frac{1}{(\int_0^L a^{-1}e^{2Mx_5})^{1/2}} \quad C_R = \frac{1}{(\int_0^L a^{-1}e^{-2Mx_5})^{1/2}}
\tag{2.39}
$$

The rest of the fields is set to zero, What we get this way is a 4D $U(1)$ gauge theory with the gauge coupling $g$ coupled to a massless charged scalar $iH(x) = (A^1_5(x) - iA^2_5(x))/\sqrt{2}$ and two massless charged fermions $\psi_L$ and $\psi_R$ with opposite charges, and a half of the scalar charge. Furtermore, fermions and the scalar interact via Yukawa term.

The effective lagrangian:

$$
\mathcal{L}_{eff} = -\frac{1}{4}F^2_{\mu\nu} + i\overline{\psi}_L\gamma^\mu D_\mu\psi_L + i\overline{\psi}_R\gamma^\mu D_\mu\psi_R + |D_\mu H|^2 - y\overline{\psi}_L H\psi_R - y\overline{\psi}_R H^\dagger\psi_L
\tag{2.40}
$$

where

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
D_\mu H &= \partial_\mu H - igA_\mu H \\
D_\mu\psi_L &= \partial_\mu\psi_L - i(g/2)A_\mu\psi_L \\
D_\mu\psi_R &= \partial_\mu\psi_R + i(g/2)A_\mu\psi_R \\
y &= g\sqrt{\frac{L\int_0^L a^{-2}}{2\int_0^L a^{-1}e^{2Mx_5}\int_0^L a^{-1}e^{-2Mx_5}}}
\end{aligned}
\tag{2.41}
$$

## 2.3 One-loop corrections

We start with the holographic CW formula (1.8). For the holographic Goldstone we can set $d = 4$ (since the potential is finite anyway). It turns out that the spectral function $S(z)$ can be written in a particularly simple form

$$
S(z,H) = 1 + F(z)\sin^2(r|H|/f)
\tag{2.42}
$$

We thus find

$$
V(H) = \sum_r \frac{N_r}{4\pi^2}\int_0^\infty dy y^3 \log\left(1 + F_s(iy)\sin^2(r|H|/f)\right)
\tag{2.43}
$$

e.hgp0

with $N_r = 3$ and $r = \sqrt{2}$ for the gauge field, and $N_r = -4N_c$, $r = 1/\sqrt{2}$ for the fermions ($N_r$ is determined by the spin, while $r$ is determined by the representation).

The derivation of this formula is postponed to Section 2.3.1 and Section 2.3.2. Before that let us savour its ingredients. Firstly, we introduced the 'decay constant':

$$f^{-1} = g L^{1/2} \left( \int_0^L dy \, a^{-2}(y) \right)^{1/2} \tag{2.44}$$

It depends on the gauge coupling and the geometry of extra dimensions only (and not on the bulk mass). Secondly, we introduced the **spectral function** $F(z)$. It is spectral in a sense that it has poles on the real axis for $z = m_n(H = 0)$. Thus it carries information about the KK spectrum in the absence of the Higgs vev.

For the gauge field contribution the spectral function can be written as

$$F_{(1)}(z) = \frac{z a_L^{-2}}{J'(L,z) Y(L,z)} \tag{2.45}$$

e.gsf

Here the base functions $J(x_5, z)$ and $Y(x_5, z)$ are definied as two independent solutions of the differential eqaution

$$\left( \partial_5^2 + 2 \frac{a'}{a} \partial_5 + \frac{z^2}{a^2} \right) f = 0 \tag{2.46}$$

e.geom

satisfying the boundary conditions $J(0,z) = 1$, $J'(0,z) = 0$, $Y(0,z) = 0$, $Y'(0,z) = z$. Note that the zeros of the denominator in eq. (2.45) indeed correspond to the spectrum in absence of the Higgs vev, cf. eq. (2.21).

In the case of the 5D fermion the spectral function is even simpler:

$$F_{(1/2)}(z) = -\frac{1}{S_M(L,z) S_{-M}(L,z)} \tag{2.47}$$

e.fsf

Here $C_M(x_5, z)$ and $S_M(x_5, z)$ are two independent solutions of the differential eqaution

$$\left[ \partial_5^2 + \left( \frac{a'}{a} + 2M \right) \partial_5 + \frac{z^2}{a^2} \right] f = 0 \tag{2.48}$$

e.feom

satisfying the boundary conditions $C_M(0,z) = 1$, $C_M'(0,z) = 0$, $S_M(0,z) = 0$, $S_M'(0,z) = z$.

Summarizing, the one-loop potential for a holographic Higgs in any 5D warped background can be calculated from a neat and clever formula

$$V(H) = \sum_r \frac{N_r}{4\pi^2} \int_0^\infty dk k^3 \log \left( 1 + F_r(ik) \sin^2(\lambda_r |H|/f) \right) \tag{2.49}$$

e.hgp

We see that in any background the Goldstone potential is periodic; the Goldstone is a phase as it should be. The spectral functions $F_{(s)}$ for a field with a given spin $s$ can be calculated analitically, provided we are able to solve the equation of motion with a given warp factor. The numerical coefficient $N_r$ depends on the spin (and multiplicity, for example the colour factor of the quarks enters there).

s.gfsH

### 2.3.1 Gauge spectrum in Higgs background

The gauge field:

$$A_\mu = A_\mu^a \frac{\sigma^a}{2} = \frac{1}{2}\left[\begin{array}{cc} A_\mu^3 & \sqrt{2}A_\mu^- \\ \sqrt{2}A_\mu^+ & -A_\mu^3 \end{array}\right] \tag{2.50}$$

where $A_\mu^\pm = A_\mu^1 \pm i A_\mu^2$. The Goldstone background:

$$A_5 \to A_5^i \frac{\sigma^i}{2} = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 0 & iH(x) \\ -iH^\dagger(x) & 0 \end{array}\right] C_H a^{-2}(x_5) \tag{2.51}$$

The quadratic gauge lagrangian in that background:

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{2}a^2(x_5)(\partial_5 A_\mu^3 + hA_\mu^+ + h^\dagger A_\mu^-)^2 + a^2(x_5)|\partial_5 A_\mu^- - hA_\mu^3|^2 \tag{2.52}$$ e.tmqglh

with the shorthand $h \equiv C_H a^{-2}(x_5) H(x)$. This can be rewritten as:

$$\frac{1}{2}A_\mu^\dagger(\partial^2\eta_{\mu\nu} - \partial_\mu\partial_\nu)A_\nu + A_\mu^\dagger D A_\mu \tag{2.53}$$

with $A = (A_\mu^3, A_\mu^+, A_\mu^-)$ and

$$D = \left[\begin{array}{ccc} -\partial_5(a^2\partial_5) + 2a^2|h|^2 & -2a^2 h\partial_5 & -2a^2 h^\dagger \partial_5 \\ 2a^2 h^\dagger \partial_5 & -\partial_5(a^2\partial_5) + a^2|h|^2 & a^2 h^2 \\ 2a^2 h\partial_5 & a^2 h^{\dagger 2} & -\partial_5(a^2\partial_5) + a^2|h|^2 \end{array}\right] \tag{2.54}$$

We perform the KK expansion

$$\begin{aligned} A_\mu^3(x,x_5) &= \sum_n f_n^3(x_5) A_\mu^{n,3}(x) \\ A_\mu^\pm(x,x_5) &= \sum_n f_n^\pm(x_5) A_\mu^{n,\pm}(x) \end{aligned} \tag{2.55}$$

The action (2.52) becomes diagonal in the KK basis if the profiles solve the eigenvalue equation

$$D\left(\begin{array}{c} f_n^3 \\ f_n^+ \\ f_n^- \end{array}\right) = m_n^2 \left(\begin{array}{c} f_n^3 \\ f_n^+ \\ f_n^- \end{array}\right) \tag{2.56}$$

subject to the boundary conditions

$$\partial_5 f_n^3|_{0,L} = f_n^\pm|_{0,L} = 0 \tag{2.57}$$ e.bcaux1

To simplify equation we can change variables as follows

$$\tilde{f}^a \frac{\sigma^a}{2} = \omega f^a \frac{\sigma^a}{2}\omega^\dagger \tag{2.58}$$

where $\omega$ is the gauge transformation removing the Goldstone vev.

$$\omega = \left(\begin{array}{cc} \cos x & \frac{H}{|H|}\sin x \\ -\frac{H^\dagger}{|H|}\sin x & \cos x \end{array}\right) \qquad x = \frac{C_H|H|}{\sqrt{2}}\int_0^{x_5} dy\, a^{-2}(y) \tag{2.59}$$

Explicitly, the change of variables acts as:

$$
\begin{aligned}
\tilde{f}_n^3 &= \cos 2x f_n^3 + \frac{1}{\sqrt{2}} \sin 2x (f_n^- H^\dagger/|H| + f_n^+ H/|H|) \\
\tilde{f}_n^+ &= \cos^2 x f_n^+ - (H^{\dagger 2}/|H|^2) \sin^2 x f_n^- - \frac{1}{\sqrt{2}} \sin 2x (H^\dagger/|H|) f_n^3 \\
\tilde{f}_n^- &= \cos^2 x f_n^- - (H^2/|H|^2) \sin^2 x f_n^+ - \frac{1}{\sqrt{2}} \sin 2x (H/|H|) f_n^3
\end{aligned}
\tag{2.60}
$$

The inverse transformation $f_n(\tilde{f}_n)$ is analogous with $x \to -x$.

In the twiddle basis the eigenvalue equation is very simple

$$
\left[\partial_5(a^2\partial_5) + m_n^2\right] \begin{pmatrix} \tilde{f}_n^3 \\ \tilde{f}_n^+ \\ \tilde{f}_n^- \end{pmatrix}
\tag{2.61}
$$

Using base solutions of eq. (2.46) we can formally write down the solutions as

$$
\begin{aligned}
\tilde{f}_n^3(x_5) &= C_n^3 J(x_5, m_n) \\
\tilde{f}_n^+(x_5) &= C_n^+ Y(x_5, m_n) \\
\tilde{f}_n^-(x_5) &= C_n^- Y(x_5, m_n)
\end{aligned}
\tag{2.62}
$$

The UV boundary condition eq. (2.57) are satisfied, as $f_n(0) = \tilde{f}_n(0)$. The IR boundary conditions:

$$
\begin{aligned}
0 &= \cos 2x_L \partial_5 \tilde{f}_n^3(L) - \frac{1}{\sqrt{2}} \sin 2x_L (\partial_5 \tilde{f}_n^-(L) H^\dagger/|H| + \partial_5 \tilde{f}_n^+(L) H/|H|) \\
0 &= \cos^2 x_L \tilde{f}_n^+(L) - (H^{\dagger 2}/|H|^2) \sin^2 x_L \tilde{f}_n^-(L) + \frac{1}{\sqrt{2}} \sin 2x_L (H^\dagger/|H|) \tilde{f}_n^3(L) \\
0 &= \cos^2 x_L \tilde{f}_n^-(L) - (H^2/|H|^2) \sin^2 x_L \tilde{f}_n^+(L) + \frac{1}{\sqrt{2}} \sin 2x_L (H/|H|) \tilde{f}_n^3(L)
\end{aligned}
\tag{2.63}
$$

become:

$$
\begin{bmatrix}
\cos 2x_L J'(L, m_n) & -\frac{1}{\sqrt{2}}\frac{H}{|H|} \sin 2x_L Y'(L, m_n) & -\frac{1}{\sqrt{2}}\frac{H^\dagger}{|H|} \sin 2x_L Y'(L, m_n) \\
\frac{1}{\sqrt{2}}\frac{H^\dagger}{|H|} \sin 2x_L J(L, m_n) & \cos^2 x_L Y(L, m_n) & -\frac{H^{\dagger 2}}{|H|^2} \sin^2 x_L Y(L, m_n) \\
\frac{1}{\sqrt{2}}\frac{H}{|H|} \sin 2x_L J(L, m_n) & -\frac{H^2}{|H|^2} \sin^2 x_L Y(L, m_n) & \cos^2 x_L Y(L, m_n)
\end{bmatrix}
\begin{pmatrix} C_n^3 \\ C_n^+ \\ C_n^- \end{pmatrix} = 0
\tag{2.64}
$$

There is one eigenvector $(0, H, -H^\dagger)$ corresponding to the quantization condition $Y(L, m_n) = 0$ (this is the same as the quantization condition for the Dirichlet tower without the Goldstone vev). This mode's mass does not feel the Goldstone vev, therefore it does not contribute to the Goldstone potential. The remaining two eigenvectors correspond to the quantization condition:

$$
1 + F(m_n) \sin^2 \left[\sqrt{2}|H| g_5 \sqrt{\int_0^L a^{-2}}\right] = 0 \qquad F(m_n) = \frac{Y'(L, m_n) J(L, m_n)}{J'(L, m_n) Y(L, m_n)} - 1
\tag{2.65}
$$

e.tmgsp

Using the Wronskian (4.102) we can further simplify the form factor

$$F(z) = \frac{z a_L^{-2}}{J'(L,m_n) Y(L,m_n)} \tag{2.66}$$

We also rewrite the spectral formula as

$$1 + F(z)\sin^2\left(\frac{\sqrt{2}|H|}{f}\right) \qquad f^{-1} = g_5 \sqrt{\int_0^L a^{-2}} \tag{2.67}$$

s.fsH

### 2.3.2 Fermionic spectrum in Higgs background

The quadratic fermionic lagrangian in the presence of non-zero vev $H(x) = H$ (reads)

$$\begin{aligned} \mathcal{L} = & \, i a^3 \overline{\psi}^{\pm}_{L,R} \gamma_\mu \partial_\mu \psi^{\pm}_{L,R} - a^4 \overline{\psi}^{\pm}_{L,R} \left(\gamma_5 \partial_5 + \gamma_5 2 \tfrac{a'}{a} + M\right) \psi^{\pm}_{R,L} \\ & - \tfrac{C_H a^2}{\sqrt{2}} \left( H \overline{\psi}^{+}_{L} \psi^{+}_{R} - H \overline{\psi}^{-}_{R} \psi^{-}_{L} - H^\dagger \overline{\psi}^{-}_{L} \psi^{-}_{R} + H^\dagger \overline{\psi}^{+}_{R} \psi^{+}_{L} \right) \end{aligned} \tag{2.68}$$

We expand into KK modes as

$$\begin{aligned} \psi^{\pm}_L &= a^{-2} \sum_n f^{\pm}_{L,n}(x_5) \psi_{L,n}(x) \\ \psi^{\pm}_R &= a^{-2} \sum_n f^{\pm}_{R,n}(x_5) \psi_{R,n}(x) \end{aligned} \tag{2.69}$$

The action takes the diagonal form

$$\mathcal{L} = \sum_n \left\{ i \overline{\psi}_{L,n} \gamma_\mu \partial_\mu \psi_{L,n} + i \overline{\psi}_{R,n} \gamma_\mu \partial_\mu \psi_{R,n} - m_n(H) \overline{\psi}_{L,n} \psi_{R,n} - m_n(H) \overline{\psi}_{R,n} \psi_{L,n} \right\} \tag{2.70}$$

provided the KK profiles satisfy

$$\begin{aligned} (\partial_5 + M) f^-_R + \frac{h}{\sqrt{2}} f^+_R &= m_n(H) a^{-1} f^+_L \\ (-\partial_5 + M) f^+_L - \frac{h}{\sqrt{2}} f^-_L &= m_n(H) a^{-1} f^-_R \\ (\partial_5 + M) f^+_R - \frac{h^\dagger}{\sqrt{2}} f^+_R &= m_n(H) a^{-1} f^-_L \\ (-\partial_5 + M) f^-_L + \frac{h^\dagger}{\sqrt{2}} f^+_L &= m_n(H) a^{-1} f^+_R \end{aligned} \tag{2.71}$$

(with the shorthand $h \equiv C_H a^{-2}(x_5) H(x)$), and the boundary conditions

$$f^-_L|_{0,L} = f^-_R|_{0,L} = 0 \tag{2.72}$$

This is a complicated system of coupled equations. But we can simplify them by defining

$$\begin{aligned} \tilde{f}^+_R &= \cos x f^+_R - \frac{H^\dagger}{|H|} \sin x f^-_R \\ \tilde{f}^-_R &= \cos x f^-_R + \frac{H}{|H|} \sin x f^+_R \\ \tilde{f}^-_L &= \cos x f^-_L - \frac{H^\dagger}{|H|} \sin x f^+_L \\ \tilde{f}^+_L &= \cos x f^+_L + \frac{H}{|H|} \sin x f^-_L \end{aligned} \tag{2.73}$$

where

$$x = \frac{C_H|H|}{\sqrt{2}} \int_0^{x_5} dy\, a^{-2}(y) \tag{2.74}$$

e.xd

This is, of course, the $A_5$ vev removing gauge transformation with $\omega$

$$\omega = \begin{pmatrix} \cos x & \frac{H}{|H|}\sin x \\ -\frac{H^\dagger}{|H|}\sin x & \cos x \end{pmatrix} \tag{2.75}$$

acting on $SU(2)$ fermions.

In fact, the lower integration variable could be anything, but this choice is the most convenient. In the twiddle variables we get two decoupled sets of equations.

$$\begin{aligned}
(\partial_5 + M)\tilde{f}_R^- &= m_n(H)a^{-1}\tilde{f}_L^+ \\
(-\partial_5 + M)\tilde{f}_L^+ &= m_n(H)a^{-1}\tilde{f}_R^- \\
(\partial_5 + M)\tilde{f}_R^+ &= m_n(H)a^{-1}\tilde{f}_L^- \\
(-\partial_5 + M)\tilde{f}_L^- &= m_n(H)a^{-1}\tilde{f}_R^+
\end{aligned} \tag{2.76}$$

e.fmeH

These are equations of motion with no $H$ vev, cf. eq. (2.28). The UV boundary conditions remain simple

$$\tilde{f}_L^-(0) = \tilde{f}_R^-(0) = 0 \tag{2.77}$$

thanks to our choice of the lower integration variable in eq. (2.74). The memory of $H$ is now stored in the boundary condtions at the IR brane.

$$\begin{aligned}
\cos x_L \tilde{f}_L^-(L) - \frac{H^\dagger}{|H|}\sin x_L \tilde{f}_L^+(L) &= 0 \\
\cos x_L \tilde{f}_R^-(L) - \frac{H}{|H|}\sin x_L \tilde{f}_R^+(L) &= 0
\end{aligned} \tag{2.78}$$

with $x_L = x(x_5 = L)$. The solution to eq. (2.76) that satisfies the boundary conditions on the UV brane can be formally written as

$$\begin{aligned}
f_{R,n}^- &= c_{R,n}e^{-Mx_5}S_{-M}(x_5, m_n) \\
f_{L,n}^+ &= c_{R,n}e^{-Mx_5}\frac{a}{m_n}S'_{-M}(x_5, m_n) \\
f_{L,n}^- &= c_{L,n}e^{Mx_5}S_M(x_5, m_n) \\
f_{R,n}^+ &= -c_{L,n}e^{Mx_5}\frac{a}{m_n}S'_M(x_5, m_n)
\end{aligned} \tag{2.79}$$

where we define the base functions $C_M(x_5, z)$ and $S_M(x_5, z)$ as the two independent solutions of the differential equation:

$$\left[\partial_5^2 + \left(\frac{a'}{a} + 2M\right)\partial_5 + \frac{z^2}{a^2}\right] f = 0 \tag{2.80}$$

satisfying the boundary conditions

$$C_M(0,z) = 1 \qquad C'_M(0,z) = 0 \qquad S_M(0,z) = 0 \qquad S'_M(0,z) = z \tag{2.81}$$

The boundary conditions on the IR brane determine the ratio $c_L/c_R$ and the quantization condition:

$$1 + F(m_n)\sin^2\left(\frac{|H|}{\sqrt{2}f}\right) = 0 \qquad F(m_n) = -\frac{a_L^2 S'_M(L,m_n)S'_{-M}(L,m_n)}{z^2 S_M(L,m_n)S_{-M}(L,m_n)} - 1 \tag{2.82}$$ e.mqc

Using the crowian (4.113) we can further simplify the form factor:

$$F(m_n) = -\frac{1}{S_M(L,m_n)S_{-M}(L,m_n)} \tag{2.83}$$ e.fsf1

Equation (2.82) can be solved for $m_n(H)$. This $m_n(H)$ can be then plugged into the Coleman-Weinberg formula to compute the effective potential for $H$.

# 3 Generalization to arbitrary gauge groups

We generalize the computation of the holographic Goldstone potential to 5D theories with a gauge group $G$ broken by boundary condition to its subgroup $H$. Our analysis is not fully general yet:

1. We consider only orbifold-like breaking $G \to H$, that is, we assume that one can assign $Z_2$ quantum numbers to the generators of $G$ such that the generators of $H$ have positive parity and the generators of $G/H$ have negative parity. I think this excludes breaking the rank of $G$.

2. We assume the same pattern of breaking on both boundaries.

3. We do not consider any boundary terms (in phenomenological models we might need boundary mass terms for the gauge fields, or boundary gauge kinetic terms to get the Weinberg angle right).

For the time being, however, we should be happy with what we have ;-)

We denote generators of the fundamental represantotion of $G$ by $T^\alpha$, the generators of $H$ by $T^a$ and the coset generators by $T^{\hat a}$. The existence of $Z_2$ implies the commutation relations

$$[T^a, T^b] = i f^{abc} T^c \quad [T^a, T^{\hat b}] = i f^{a\hat b\hat c} T^{\hat c} \quad [T^{\hat a}, T^{\hat b}] = i f^{\hat a\hat b c} T^c \tag{3.84}$$

The fermions $\psi = (\psi^m, \psi^{\hat n})$ are assumed to transform under representation $t^\alpha$. To preserve $Z_2$ we need to assume that $(t^a)_{m\hat n} = (t^a)_{\hat m n} = 0$ and $(t^{\hat a})_{mn} = (t^a)_{\hat m\hat n} = 0$.

The zero modes reside in $A_\mu^a$ and in $A_5^{\hat a}$, the latter is our holographic Goldstone. We can obtain the effective theory by inserting the profiles in the 5D action:

$$\begin{array}{rcl} A_\mu^a(x,x_5) & \to & g A_\mu^a(x) \\ A_5^{\hat a}(x,x_5) & \to & C_H a^{-2}(x_5) A_5^{\hat a}(x) \\ \psi_L^m(x,x_5) & \to & C_L e^{Mx_5} a^{-2}(x_5)\psi_L^m(x) \\ \psi_R^{\hat m}(x,x_5) & \to & C_R e^{-Mx_5} a^{-2}(x_5)\psi_R^{\hat m}(x) \end{array} \tag{3.85}$$

where the normalization constants are

$$g = \frac{g_5}{\sqrt{L}} \quad C_H = g\frac{L^{1/2}}{(\int_0^L a^{-2})^{1/2}} \quad C_L = -\frac{1}{(\int_0^L a^{-1}e^{2Mx_5})^{1/2}} \quad C_R = \frac{1}{(\int_0^L a^{-1}e^{-2Mx_5})^{1/2}} \tag{3.86}$$

and integrating over the 5th dimension. Doing so we obtain

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}(\partial_\mu A_5^{\hat a} - igA^c_\mu (t^c_H)_{\hat a\hat b}A_5^{\hat b})^2 \\ &+i\overline{\psi}^m_L\gamma^\mu D_\mu \psi^m_L + i\overline{\psi}^{\hat m}_R\gamma^\mu D_\mu\psi^{\hat m}_R \\ &-yA_5^{\hat a}\overline{\psi}^m_L(t_F^{\hat a})_{m\hat n}\psi^{\hat n}_R - yA_5^{\hat a}\overline{\psi}^{\hat m}_R(t_F^{\hat a})_{\hat m n}\psi^n_L\end{aligned} \tag{3.87}$$

where

$$\begin{aligned}F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\mu \\ D_\mu H &= \partial_\mu H - igA_\mu H \\ D_\mu \psi^m_L &= \partial_\mu \psi^m_L - igA^a_\mu (t^a_L)_{mn}\psi^n_L \\ D_\mu \psi^{\hat m}_R &= \partial_\mu \psi^{\hat m}_R - igA^a_\mu (t^a_R)_{\hat m\hat n}\psi^{\hat n}_R \\ y &= g\sqrt{\frac{L\int_0^L a^{-2}}{\int_0^L a^{-1}e^{2Mx_5}\int_0^L a^{-1}e^{-2Mx_5}}} \\ (t^c_H)_{\hat a\hat b} &= if^{c\hat b\hat a} \\ (t^a_L)_{mn} &= (t^a)_{mn} \\ (t^a_R)_{\hat m\hat n} &= (t^a)_{\hat m\hat n}\end{aligned} \tag{3.88}$$

Note that by virtue of $Z_2$ symmetry obeyed by the generators (combined with the original $G$ commutation relations) $t^a_L$, $(t^a_R)$ and $t^a_H$ indeed furnish a represenation of $H$: $[t^a, t^b] = if^{abc}t^c$.

Now, let us look at the diagonalization of this more general theory in the presence of the Goldstone background $\langle A^a_5\rangle$. As usually, we expand

$$\begin{aligned}A^a_\mu(x, x_5) &= \sum_n f^a_n(x_5)A^{n,a}_\mu(x) \\ A^\pm_\mu(x, x_5) &= \sum_n f^{\hat a}_n(x_5)A^{n,\hat a}_\mu(x)\end{aligned} \tag{3.89}$$

The profiles satisfy the equation of motion in the presence of the background Goldstone field. We go to the twiddle basis, in which the Goldstone vev vanishes. This is done by means of the transformation,

$$\tilde f^\alpha(x_5)T^\alpha = \omega f^\alpha(x_5)T^\alpha\omega^{-1} \tag{3.90}$$

e.vrt

where $\omega$ is the gauge transformation removing the vev,

$$\omega(x_5) = \exp(-iC_H A_5^{\hat a}T^a\int_0^{x_5} dya^{-2}(y)) \tag{3.91}$$

In the twiddle basis we find a solution satisfying the UV boundary condtions

$$\begin{aligned}\tilde f^a_n &= C_{n,a}J(x_5, m_n) \\ \tilde f^{\hat a}_n &= C_{n,\hat a}Y(x_5, m_n)\end{aligned} \tag{3.92}$$

To unravel the IR boundary conditions we need the relation between $f(L)$ and $\tilde{f}(L)$. From eq. (3.90), infinitesimaly,

$$\begin{aligned} f^a(L) &= \tilde{f}^a(L) - \frac{1}{2}\frac{A_5^{\hat{a}}A_5^{\hat{b}}}{f^2} f^{a\hat{a}\hat{c}} f^{b\hat{b}\hat{c}} \tilde{f}^b(L) - f^{a\hat{b}\hat{c}}\frac{A_5^{\hat{b}}}{f} f^{\hat{c}}(L) \\ f^{\hat{a}}(L) &= \tilde{f}^{\hat{a}}(L) - f^{\hat{a}\hat{b}c}\frac{A_5^{\hat{b}}}{f} f^c(L) \end{aligned} \tag{3.93}$$

(for our purpose it is enough to expand the second to linear order only). So that the IR boundary conditions

$$\begin{aligned} 0 &= J'(L,m_n)C^a - \frac{1}{2}\frac{A_5^{\hat{a}}A_5^{\hat{b}}}{f^2} f^{a\hat{a}\hat{c}} f^{b\hat{b}\hat{c}} C^b - f^{a\hat{b}\hat{c}}\frac{A_5^{\hat{b}}}{f} Y'(L,m_n)C^{\hat{c}} \\ 0 &= Y(L,m_n)C^{\hat{a}} - f^{\hat{a}\hat{b}c}\frac{A_5^{\hat{b}}}{f} J(L,m_n)C^c \end{aligned} \tag{3.94}$$

The quantization condition follows

$$\frac{J'(L,m_n)Y(L,m_n)}{J(L,m_n)Y'(L,m_n)}(C_a - \frac{1}{2}\frac{A_5^{\hat{a}}A_5^{\hat{b}}}{f^2} f^{a\hat{a}\hat{c}} f^{b\hat{b}\hat{c}} C^b) = -(t_H^a)_{\hat{a}\hat{c}}(t_H^b)_{\hat{c}\hat{b}}\frac{A_5^{\hat{a}}A_5^{\hat{b}}}{f^2} C_b \tag{3.95}$$

We introduce the eigenvalues of the gauge boson mass matrix:

$$[(t_H^a)_{\hat{a}\hat{c}}(t_H^b)_{\hat{c}\hat{b}} A_5^{\hat{a}}A_5^{\hat{b}}]C_b = \lambda_r^2 h^2 C_a \qquad h^2 = A_5^{\hat{a}}A_5^{\hat{a}} \tag{3.96}$$

Note that some of the $\lambda_r$ may vanish; those eigenvector do not contribute to the effective potential for the Goldstone. The quantization condtion can be rewritten as:

$$\frac{J'(L,m_n)Y(L,m_n)}{J(L,m_n)Y'(L,m_n)}(1 - \frac{\lambda_r^2 h^2}{2f^2}) = -\frac{\lambda_r^2 h^2}{f^2} \tag{3.97}$$

We can guess that generalization to higher orders in $h^2$ leads to

$$1 + F_{(1)}(m_n)\sin^2\left(\frac{\lambda_r h}{f}\right) \qquad F_{(1)}(z) = \frac{J(L,z)Y'(L,z)}{J'(L,z)Y(L,z)} - 1 \tag{3.98}$$

For fermions, analogously we get

$$1 + F_{(1/2)}(m_n)\sin^2\left(\frac{\lambda_r h}{f}\right) \qquad F_{(1/2)}(z) = -\frac{1}{S_M(L,z)S_{-M}(L,z)} \tag{3.99}$$

where $\lambda_r$, this time, are related to the eigenvalues equation

$$(t^{\hat{a}})_{m\hat{n}}(t^{\hat{b}})_{\hat{n}n} A_5^{\hat{a}}A_5^{\hat{b}} C_n = \lambda_r^2 h^2 C_m \tag{3.100}$$

# 4 Properties of the form factors

aka spectral functions, aka propagation functions

## 4.1 General properties

Even without assuming a specific form of the background and finding explicit solutions to the equations of motion we are able to say surprisingly much about the spectral function.

Gauge spectral functions

The functions $J(x_5, z)$ and $Y(x_5, z)$ solving eq. (2.46) enjoy :-) the following properties

1. **Normalization** We can always choose the solutions such that

$$\begin{array}{ll} J(0,z) = 1 & J'(0,z) = 0 \\ Y(0,z) = 0 & Y'(0,z) = z \end{array} \tag{4.101}$$

e.ns

   and we will stick to this normalization henceforth. We also fix, that the solutions are real on the real $z$ axis.

2. **Wronskian.** As usually for solutions of the second order differential equation there is a gospodin Wronski relation between the two solutions

$$Y'(x_5, z)J(x_5, z) - J'(x_5, z)Y(x_5, z) = a^{-2}(x_5)W(z) \tag{4.102}$$

e.gbfw

   We will normalize our solutions such that $W(z) = z$. Note, that normalization is irrelevant for computing the spectral function, as all normalization factors cancel in eq. (2.45).

3. **Low energy.** For small $z$ we can approximately solve eq. (2.46) as

$$\begin{aligned} J(x_5, z) &= 1 - z^2 \int_0^{x_5} (y a^{-2}(y)) + \mathcal{O}(z^4) \\ Y(x_5, z) &= z \left\{ \int_0^{x_5} a^{-2}(y) - z^2 \int_0^{x_5} a^{-2}(y) \int^y \int^{y'} a^{-2}(y'') + \mathcal{O}(z^4) \right\} \end{aligned} \tag{4.103}$$

e.jysz

   The normalization is chosen that eq. (4.101) is satisfied.

4. **High energy.** For large enoguh $z$ the $z^2/a^2$ term dominates the $2(a'/a)\partial_5$ in the equation of motion, for all values of $x_5$. We refer to this range of $z$ as the *flat regime*. The approximate solutions can be written as

$$\begin{aligned} J(x_5, z) &= a^{-1/2}(x_5) \cos(z \int_0^{x_5} a^{-1}(y)) \\ Y(x_5, z) &= a^{-1/2}(x_5) \sin(z \int_0^{x_5} a^{-1}(y)) \end{aligned} \tag{4.104}$$

e.jylz

   The Karenina factor $a^{-1/2}$ in front is for satisfying Wronski. We also imposed the normalization condtions (4.101), as here we assume that $z$ is large enough so that this solution is valid at $x_5 = 0$.

5. **Intermediate energy.** We can also consider the range of $z$ such that the $z^2/a^2$ term dominates close to $x_5 \sim L$, while it is a small perturbation for $x_5 \sim 0$. We refer to this range of $z$ as the *warped regime*. Close to the IR brane, where $z^2/a^2$ is sizable we can write the solutions as

$$\begin{aligned} J(x_5, z) &= a^{-1/2}(x_5)|\alpha| \cos(z \int_0^{x_5} a^{-1}(y) + \phi_\alpha) \\ Y(x_5, z) &= a^{-1/2}(x_5)|\beta| \sin(z \int_0^{x_5} a^{-1}(y) + \phi_\beta) \end{aligned} \quad (4.105) \quad \boxed{\texttt{e.jyiz}}$$

The introduced the complex number coefficients $\alpha = |\alpha|e^{i\phi_\alpha}$, $\beta = |\beta|e^{i\phi_\beta}$. For fun, i'm going to call them the Bogoliubov coefficient. They are, in general, dependent on $z$. The phase shift appears because we cannot impose the normalization (4.101) now ($x_5 = 0$ is outside of the range of validity of the solution). With our normalization of the Wronskian the Bogoliubov coefficients satisfy the relation

$$|\beta\alpha| \cos(\phi_\beta - \phi_\alpha) = 1 \quad (4.106) \quad \boxed{\texttt{e.bw}}$$

Using the Wronski relation and the our choice of normalization we can greatly compactify the expression (2.45) for the gauge spectral function.

$$F_{(1)}(z) = \frac{z}{a_L^2 J'(L, z) Y(L, z)} \quad (4.107) \quad \boxed{\texttt{e.gsfss}}$$

Neat, isn't it :-)

Inserting this solutions into the general form of the spectral function (4.107) we easily obtain the asymptotics:

$$F_{(1)}(z) \approx -\frac{1}{L \int_0^L a^{-2}(y) z^2} \qquad z \to 0 \quad (4.108) \quad \boxed{\texttt{e.gsfsz}}$$

$$F_{(1)}(z) \approx -\frac{1}{\sin^2(\int_0^L a^{-1}(y) z)} \qquad z \to \infty \quad (4.109) \quad \boxed{\texttt{e.gsflz}}$$

In the warped regime we find:

$$F_{(1)}(z) \approx -\frac{\cos(\phi_\alpha - \phi_\beta)}{\sin(\int_0^L a^{-1}(y) z + \phi_\alpha) \sin(\int_0^L a^{-1}(y) z + \phi_\beta)} \quad (4.110)$$

The pole at $z \to 0$ is of course a consequence of the presence of a massless gauge boson in the spectrum. As a consequence, the Higgs potential (2.49) cannot be expanded in powers of $\sin^2(2H/f)$: the coefficients of $\sin^4(2H/f)$ and higher terms are IR divergent. There is nothing strange in it; the Higgs potential is non-analytic in the presence of massless particles.

$F(iy)$ at $y \to \infty$ is exponentially suppressed, $F(iy) \sim e^{-|c|y}$. This ensures that the $y$ integral in eq. (2.49) is UV convergent. It's also IR convergent unless we try expand in $\sin^2(2H/f)$. So we can see (what we have already known anyway) that the holographic Higgs potential is finite and calculable.

Fermionic spectral functions

The functions $J_M(x_5, z)$ and $Y_M(x_5, z)$ solving eq. (2.48) enjoy the following properties

1. **Normalization.** We can always choose the solutions such that

$$
\begin{array}{ll}
J_M(0,z) = 1 & J_M'(0,z) = 0 \\
Y_M(0,z) = 0 & Y_M'(0,z) = z
\end{array}
\tag{4.111}
$$
e.nsf

   We also fix, that the solutions are real on the real $z$ axis.

2. **Wronskian.**

   As usually for solutions of the second order differential equation there is a gospodin Wronski relation between the two solutions

$$
Y_M'(x_5,z)J_M(x_5,z) - J_M'(x_5,z)Y_M(x_5,z) = a^{-1}(x_5)e^{-2Mx_5}W(M,z) \tag{4.112}
$$
e.fbfw

   With our normalization, $W(M,z) = z$.

3. **Crowian.** There is another funny (and useful!) relation, this time between the solutions with opposite $M$,

$$
a^2(x_5)Y_M'(x_5,z)Y_{-M}'(x_5,z) + z^2 Y_M(x_5,z)Y_{-M}(x_5,z) = K(M,z) \tag{4.113}
$$
e.fbfc

   where the rhs. is independent of $x_5$. In parallel to the wronskian $W(M,z)$ the factor $K(M,z)$ will be called crowian. With our normalization $K(M,z) = z^2$.

4. **Low energy.**

   For small $z$ we can approximately solve eq. (2.48) as

$$
\begin{aligned}
J_M(x_5,z) &= 1 - z^2 \int_0^{x_5} e^{-2My}a^{-1}(y) \int_0^y e^{2My'}a^{-1}(y') + \mathcal{O}(z^4) \\
Y_M(x_5,z) &= z\left\{ \int_0^{x_5} a^{-1}(y)e^{-2My} \right. \\
&\left. - z^2 \int_0^{x_5} a^{-1}(y)e^{-2My} \int_0^y a^{-1}(y')e^{2My'} \int_0^{y'} a^{-1}(y'')e^{-2My''} + \mathcal{O}(z^4) \right\}
\end{aligned}
\tag{4.114}
$$
e.jmymsz

   The normalization is chosen such that indeed $W(z) = z$.

5. **High energy.** For large enoguh $z$ the $z^2/a^2$ term dominates the $((a'/a) - 2M)\partial_5$ in the equation of motion, for all values of $x_5$. We refer to this range of $z$ as the *flat regime*. The approximate solutions can be written as

$$
\begin{aligned}
J_M(x_5,z) &= e^{-Mx_5}\cos(z\int_0^{x_5} a^{-1}(y)) \\
Y_M(x_5,z) &= e^{-Mx_5}\sin(z\int_0^{x_5} a^{-1}(y))
\end{aligned}
\tag{4.115}
$$
e.jmymlz

   The factor $e^{-Mx_5}$ in front is for satisfying Wronski. We also imposed the normalization condtions (4.111), as here we assume that $z$ is large enough so that this solution is valid at $x_5 = 0$.

6. **Intermediate energy.** We can also consider the range of $z$ such that the $z^2/a^2$ term dominates close to $x_5 \sim L$, while it is a small perturbation for $x_5 \sim 0$. We refer to this range of $z$ as the *warped regime*. Close to the IR brane, where $z^2/a^2$ is sizable we can write the solutions as

$$
\begin{aligned}
J_M(x_5, z) &= |\alpha| e^{-M x_5} \cos(z \int_0^{x_5} a^{-1}(y) + \phi_\alpha) \\
Y_M(x_5, z) &= |\beta| e^{-M x_5} \sin(z \int_0^{x_5} a^{-1}(y) + \phi_\beta)
\end{aligned}
\tag{4.116}
$$

The introduced the complex number coefficients $\alpha = |\alpha| e^{i\phi_\alpha}$, $\beta = |\beta| e^{i\phi_\beta}$. For fun, i'm going to call them the Bogoliubov coefficient. They are, in general, dependent on $z$. The phase shift appears because we cannot impose the normalization (4.101) now ($x_5 = 0$ is outside of the range of validity of the solution). With our normalization of the Wronskian the Bogoliubov coefficients satisfy the relation

$$
|\beta\alpha| \cos(\phi_\beta - \phi_\alpha) = 1 \tag{4.117}
$$

The compactified formula for the fermionic spectral function

$$
F_{(1/2)}(z) = -\frac{1}{Y_M(L, z) Y_{-M}(L, z)} \tag{4.118}
$$

e.fsfs

This one is reallllly neat :-)

Inserting this solutions into the general form of the spectral function (4.118) we easily obtain the asymptotic :

$$
F_{(1/2)}(z) \approx -\frac{1}{\int_0^L a^{-1}(y) e^{-2My} \int_0^L a^{-1}(y) e^{2My} z^2} \qquad z \to 0 \tag{4.119}
$$

e.fsfsz

$$
F_{(1/2)}(z) \approx -\frac{1}{\sin^2(\int_0^L a^{-1}(y) z)} \qquad z \to \infty \tag{4.120}
$$

e.fsflz

In the warped regime:

$$
F_{(1/2)}(z) \approx \frac{1}{|\beta_M||\beta_{-M}| \sin(z \int_0^{x_5} a^{-1}(y) + \phi_{\beta_M}) \sin(z \int_0^{x_5} a^{-1}(y) + \phi_{\beta_{-M}})} \tag{4.121}
$$

## 4.2 Particular form factors

Let us now investigate the gauge spectral function in concrete backgrounds.

### 4.2.1 Flat model

For flat space $a(x_5) = 1$ and everything is easy. The solutions to eq. (2.46) are $e^{\pm i z x_5 5}$. We pick up the following combinations as the gauge base functions:

$$
J(x_5, z) = \cos(z x_5) \qquad Y(x_5, z) = \sin(z x_5) \tag{4.122}
$$

There is no warped regime in the case, so the Bogoliubov coefficients are simply $\alpha = \beta = 1$. Inserting the solutions into the mother formula eq. (4.107) we get

$$F_{(1)}(z) = -\frac{1}{\sin^2(zL)} \tag{4.123}$$

The fermionic base functions:

$$J_M(x_5, z) = e^{-Mx_5}\cos(\sqrt{z^2 - M^2}x_5) \qquad Y_M(x_5, z) = \frac{z}{\sqrt{z^2 - M^2}}e^{-Mx_5}\sin(\sqrt{z^2 - M^2}x_5) \tag{4.124}$$

From eq. (4.118), the fermionic propagation function:

$$F_{(1/2)}(z) = -\frac{z^2 - M^2}{z^2}\frac{1}{\sin^2(\sqrt{z^2 - M^2}L)} \tag{4.125}$$

So in the flat space, the general large $z$ asymptotic formulae remain valid for all $z$.

### 4.2.2 Raman-Sundrum model

For AdS$_\mathbf{5}$space we insert $a(x_5) = e^{-kx_5}$ and almost as easily solve eq. (2.46) in terms of Bessel functions. The solution is $a^{-1}Z_1(m/ka)$ (recall also that $[a^{-1}Z_1(m/ka)]' = za^{-2}Z_0(m/ka)$) and we pick up the following combinations

$$\begin{aligned} J(x_5, z) &= \frac{\pi z}{2k}a^{-1}(x_5)\left[Y_0\left(\frac{z}{ka_0}\right)J_1\left(\frac{z}{ka(x_5)}\right) - J_0\left(\frac{z}{ka_0}\right)Y_1\left(\frac{z}{ka(x_5)}\right)\right] \\ Y(x_5, z) &= \frac{\pi z}{2k}a^{-1}(x_5)\left[J_1\left(\frac{z}{ka_0}\right)Y_1\left(\frac{z}{ka(x_5)}\right) - Y_1\left(\frac{z}{ka_0}\right)J_1\left(\frac{z}{ka(x_5)}\right)\right] \end{aligned} \tag{4.126}$$

For large $z$

$$\begin{aligned} J(x_5, z) &\approx a^{-1/2}\cos\left(z\frac{a^{-1} - 1}{k}\right) \\ Y(x_5, z) &\approx a^{-1/2}\sin\left(z\frac{a^{-1} - 1}{k}\right) \end{aligned} \tag{4.127}$$

in agreement with eq. (4.104). For intermediate $z$:

$$\begin{aligned} J(x_5, z) &\approx \sqrt{\frac{\pi z}{2k}}\frac{1}{\sin\delta}a^{-1/2}\cos\left(\frac{z}{ka} - \frac{3\pi}{4} + \delta\right) \\ Y(x_5, z) &\approx \sqrt{\frac{2k}{\pi z}}a^{-1/2}\sin\left(\frac{z}{ka} - \frac{\pi}{4}\right) \end{aligned} \tag{4.128}$$

where $\cot\delta = (2/\pi)\log(ze^\gamma/2k)$ (if the log is large then $\delta^{-1}$ is equal to $2/pi\log$). This is in agreement with eq. (4.105), identifying the Bogoliubov coefficients as

$$\alpha = \sqrt{\frac{\pi z}{2k}}\sin^{-1}\delta e^{-3i\pi/4+\delta} \qquad \beta = \sqrt{\frac{2k}{\pi z}}e^{-i\pi/4} \tag{4.129}$$

Note that eq. (4.106) is satisfied.

The spectral function:

$$F_{(1)}(z) = -\frac{4k^2 a_L}{\pi^2 z^2} \frac{1}{\left[J_0\left(\frac{z}{k}\right) Y_0\left(\frac{z}{ka_L}\right) - Y_0\left(\frac{z}{k}\right) J_0\left(\frac{z}{ka_L}\right)\right]\left[J_1\left(\frac{z}{k}\right) Y_1\left(\frac{z}{ka_L}\right) - Y_1\left(\frac{z}{k}\right) J_1\left(\frac{z}{ka_L}\right)\right]} \tag{4.130}$$

e.gsfrsw

Expansion at small $z$:

$$F_{(1)}(z) \approx -\frac{2k}{L(a_L^{-2} - 1)} \frac{1}{z^2} \tag{4.131}$$

in concordance with eq. (4.108). At large $z$:

$$F_{(1)}(z) \approx -\frac{1}{\sin^2\left(\frac{z}{k}(a_L^{-1} - 1)\right)} \tag{4.132}$$

in agreement with eq. (4.109). For intermediate $ka_L < z < k$:

$$F_{(1)}(z) \approx -\frac{\sin\delta}{\sin(\frac{z}{ka_L} - 3\pi/4 + \delta)\sin(\frac{z}{ka_L} - \pi/4)} \qquad \tan\delta = \frac{\pi}{2}\log^{-1}(ze^{\gamma}/2k) \tag{4.133}$$

On the imaginary axis

$$F_{(1)}(iy) = \frac{k^2 a_L}{y^2} \frac{1}{\left[I_0\left(\frac{y}{k}\right) K_0\left(\frac{y}{ka_L}\right) - K_0\left(\frac{y}{k}\right) I_0\left(\frac{y}{ka_L}\right)\right]\left[I_1\left(\frac{y}{k}\right) K_1\left(\frac{y}{ka_L}\right) - K_1\left(\frac{y}{k}\right) I_1\left(\frac{y}{ka_L}\right)\right]} \tag{4.134}$$

e.gsfrse

The superapproximate:

$$F_{(1)}(iy) \approx \frac{2}{kL(a_L^{-2} - 1)} \frac{(a_L^{-1} - 1)^2}{\sinh^2\left(y\frac{(a_L^{-1} - 1)}{k}\right)} \tag{4.135}$$

The fermionic base functions, for $M = ck$

$$\begin{aligned} J_M(x_5, z) &= \frac{\pi z}{2k} a^{c-1/2}(x_5)\left[Y_{-c-1/2}\left(\frac{z}{k}\right) J_{-c+1/2}\left(\frac{z}{ka(x_5)}\right) - J_{-c-1/2}\left(\frac{z}{k}\right) Y_{-c+1/2}\left(\frac{z}{ka(x_5)}\right)\right] \\ Y_M(x_5, z) &= \frac{\pi z}{2k} a^{c-1/2}(x_5)\left[J_{-c+1/2}\left(\frac{z}{k}\right) Y_{-c+1/2}\left(\frac{z}{ka(x_5)}\right) - Y_{-c+1/2}\left(\frac{z}{k}\right) J_{-c+1/2}\left(\frac{z}{ka(x_5)}\right)\right] \end{aligned} \tag{4.136}$$

For small $z$

$$Y_M \approx \frac{1 - a^{2c-1}}{(2c - 1)k} z \tag{4.137}$$

in agreement with eq. (4.114). For large $z$

$$Y_M(x_5, z) \approx a^c \sin\left(z\frac{a^{-1} - 1}{k}\right) \tag{4.138}$$

in agreement with eq. (4.115). For intermediate $z$:

$$Y_M(x_5,z) \approx \sqrt{\frac{\pi z}{2k}} \frac{a^c}{\cos(\pi c)} \left\{ \frac{(z/2k)^{-c+1/2}}{\Gamma(-c+3/2)} \cos\left(\frac{z}{ka} - \frac{c\pi}{2}\right) - \frac{(z/2k)^{c-1/2}}{\Gamma(c+1/2)} \cos\left(\frac{z}{ka} - \frac{(c-1)\pi}{2}\right)\right\} \tag{4.139}$$

The first terms in curly brackets dominates for $c > 1/2$, the other for $c < 1/2$.

The fermionic form factor:

$$F_{(1/2)}(z) = -\frac{4k^2 a_L}{\pi^2 z^2} \cdot \frac{1}{J_{-c+1/2}\left(\frac{z}{k}\right) Y_{-c+1/2}\left(\frac{z}{ka_L}\right) - Y_{-c+1/2}\left(\frac{z}{k}\right) J_{-c+1/2}\left(\frac{z}{ka_L}\right)} \cdot \frac{1}{J_{c+1/2}\left(\frac{z}{k}\right) Y_{c+1/2}\left(\frac{z}{ka_L}\right) - Y_{c+1/2}\left(\frac{z}{k}\right) J_{c+1/2}\left(\frac{z}{ka_L}\right)} \tag{4.140}$$

e.fsfrsw

At low $z$:

$$F_{(1/2)}(z) \approx -\frac{1}{z^2} \frac{k^2(2c+1)(-2c+1)}{(a_L^{2c-1}-1)(a_L^{-2c-1}-1)} \tag{4.141}$$

At large $z$

$$F_{(1/2)}(z) \approx -\frac{1}{\sin^2\left(z\frac{a_L^{-1}-1}{k}\right)} \tag{4.142}$$

On the imaginary axis

$$F_{(1/2)}(iy) = \frac{k^2 a_L}{y^2} \frac{1}{\left[I_{-c+1/2}\left(\frac{y}{k}\right) K_{-c+1/2}\left(\frac{y}{ka_L}\right) - K_{-c+1/2}\left(\frac{y}{k}\right) I_{-c+1/2}\left(\frac{y}{ka_L}\right)\right]} \frac{1}{\left[I_{c+1/2}\left(\frac{y}{k}\right) K_{c+1/2}\left(\frac{y}{ka_L}\right) - K_{c+1/2}\left(\frac{y}{k}\right) I_{c+1/2}\left(\frac{y}{ka_L}\right)\right]} \tag{4.143}$$

e.fsfrse

Superapprox:

$$F_{(1/2)}(iy) = \frac{(2c+1)(-2c+1)}{(a_L^{2c-1}-1)(a_L^{-2c-1}-1)} \frac{(a_L^{-1}-1)^2}{\sinh^2\left(y\frac{a_L^{-1}-1}{k}\right)} \tag{4.144}$$

It is almost perfect for $|c| \le 1/2$, while it deviates a little for $|c| > 1/2$ (this is not an important case as then the form factor is very much suppressed).

The funny thing is that for $c = 0$ (no bulk mass) the form factor simplifies surprisingly:

$$F_{(1/2)}(iy) = \frac{1}{\sinh^2\left(z\frac{a_L^{-1}-1}{k}\right)} \tag{4.145}$$

and looks almost like the spectral function in the flat space with $L \to ka_L$. For $c = 0$ the superapproximate is equal to the true form factor.

### 4.2.3 Power Model

The warp factor is chosen as

$$a(x_5) = \left(1 - \frac{kx_5}{\gamma - 1}\right)^{\gamma} \tag{4.146}$$

`e.plwf`

For $\gamma \to \infty$ we recover the exponential warp factor of the RS model. For definiteness we assume $\gamma \neq 1$.

This power-law warp factor arises as a solution in 5D gravity-dilaton theories, see Oceansize for a more extended discussion.

With the power-law warp factor the solutions to eq. (2.46) can be written, simalarly as in the $\mathrm{AdS}_5$case, in terms of the Bessel functions,

$$\begin{aligned} J(x_5, z) &= \frac{\pi z}{2k} a^{-1+1/2\gamma} \left[ Y_\epsilon\left(\frac{z}{k}\right) J_{1+\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right) - J_\epsilon\left(\frac{z}{k}\right) Y_{1+\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right)\right] \\ Y(x_5, z) &= \frac{\pi z}{2k} a^{-1+1/2\gamma} \left[ J_{1+\epsilon}\left(\frac{z}{k}\right) Y_{1+\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right) - Y_{1+\epsilon}\left(\frac{z}{k}\right) J_{1+\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right)\right] \end{aligned} \tag{4.147}$$

`e.plwf`

where we introduced the shorthand: $\epsilon = \frac{1}{2(\gamma-1)}$. We have the following formula:

$$[a^{-1+1/2\gamma} Z_{1+\epsilon}(z/ka^{1-1/\gamma})]' = za^{-1}a^{-1+1/2\gamma} Z_\epsilon(z/ka^{1-1/\gamma}) \tag{4.148}$$

Thus:

$$\begin{aligned} J'(x_5, z) &= \frac{\pi z^2}{2k} a^{-2+1/2\gamma} \left[ Y_\epsilon\left(\frac{z}{k}\right) J_{\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right) - J_\epsilon\left(\frac{z}{k}\right) Y_{\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right)\right] \\ Y'(x_5, z) &= \frac{\pi z^2}{2k} a^{-1+1/2\gamma} \left[ J_{1+\epsilon}\left(\frac{z}{k}\right) Y_{\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right) - Y_{1+\epsilon}\left(\frac{z}{k}\right) J_{\epsilon}\left(\frac{z}{ka^{1-1/\gamma}}\right)\right] \end{aligned} \tag{4.149}$$

At low $z$:

$$\begin{aligned} Y(x_5, z) &\approx z\frac{1}{k}\frac{\gamma-1}{2\gamma-1}(a^{-2+1/\gamma} - 1) \\ J'(x_5, z) &\approx z^2 a^{-2} x_5 \end{aligned} \tag{4.150}$$

$$F_{(1)}(z) = -\frac{4k^2 a_L^{1-1/\gamma}}{\pi^2 z^2} \cdot \frac{1}{\left[J_\epsilon\left(\frac{z}{k}\right) Y_\epsilon\left(\frac{z}{ka_L^{1-1/\gamma}}\right) - Y_\epsilon\left(\frac{z}{k}\right) J_\epsilon\left(\frac{z}{ka_L^{1-1/\gamma}}\right)\right]} \cdot \frac{1}{\left[J_{1+\epsilon}\left(\frac{z}{k}\right) Y_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{z}{ka_L^{1-1/\gamma}}\right) - Y_{1+\epsilon}\left(\frac{z}{k}\right) J_{1+\epsilon}\left(\frac{z}{ka_L^{1-1/\gamma}}\right)\right]} \tag{4.151}$$

`e.gsfp`

At small $z$:

$$F_{(1)}(z) \approx -\frac{1}{z^2}\frac{2\gamma-1}{\gamma-1}\frac{k}{L(a_L^{-2+1/\gamma} - 1)} \tag{4.152}$$

At large $z$:

$$F_{(1)}(z) \approx -\frac{1}{\sin^2\left(z\frac{a_L^{-1+1/\gamma}-1}{k}\right)} \tag{4.153}$$

On the imaginary axis

$$F_{(1)}(iy) = \frac{k^2 a_L^{1-1/\gamma}}{y^2} \cdot \frac{1}{\left[I_{\frac{1}{2\gamma-2}}\left(\frac{y}{k}\right) K_{\frac{1}{2\gamma-2}}\left(\frac{y}{k a_L^{1-1/\gamma}}\right) - K_{\frac{1}{2\gamma-2}}\left(\frac{y}{k}\right) I_{\frac{1}{2\gamma-2}}\left(\frac{y}{k a_L^{1-1/\gamma}}\right)\right]} \cdot \frac{1}{\left[I_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{y}{k}\right) I_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{y}{k a_L^{1-1/\gamma}}\right) - K_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{y}{k}\right) I_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{y}{k a_L^{1-1/\gamma}}\right)\right]} \tag{4.154}$$

e.gsfpe

The superapprox:

$$F_{(1)}(z) \approx \frac{2\gamma-1}{\gamma-1}\frac{1}{kL(a_L^{-2+1/\gamma}-1)}\frac{(a_L^{-1+1/\gamma}-1)^2}{\sinh^2\left(y\frac{a_L^{-1+1/\gamma}-1}{k}\right)} \tag{4.155}$$

# Appendix A Fun with Bessel

a.b

The Bessel equation:

$$x^2 f''(x) + x f'(x) + (x^2 - n^2) f(x) = 0 \tag{A.1}$$

has two independent solutions called the Bessel $J_n(x)$ and the Neumann $Y_n(x)$. Furthermore, the equation

$$x^2 f''(x) - \alpha x f'(x) + ((bx)^2 - \beta) f(x) = 0 \tag{A.2}$$

is solved by

$$f(x) = x^{(1+\alpha)/2} Z_n(bx) \tag{A.3}$$

where $Z_n = J_n, Y_n$ and $n^2 = (1+\alpha)^2/4 + \beta$. Yet another incarnation of the Bessel equation is useful. The equation

$$\left[\partial_5^2 - \alpha k \partial_5 + (z^2 a^{-2} - \beta k^2)\right] f = 0 \tag{A.4}$$

with $a = e^{-kx_5}$ is solved by

$$f = a^{-\alpha/2} Z_n\left(\frac{z}{ka}\right) \qquad n^2 = \alpha^2/4 + \beta \tag{A.5}$$

The modified Bessels $I_n$ and $K_n$ are two independent solution of the Euclidean Bessel equation

$$x^2 f''(x) + x f'(x) - (x^2 + n^2) f(x) = 0 \tag{A.6}$$

They are related to ordinary Bessels as

$$I_n(x) = i^{-n} J_n(ix) \qquad K_n(x) = -\frac{\pi}{2} i^n [Y_n(ix) - iJ_n(ix)] \tag{A.7}$$

Asymptotic behaviour for $x \ll 1$

$$\begin{aligned} J_n(x) &\approx \frac{(x/2)^n}{\Gamma(n+1)} & I_n(x) &\approx \frac{(x/2)^n}{\Gamma(n+1)} \\ Y_n(x) &\approx -\frac{\Gamma(n)2^n}{\pi x^n} & K_n(x) &\approx \frac{\Gamma(n)2^{n-1}}{x^n}(?) \\ Y_0(x) &\approx \frac{2}{\pi}(\log(x/2) + \gamma) & K_0(x) &\approx -(\log(x/2) + \gamma) \end{aligned} \tag{A.8}$$

Asymptotic behaviour of the linear combinations

$$\begin{aligned} J_n(x)Y_n(x/a) - Y_n(x)J_n(x/a) &\approx \frac{a^{-n} - a^n}{\pi n} \\ J_n(x)Y_{n-1}(x/a) - Y_n(x)J_{n-1}(x/a) &\approx \frac{2}{\pi x} a^{1-n} + \mathcal{O}(x a^{n-1}) \end{aligned} \tag{A.9}$$

Asymptotic behaviour for $x \gg 1$

$$\begin{aligned} J_n(x) &\approx \sqrt{\frac{2}{\pi x}} \cos(x - n\pi/2 - \pi/4) & I_n(x) &\approx \frac{1}{\sqrt{2\pi x}} e^x \\ Y_n(x) &\approx \sqrt{\frac{2}{\pi x}} \sin(x - n\pi/2 - \pi/4) & K_n(x) &\approx \sqrt{\frac{\pi}{2x}} e^{-x} \end{aligned} \tag{A.10}$$

The Wronskian formula

$$\begin{aligned} Y_n(x)J_{n+1}(x) - J_n(x)Y_{n+1}(x) &= 2/(\pi x) \\ K_n(x)I_{n+1}(x) - I_n(x)K_{n+1}(x) &= 1/x \end{aligned} \tag{A.11}$$

e.wmf

Recursion relations:

$$\begin{aligned} J'_n(x) + \frac{n}{x} J_n(x) &= J_{n-1}(x) \\ -J'_n(x) + \frac{n}{x} J_n(x) &= J_{n+1}(x) \end{aligned} \tag{A.12}$$

The same formulae hold for $Y_n$ and $I_n$. However some dumbhead defined the McDonald function $K_n$ such that the first of the recursion relation has a flipped sign:

$$\begin{aligned} K'_n(x) + \frac{n}{x} K_n(x) &= -K_{n-1}(x) \\ -K'_n(x) + \frac{n}{x} K_n(x) &= K_{n+1}(x) \end{aligned} \tag{A.13}$$

Kill McDonald, indeed.

# References

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