Note on warped models with KK parity

(last compiled December 7, 2007)

We consider warped models with Z_2 symmetry. We glue two copies of Randall and Sundrum at the UV brane (the IR-UV-IR set-up with three branes). We will use an equivalent and more convenient description with two branes. This is the usual RS, with the exception that *both Neumann and Dirichlet BC at the UV brane are allowed for the same field*. The even modes are those with Neumann, while the odd modes are the Dirichlet ones. Odd modes couple to the even ones only in pairs (in the full IR-UV-IR setup this is explicit, in the two brane formulation we set these couplings to zero by hand).

For most of the calculation I use the background independent formalism just because i find it simpler to manipulate. At the end I insert the AdS_5 warp factor.

Toy

As a warm-up consider just a U(1) model with brane kinetic terms on both branes. The lagrangian for a gauge field in a warped background:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 \left(1 + r_0 \delta(0) + r_L \delta(L)\right) + \frac{1}{2} a^2 (\partial_5 A_\mu)^2 \tag{1}$$

We expand $A_\mu = A_{\mu,n}(x) f_n(x_5)$ where the profile satisfies the equation of motion

$$\partial_5 (a^2 \partial_5 f_n) + m_n^2 f_n = 0 \tag{2}$$

and the boundary conditions

$$\begin{aligned} a^2(L) \partial_5 f_n(L) &= m_n^2 r_L f_n(L) \\ \partial_5 f_n(0) &= -m_n^2 r_0 f_n(0) \end{aligned} \tag{3}$$

The normalization is $\int_0^L f_n^2 (1 + r_0 \delta(0) + r_L \delta(L)) = 1$.

The even and odd modes have profiles that can be concisely written using the warped trigs defined in Appendix B

$$\begin{aligned} f_{n_+}(y) &= \alpha_{n_+} \left(C(y, m_{n_+}) - r_0 m_n^+ S(y, m_{n_+}) \right) \\ f_{n_-}(y) &= \alpha_{n_-} S(y, m_{n_-}) \end{aligned} \tag{4}$$

By the properties of the warped trigs, they satisfy the UV boundary conditions. The IR boundary conditions fix the quantization condition:

$$\begin{aligned} a_L^2 \left(C'(L, m_{n_+}) - r_0 m_n^+ S'(L, m_{n_+}) \right) &= m_{n_+}^2 r_L \left(C(L, m_{n_+}) - r_0 m_n^+ S(L, m_{n_+}) \right) \\ a_L^2 S'(L, m_{n_-}) &= m_{n_-}^2 r_L S(L, m_{n_-}) \end{aligned} \tag{5}$$

Without brane kinetic terms the quantization condition would be

$$C'(L, m_{n_+}) = 0 \qquad S'(L, m_{n_-}) = 0 \tag{6}$$

In warped spaces where the warp factor decreseas steeply towards the IR there's an approximate relation $C'(L,m) \approx -mLS'(L,m)$. As a result, the even and odd modes have almost degenerate masses.

In order to get a mass splitting between m_{1+} and m_{1-} we need to go to large brane kinetic terms. For $r_L > L$, using the perturbative expansion of the trigs in eq. (B.27) we find the approximate expression

$$
\begin{aligned}
m_{1-}^2 &\approx \frac{1}{r_L}\frac{1}{\int_0^L a^{-2}} \\
m_{1+}^2 &\approx \frac{r_0 + r_L + L}{r_L \int_0^L y a^{-2} + r_L r_0 \int_0^L y a^{-2}}
\end{aligned}
\tag{7}
$$

In AdS$_5$ we obtain

$$
\begin{aligned}
m_{1-}^2 &\approx \frac{2}{k r_L}(k a_L)^2 \\
m_{1+}^2 &\approx \frac{r_0 + r_L + L}{(r_0 + L)}\frac{2}{k r_L}(k a_L)^2
\end{aligned}
\tag{8}
$$

In fact, in both even and odd towers the lightest KK modes are suppressed wrt the KK scale $k a_L$ where the tower of equally spaced KK modes begins. The odd mode is light (suppressed roughly by $k r_L$ with respect to the KK scale), while the even is half-light (suppressed roughly by kL). Thus, we have the possibility to split the lightest even and odd modes:

$$
\frac{m_{1+}}{m_{1-}} \approx \sqrt{1 + \frac{r_L}{r_0 + L}}
\tag{9}
$$

We need r_L to be much larger than L to achieve sizable splitting. Picking negative r_0 helps a little.

Using again eq. (B.27), the profile of the lighest modes can be approximated by

$$
\begin{aligned}
f_0(y) &\approx \frac{1}{\sqrt{L + r_0 + r_L}} \approx \frac{1}{\sqrt{r_L}} \\
f_{1-}(y) &\approx \frac{1}{\sqrt{r_L}}\frac{\int_0^y a^{-2}}{\int_0^L a^{-2}} \\
f_{1+}(y) &\approx \frac{1}{\sqrt{L + r_0}}\left(1 - m_{1+}^2 (y + r_0)\int_0^y a^{-2}\right)
\end{aligned}
\tag{10}
$$

On the IR brane the zero mode and the first odd mode couple with similar strength, $f_0(L) \approx f_{1-}(L) \approx \frac{1}{\sqrt{r_L}}$, while the half-light even mode coupling is $f_{1-}(L) \approx \frac{1}{\sqrt{r_L}}\sqrt{\frac{r_0+L}{r_L}}$ so it is suppressed by m_{1-}/m_{1+}.

On the UV brane, the odd mode of course does not couple. The zero mode couples again via $f_0(0) \approx \frac{1}{\sqrt{r_L}}$. The half-light mode coupling is **enhanced** by m_{1+}/m_{1-} this time: $f_{1-}(0) \approx \frac{1}{\sqrt{r_L}}\sqrt{\frac{r_L}{r_0+L}}$.

One problem with large r_L is that the zero mode coupling is now $1/g_0^2 = (L + r_0 + r_L)/g_5^2 \approx r_L/g_5^2$. If we want $g_0 \sim g_{\rm SM}$, we need g_5 to be large, $g_5^2 \sim g_{\rm SM}^2 r_L$. This leads to the cutoff scale of order k.

Another problem is the enhanced coupling of the half-light mode which is correlated with the mass splitting m_{1+}/m_{1-}.....

Full Monty

We move to the full-fledged phenomenological model with $SU(2)_L \times SU(2)_R \times U(1)_X$ (color ignored). For simplicity, below we ignore UV kinetic terms. On the IR brane we may put three different brane kinetic terms for each of the group factors. We denote them r_L, r_R and r_X. The bulk gauge couplings are denoted as $\tilde{g}_L\sqrt{L}$, $\tilde{g}_R\sqrt{L}$ and $\tilde{g}_X\sqrt{L}$.

The 5D action:

$$\begin{aligned} S = & \int d^4x \int_0^L dx_5 \sqrt{g} \left\{ -\tfrac{1}{2}\mathrm{Tr}\{L_{MN}L_{MN}\}\left(1 + r_L\delta(L)\right) - \tfrac{1}{2}\mathrm{Tr}\{R_{MN}R_{MN}\}\left(1 + r_R\delta(L)\right) \right. \\ & \left. -\tfrac{1}{4}X_{MN}X_{MN}\left(1 + r_X\delta(L)\right)\right\} \\ & + \int d^4x dx_5 \sqrt{g_4}\delta(L)\left(\tfrac{1}{4}\mathrm{Tr}|D_\mu\Phi|^2 - V(\Phi^\dagger\Phi)\right) \end{aligned} \tag{11}$$

The higgs field acquires the vev $\langle\Phi\rangle = a_L^{-1}\tilde{v}I_{2\times 2}$ that spontaneously breaks $SU(2)_L \times SU(2)_R$ to $SU(2)_V$ on the IR brane.

There is some problem how to impose correct UV boundary conditions in this case. We need the UV brane to break $SU(2)_R \times U(1)_X$ down to $U(1)_Y$. Usually, it is done by imposing Dirichlet boundary conditions for R_μ^i and one linear combination of X_μ and R_μ^3, while the remaining fields are Neumann. In the present framework each field can have both Neumann and Dirichlet boundary conditions in UV. However, there's is a way out. We can think of some Higgs mechanism localized in UV that gives very large (Planck scale) masses to R_μ^i and $B'_\mu = -c_x R_\mu^3 + s_x X_\mu$ breaking the gauge symmetry down to the SM. In the limit these masses are infinite, the UV boundary conditions for R_μ^i and B'_μ are always Dirichlet, regardless whether we consider even or odd modes. I guess this is consistent.

We write the profiles for even modes as

$$\begin{aligned} f^a_{L,n_+}(x_5) &= \alpha^a_{L,n_+} C(x_5, m_{n_+}) \\ f^i_{R,n_+}(x_5) &= \alpha^i_{R,n_+} S(x_5, m_{n_+}) \\ f^3_{R,n_+}(x_5) &= \alpha_{N,n_+} s_x C(x_5, m_{n_+}) - \alpha_{D,n_+} c_x S(x_5, m_{n_+}) \\ f_{X,n_+}(x_5) &= \alpha_{N,n_+} c_x C(x_5, m_{n_+}) + \alpha_{D,n_+} s_x S(x_5, m_{n_+}) \end{aligned} \tag{12}$$

where i again use my annoying warped trig notation. We also denote $s_x = \tilde{g}_X/\sqrt{\tilde{g}_X^2 + \tilde{g}_R^2}$ and $c_x = \tilde{g}_R/\sqrt{\tilde{g}_X^2 + \tilde{g}_R^2}$ This part is the same as in the standard Agashe et al. [1, 3]. In addition, we have the odd modes

$$\begin{aligned} f^a_{L,n_-}(x_5) &= \alpha^a_{L,n_-} S(x_5, m_{n_-}) \\ f^a_{R,n_-}(x_5) &= \alpha^a_{R,n_-} S(x_5, m_{n_-}) \\ f_{X,n_-}(x_5) &= \alpha_{X,n_-} S(x_5, m_{n_-}) \end{aligned} \tag{13}$$

The IR boundary conditions:

$$\begin{aligned} a_L^2\partial_5 f_{X,n}(L) - m_n^2 r_X f_{X,n}(L) &= 0 \\ a_L^2\partial_5 f^a_{L,n}(L) - m_n^2 r_L f^a_{L,n}(L) &= -\frac{1}{4}\tilde{g}_L L\tilde{v}^2(\tilde{g}_L f^a_{L,n}(L) - \tilde{g}_R f^a_{R,n}(L)) \\ a_L^2\partial_5 f^a_{R,n}(L) - m_n^2 r_R f^a_{R,n}(L) &= +\frac{1}{4}\tilde{g}_R L\tilde{v}^2(\tilde{g}_L f^a_{L,n}(L) - \tilde{g}_R f^a_{R,n}(L)) \end{aligned} \tag{14}$$

A sometimes useful combination of the last two is

$$\tilde{g}_R(a_L^2\partial_5 f_{L,n}^a(L) - m_n^2 r_L f_{L,n}^a(L)) + \tilde{g}_L(a_L^2\partial_5 f_{R,n}^a(L) - m_n^2 r_R f_{R,n}^a(L)) = 0 \tag{15}$$

The mass eigenstates and the mass eigenvalues are found by inserting the profiles into the IR boundary conditions.

Ignoring EW breaking for a moment, we have four massless gauge bosons. The SU(2) gauge bosons L_μ with the profiles

$$f_{L,L}^a = \frac{1}{\sqrt{L + r_L}} \tag{16}$$

and the hyperchage gauge bosons B_μ:

$$\begin{aligned} f_{R,B}^3 &= \frac{s_x}{\sqrt{L + s_x^2 r_R + c_x^2 r_X}} \\ f_{X,B} &= \frac{c_x}{\sqrt{L + s_x^2 r_R + c_x^2 r_X}} \end{aligned} \tag{17}$$

Thus the SM gauge couplings are related to the 5D parameters as:

$$g_L = \frac{\tilde{g}_L}{\sqrt{1 + r_L/L}} \qquad g_Y = \frac{s_x \tilde{g}_R}{\sqrt{1 + s_x^2 r_R/L + c_x^2 r_X/L}} \tag{18}$$

and the hypercharge is realized as $Y = T_R^3 + X$.

We turn to profiles and masses of the lightest KK states. Ignoring EW breaking, it is clear the eigenstates are just along L, R or X. The quantization condition for both charged and neutral states is

$$m_{n_-}^2 = \frac{a_L^2 S'(L, m_{n_-})}{r_A S(L, m_{n_-})} \tag{19}$$

where $A = L, R, X$. For large BKT we find a light state:

$$m_{1-}^2 \approx \frac{1}{r_A} \frac{1}{\int_0^L a^{-2}} \tag{20}$$

just like in the toy model. Up to EW breaking:

$$f_{A,1-}(y) \approx \frac{1}{\sqrt{r_A}} \frac{\int_0^y a^{-2}}{\int_0^L a^{-2}} \tag{21}$$

Playing with r_A we can choose whether LKP lies in L^3, R^3 or X (at tree-level the charged and neutral states are degenerate, but i assume loop effects split them in the correct way).

Now the light even modes. Those are slightly more complicated. Since even modes include fields with both Neumann and Dirichlet boundary conditions, there may be both light and half-light states. In the charged sector we have the light state along R^i:

$$m_{R,1+}^2 \approx \frac{1}{r_L} \frac{1}{\int_0^L a^{-2}} \qquad f_{R,1+}^i \approx \frac{1}{\sqrt{r_L}} \frac{\int_0^y a^{-2}}{\int_0^L a^{-2}} \tag{22}$$

and a half-light state along L^i:

$$m^2_{L,1+} \approx \frac{L + r_L}{L r_L} \frac{1}{\int_0^L a^{-2}} \qquad f_{L,1+} \approx \frac{1}{\sqrt{L}} \left(1 - m^2_{L,1+} y \int_0^y a^{-2}\right) \tag{23}$$

In the neutral sector it is even more tricky. We again have a half-light state along L^3. We also have a light state (combination of R^3 and X) with a mass

$$m^2_{1+} \approx \frac{L + s_x^2 r_R + c_x^2 r_X}{r_R r_X + L(r_R + r_X)} \frac{1}{\int_0^L a^{-2}} \tag{24}$$

In fact, it is light only when both r_X and r_R exceed L. There is one more half-light combination of R^3 and X whose mass i didnt bother to compute.

Electroweak Precision

We use the formalism of ref. [3]. In this formalism all gauge bosons are mass eigenstates and don't mix, even when electroweak breaking is included. In particular, there is no tree-level contribution to the S parameter (as S implies mixing of photon and Z). There is also no p^4 corrections to the EW gauge boson propagators at tree level. There are however vertex corrections to interactions of W and Z with the fermions and four-fermion operators. EW breaking tilts the profiles of W and Z, so that their coupling to fermions gets modified. Integrating out the heavy modes induces four-fermion operators. Corrections to tri- and four-linear boson vertices that also arise are being carefully ignored here.

Below we assume that light fermions are sharply localized on the UV brane. In such a case, the vertex corrections and four-fermions for the light fermions are universal and can be removed by a field redefinition This very redefinition generates S and other oblique parameters.

The drawback of this formalism is that it is not very illustrative. The advantage is that it is hard to miss anything; all tree-level contributions from all KK modes are automatically included. To compute the vertex corrections we have to do is to find the profiles of zero mode gauge bosons in the presence of electroweak breaking. To compute the effective four-fermion operators we have to integrate out the heavy KK mode exchange.

We parametrize the universal vertex correction and the four-fermion oepratrors as:

$$\begin{aligned}\mathcal{L}_{eff} = &+\tfrac{g_L}{\sqrt{2}}(1 + m_W^2\beta)\overline{\psi_j}\gamma_\mu t^\pm \psi_j W^\pm_\mu \\ &+\tfrac{1}{\sqrt{g_L^2+g_Y^2}}(g_L^2(1+\beta m_Z^2)t^3_j - g_Y^2(1+\gamma m_Z^2)Y_j)\overline{\psi_j}\gamma_\mu\psi_j Z_\mu \\ &-c g_Y^2 \overline{\psi}_i\gamma_\mu Y\psi_i\overline{\psi}_j\gamma_\mu Y\psi_j - d g_L^2\overline{\psi}_i\gamma_\mu t^a\psi_i\overline{\psi}_j\gamma_\mu t^a\psi_j\end{aligned} \tag{25}$$

After redefining away the vertex correction (see Appendix A), the S parameter that pops out is given by:

$$S = 4\pi v^2(2c + 2d + \beta + \gamma) \tag{26}$$

e.svv

We also generate the other oblique parameters coresponding to higher derivatives of the gauge propagators. Using the set defined in ref. [4]

$$\hat{S} = m_W^2(2c + 2d + \beta + \gamma) \qquad W = 2m_W^2 d \qquad Y = 2m_W^2 c \tag{27}$$

The vertex corrections can be determined by expanding the W and Z profiles at the UV brane in powers of m_W^2 and m_Z^2:

$$\begin{aligned}
\tilde{g}_L\sqrt{L}f^i_{L,W}(0) &= g_L\left(1+\beta m_W^2\right)\\
\tilde{g}_L\sqrt{L}f^3_{L,Z}(0) &= g_L\cos\theta_W\left(1+\beta m_Z^2\right)\\
\tilde{g}_R\sqrt{L}f^3_{R,Z}(0) &= -g_Y\sin\theta_W\left(1+\gamma m_Z^2\right)\\
\tilde{g}_X\sqrt{L}f_{X,Z}(0) &= -g_Y\sin\theta_W\left(1+\gamma m_Z^2\right)
\end{aligned}\tag{28}$$

Here $\cos\theta_W = g_L/\sqrt{g_L^2+g_Y^2}$.

In our set-up. The four-fermions are dominated by the exchange of the half-light mode. The rest of the KK tower is much heavier and couples less strongly (i guess). Approximately, the coefficients are given by

$$d \approx r_L\frac{(f_{L,1+}(0))^2}{2m^2_{L,1+}} \qquad c \approx r_L\frac{(f_{B,1+}(0))^2}{2m^2_{B,1+}}\tag{29}$$

(where $f_B = c_x f_X + s_X f_R$).

The W boson is a combination of L^i and R^i:

$$\begin{aligned}
f^i_{L,W}(x_5) &= \alpha_{L,W}C(x_5,m_W)\\
f^i_{R,W}(x_5) &= \alpha_{R,W}S(x_5,m_W)
\end{aligned}\tag{30}$$

while the Z boson is a combination of L^3, R^3 and X:

$$\begin{aligned}
f^3_{L,Z}(x_5) &= \alpha_{L,Z}C(x_5,m_Z)\\
f^3_{R,Z}(x_5) &= s_x\alpha_{N,Z}C(x_5,m_Z) - c_x\alpha_{D,Z}S(x_5,m_Z)\\
f_{X,Z}(x_5) &= c_x\alpha_{N,Z}C(x_5,m_Z) + s_x\alpha_{D,Z}S(x_5,m_Z)
\end{aligned}\tag{31}$$

Thus, $f^i_{L,W}(0) = \alpha_{L,W}$ and $f^3_{R,Z}(0) = s_x\alpha_{N,Z}$. We need to extract these constants from the IR boundary conditions and the normalization.

The IR boundary conditions yield:

$$\begin{aligned}
\alpha_{R,W} &\approx -\tfrac{\tilde{g}_R}{\tilde{g}_L}\tfrac{a_L^2C'(L,m_W)-m_W^2r_LC(L,m_W)}{a_L^2S'(L,m_W)-m_W^2r_RS(L,m_W)}\alpha_{L,W}\\
\alpha_{D,Z} &\approx -\tfrac{c_x}{s_x}\tfrac{a_L^2C'(L,m_Z)-m_Z^2r_XC(L,m_Z)}{a_L^2S'(L,m_Z)-m_Z^2r_XS(L,m_Z)}\alpha_{N,Z}\\
s_x\tilde{g}_R\alpha_{L,Z}(a_L^2C'(L,m_Z) - m_Z^2r_LC(L,m_Z)) &+ \tilde{g}_L\alpha_{N,Z}(a_L^2C'(L,m_Z) - m_Z^2(s_x^2r_R + c_x^2r_X)C(L,m_Z))\\
&+\tilde{g}_L\alpha_{D,Z}c_xs_xm_Z^2(r_R - r_X) = 0
\end{aligned}\tag{32}$$

For the rest of the discussion we restrict to two cases:

1. Universal: $r_L = r_R = r_X = r$, $r \gg L$
2. Lefty: $r_R = r_X = 0$, $r_L = r$, $r \gg L$

To make things even more simple we take $\tilde{g}_L = \tilde{g}_R \equiv \tilde{g}$. There is not much theoretical justification for all these simplifications, but the general case is a nightmare.

Universal. The SM gauge couplings are given by

$$g_L = \frac{\tilde{g}}{\sqrt{1+r/L}} \qquad g_Y = \frac{s_x \tilde{g}}{\sqrt{1+r/L}} \tag{33}$$

Note that $s_x = g_Y/g_L$.

The IR boundary conditions fix

$$\begin{aligned}
\alpha_{R,W} &= -\alpha_{L,W} \tfrac{a_L^2 C'(L,m_W) - m_W^2 r C(L,m_W)}{a_L^2 S'(L,m_W) - m_W^2 r S(L,m_W)} \approx m_W (L+r) \alpha_{L,W} \\
\alpha_{D,Z} &= -\alpha_{N,Z} \tfrac{c_x}{s_x} \tfrac{a_L^2 C'(L,m_Z) - m_Z^2 r C(L,m_z)}{a_L^2 S'(L,m_Z) - m_Z^2 r S(L,m_Z)} \approx m_Z (L+r) \tfrac{c_x}{s_x} \alpha_{N,Z} \\
\alpha_N &= -s_x \alpha_{L,Z}^3
\end{aligned} \tag{34}$$

From normalization:

$$\begin{aligned}
1 = &\int_0^L dy (1 + r\delta(L)) \left(\alpha_{L,W}^2 C^2(y, m_W) + \alpha_{R,W}^2 S^2(y, m_W) \right) \\
\approx &\ \alpha_{L,W}^2 \left(L + r - 2m_W^2 \left[L \textstyle\int\int a^{-2} + rL \int a^{-2} \right] \right) + \mathcal{O}(m_W^4)
\end{aligned} \tag{35}$$

$$\begin{aligned}
1 = &\int_0^L dy (1 + r\delta(L)) \left(\alpha_{L,Z}^2 C^2(y, m_Z) + \alpha_{N,Z}^2 C^2(y, m_z) + \alpha_{D,Z}^2 S^2(y, m_z) \right) \\
\approx &\ \alpha_{L,Z}^2 (1 + s_x^2) \left(L + r - 2m_W^2 \left[L \textstyle\int\int a^{-2} + rL \int a^{-2} \right] \right) + \mathcal{O}(m_W^4)
\end{aligned} \tag{36}$$

Here there and everywhere we pulled out y from the integrals: $\int y a^{-2} \to L \int a^{-2}$ which is fine if the warp factor is sharply peaked in IR. Thus, the profiles at the UV brane

$$\begin{aligned}
f_{L,W}^i(0) &\approx \frac{1}{L+r} \left(1 + m_W^2 \frac{L}{L+r} \left[\int\int a^{-2} + r \int a^{-2} + \mathcal{O}(m_W^4) \right] \right) \\
f_{R,Z}^3(0) &\approx -\frac{\sin\theta_W}{L+r} s_x \left(1 + m_Z^2 \frac{L}{L+r} \left[\int\int a^{-2} + r \int a^{-2} + \mathcal{O}(m_Z^4) \right] \right)
\end{aligned} \tag{37}$$

We readily read off

$$\beta = \gamma = \frac{L}{L+r} \left[\int\int a^{-2} + r \int a^{-2} \right] \tag{38}$$

For $r \gg L$

$$\beta \approx L \int a^{-2} \approx \frac{L+r}{r} \frac{1}{m_{1+}^2} \approx \frac{1}{2} kL \frac{1}{(ka_L)^2} \tag{39}$$

Last one valid in AdS_5 . This is an expected result: $\beta \sim 1/m_{1+}^2$. The first **even** KK mode sets the scale of vertex corrections.

To compute the four-fermion operators we need profiles of the half-light modes. The $SU(2)_L$ adjoint lives along L and couples to the brane via t_L^a, while $SU(2)_L$ adjoint lives along B and couples to the brane via the hypercharge. The former is responsible for the coefficient d, the latter for c. Both have profiles and masses

$$m_{1+}^2 \approx \frac{L + r_L}{L r_L} \frac{1}{\int_0^L a^{-2}} \qquad f_{1+} \approx \frac{1}{\sqrt{L}} \left(1 - m_{1+}^2 y \int_0^y a^{-2} \right) \tag{40}$$

which means

$$d = c \approx \frac{r_L}{L} \frac{1}{2m_{1+}^2} \approx \frac{1}{2m_{1-}^2} \tag{41}$$

We can write down the oblique parameters:

$$\hat{S} \approx 2m_W^2 \left(\frac{1}{m_{1+}^2} + \frac{1}{m_{1-}^2} \right) \qquad W = Y \approx \frac{m_W^2}{m_{1-}^2} \tag{42}$$

The S parameter appears large. Imposing $S < 0.2$ would yield a severe bound $m_{1-} > 11v \sim 2.8\,\mathrm{TeV}$. However, when the main contribution to S comes of course from the four fermion operators this is probably not the right procedure. The reason is that the LEP1 observable ϵ_3 is proportional to $\hat{S} - W - Y$ (see eq. 6c of [4]) so the largest piece depending on m_-^2 cancels out. The four-fermion operators are constrained mainly by the LEP2 data which may be translated as the constraints on W and Y. Using eq. 12 of [4] I impose $W < 0.2 \cdot 10^{-3}$ (the constraint for Y looks twice weaker). Then $m_{1-} > 22 m_W \approx 1.8\,\mathrm{TeV}$.

For comparison, take $r \to 0$ to recover the RS results:

$$\beta = \int \int a^{-2} \approx \frac{1}{4} \frac{1}{(k a_L)^2} \tag{43}$$

and negligible four-fermion operators. This is the result given in [1, 3]. In that case, the mass of the lightest resonance is of order $2.5 k a_L$. Thus $\beta \approx 1.5/m_1^2$ and

$$S = 12\pi \frac{v^2}{m_1^2} \tag{44}$$

Imposing $S < 0.2$ yields $m_1 > 14v \sim 3.4\,\mathrm{TeV}$.

Lefty. The SM gauge couplings are given by

$$g_L = \frac{\tilde{g}}{\sqrt{1 + r/L}} \qquad g_Y = s_x \tilde{g} \tag{45}$$

Note that $s_x = g_Y/g_L \sqrt{1 + r/L}$.

The IR boundary conditions fix

$$\begin{aligned} \alpha_{R,W} &= -\alpha_{L,W} \tfrac{a_L^2 C'(L,m_W) - m_W^2 r C(L,m_W)}{a_L^2 S'(L,m_W)} \approx m_W (L + r) \alpha_{L,W} \\ \alpha_{D,Z} &= -\alpha_{N,Z} \tfrac{c_x}{s_x} \tfrac{C'(L,m_Z)}{S'(L,m_Z)} \approx m_Z L \tfrac{c_x}{s_x} \alpha_{N,Z} \\ \alpha_{N,Z} &= -s_x \alpha_{L,Z} \tfrac{a_L^2 C'(L,m_Z) - m_Z^2 r C(L,m_Z)}{a_L^2 C'(L,m_Z)} \approx -s_x \alpha_{L,Z} \left(1 + r/L - m_Z^2 r \textstyle\int a^{-2} + m_Z^2 \tfrac{r}{L} \int\int a^{-2}\right) \end{aligned} \tag{46}$$

From normalization:

$$\begin{aligned} 1 = &\ \textstyle\int_0^L dy \left(\alpha_{L,W}^2 C^2(y, m_W)(1 + r\delta(L)) + \alpha_{R,W}^2 S^2(y, m_W) \right) \\ \approx &\ \alpha_{L,W}^2 \left(L + r - 2m_W^2 L \left[\textstyle\int\int a^{-2} + r \int a^{-2} \right] \right) + \mathcal{O}(m_W^4) \end{aligned} \tag{47}$$

$$\begin{aligned} 1 = &\ \textstyle\int_0^L dy \left(\alpha_{L,Z}^2 C^2(y, m_Z)(1 + r\delta(L)) + \alpha_{N,Z}^2 C^2(y, m_Z) + \alpha_{D,Z}^2 S^2(y, m_Z) \right) \\ \approx &\ \alpha_{L,Z}^2 \cos^{-2}\theta_W \left(L + r - 2m_Z^2 L \left[\textstyle\int\int a^{-2} + r \int a^{-2} \right] \right) + \mathcal{O}(m_Z^4) \end{aligned} \tag{48}$$

Thus:

$$
\begin{aligned}
\alpha_{L,W} &\approx \frac{1}{\sqrt{L+r}}\left(1+m_W^2\frac{L}{L+r}\left[\int\int a^{-2}+r\int a^{-2}\right]\right)\\
\alpha_{L,Z} &\approx \frac{\cos\theta_W}{\sqrt{L+r}}\left(1+m_Z^2\frac{L}{L+r}\left[\int\int a^{-2}+r\int a^{-2}\right]\right)\\
\alpha_{N,Z} &\approx -\frac{\sin\theta_W}{\sqrt{L}}\left(1+m_Z^2\int\int a^{-2}\right)
\end{aligned}
\tag{49}
$$

The profiles at the UV brane

$$
\begin{aligned}
\tilde{g}_L\sqrt{L}f^i_{L,W}(0) &\approx g_L\left(1+m_W^2\frac{L}{L+r}\left[\int\int a^{-2}+r\int a^{-2}\right]\right)+\mathcal{O}(m_W^4)\\
\tilde{g}_L\sqrt{L}f^3_{L,Z}(0) &\approx \cos\theta_W g_L\left(1+m_Z^2\frac{L}{L+r}\left[\int\int a^{-2}+r\int a^{-2}\right]\right)+\mathcal{O}(m_Z^4)\\
\tilde{g}_R\sqrt{L}f^3_{R,Z}(0) &\approx -\sin\theta_W g_Y\left(1+m_Z^2\int\int a^{-2}\right)+\mathcal{O}(m_Z^4)
\end{aligned}
\tag{50}
$$

We readily read off

$$
\begin{aligned}
\beta &= \frac{L}{L+r}\left[\int\int a^{-2}+r\int a^{-2}\right]\\
\gamma &= \int\int a^{-2}
\end{aligned}
\tag{51}
$$

For $r \gg L$

$$
\beta \approx L\int a^{-2} \qquad \gamma\approx 0 \tag{52}
$$

The difference is that now γ is negligibly small, because only the $SU(2)_L$ sector has light states affecting the vertex corrections.

Finally, the four-fermion operators and the half-light modes. This times there's only the $SU(2)_L$ adjoint along L. As a consequence, only d is sizable .

$$
d \approx \frac{r_L}{L}\frac{1}{2m_{1+}^2}\approx\frac{1}{2m_{1-}^2} \qquad c\approx 0 \tag{53}
$$

The oblique paramaters

$$
\hat{S}\approx m_W^2\left(\frac{1}{m_{1+}^2}+\frac{1}{m_{1-}^2}\right) \qquad W\approx\frac{m_W^2}{m_{1-}^2} \qquad Y=0 \tag{54}
$$

The dominant piece still cancels in ϵ_3. Since W is unchanged, the bound on m_{1-} similar as in the previous case. In fact, looking at the elipse in Fig. 1 of [4], the bound $W < 0.002$ is more justified when $Y=0$.

Fermions

The lagrangian for the fermions

$$
\mathcal{L} = i\bar{\psi}\Gamma^M\left(D_M-\epsilon(y)M\right)\psi \tag{55}
$$

has the Z_2 symmetry $y \to -y$ with $\psi_{L,R} \to \gamma_5 \psi_{L,R}$. For the fermions we do not need BKTs to obtain a suitable spectrum so we omit them in the following discussion. The IR boundary conditions require vanishing of one chiral component. Consider the case when the right-handed component vanishes: $\psi_R(L) = 0$; the discussion for the case $\psi_L(L) = 0$ is parallel, with $M \to -M$.

Much as the gauge field, the fermion will have two KK towers with different UV boundary conditions.

$$\begin{aligned} \psi_L &= a^{-3/2} f_{L,n+}(y) \psi_{L,n+}(x) + \epsilon(y) a^{-3/2} f_{L,n-}(y) \psi_{L,n-}(x) \\ \psi_R &= \epsilon(y) a^{-3/2} f_{R,n+}(y) \psi_{R,n+}(x) + a^{-3/2} f_{R,n-}(y) \psi_{R,n-}(x) \end{aligned} \tag{56}$$

The *even* tower corresponds to the BC $f_{R,n+}(0) = 0$, while the *odd* tower has $f_{L,n-}(0) = 0$. The profiles for $y > 0$:

$$\begin{aligned} f_{L,n_+}(y) &= \alpha_n^+ a^{-1/2} e^{-My} C_{-M}(y) & \qquad f_{L,n_-}(y) &= -\alpha_n^- a^{-1/2} e^{-My} S_{-M}(y) \\ f_{R,n_+}(y) &= \alpha_n^+ a^{-1/2} e^{My} S_M(y) & \qquad f_{R,n_-}(y) &= \alpha_n^- a^{-1/2} e^{My} C_M(y) \end{aligned} \tag{57}$$

For the even modes, the left-handed component has a symmetric profile, while the right-handed profile is actually anti-symmetric. The opposite holds for the *odd* modes. The multiplicative constants are determined from normalization: $\int_0^L f_{L,n}^2 = \int_0^L f_{R,n}^2 = 1$.

So far i used the general metric notation. The following results are for AdS_5 .

The even tower contains a chiral left-handed zero mode with the profile

$$f_{L,0} = \sqrt{\frac{k(1-2c)}{e^{(1-2c)kL} - 1}} e^{(1/2-c)ky} \qquad f_{R,0} = 0 \tag{58}$$

It is localized towards UV for $c > 1/2$ or towards IR for $c < 1/2$:

$$\begin{aligned} f_{L,0} &\approx \sqrt{k(2c-1)} e^{(1/2-c)ky} \qquad c > 1/2 \\ f_{L,0} &\approx \sqrt{k(1-2c)} e^{(1/2-c)k(y-L)} \qquad c < 1/2 \\ f_{L,0} &= \sqrt{\frac{1}{L}} \qquad c = 1/2 \end{aligned} \tag{59}$$

The massive KK tower start at $\sim M_{\mathrm{KK}} = k a_L$.

Let's consider now the **odd tower**. The quantization condition is $C_M(L) = 0$. Solving this approximatetely, we find the lightest odd state mass

$$\begin{aligned} m_{1-} &\sim M_{\mathrm{KK}} \qquad c > -1/2 \\ m_{1-} &\approx \sqrt{4c^2 - 1}\, M_{\mathrm{KK}}^2 e^{(1/2+c)kL} \qquad c < -1/2 \end{aligned} \tag{60}$$

Thus, choosing $c < -1/2$ we can generate sizable splitting between the lightest even and odd KK modes without resorting to BKTs. Note that the splitting can be achieved only when the corresponding zero mode fermions is sharply localized in IR. In practice, this is enough for our purposes. Only the top quark loops are required by naturalness to be cut off at $\lesssim 1$ TeV and top is expected to be localized near IR. For $c < -1/2$, the fermionic profiles are given by, approximately,

$$\begin{aligned} f_{L,1-} &\approx \sqrt{k|2c-1|} e^{(1/2-c)k(y-L)} \\ f_{R,1-} &\approx \sqrt{k|2c+1|} e^{(c+1/2)ky} \end{aligned} \tag{61}$$

The left profile is localized in IR and it is approximately the same as the corresponding zero mode profile. The right profile is localized in UV. Both expressions are good approximations only on the side where they are localized; recall that the left one should vanish on the UV brane, while the right one should vanish on the IR brane.

Consider zero mode fermion masses now. Take up quark, for example. u has the profiles of the first type (bulk mass denoted M_Q), while u^c has profiles of the second type (bulk mass denoted M_u). With mass term on the IR brane we get the IR Boundary conditions:

$$f_R^u(L) = y_u \tilde{v} a_L^{-1} f_R^{u^c}(L) \qquad f_L^{u^c}(L) = -y_u \tilde{v} a_L^{-1} f_L^u(L) \tag{62}$$

where y_u is the brane Yukawa coupling whose dimension is length. Quantization condition :

The quark mass that follows is:

$$m_u^2 \approx (y_t \tilde{v})^2 \frac{a_L^{-1} e^{-2M_Q L}}{\int_0^L a^{-1}(y) e^{-2M_Q y}} \frac{a_L^{-1} e^{-2M_u L}}{\int_0^L a^{-1}(y) e^{-2M_u y}} \tag{63}$$

In AdS$_5$:

To get suppressed masses we need $c > 1/2$. Thus, light SM fermions need to have $c > 1/2$, while for the 3rd generation we need $c \lesssim 1/2$.

Dark Matters

Here we discuss who is the LKP in the framework of $SU(2)_L \times SU(2)_R \times U(1)_X$ models and what are his/her interactions. The relevant players are the following terms in the lagrangian:

$$\begin{aligned} \mathcal{L}_{kin} &= -\frac{1}{4} L_{\mu\nu}^2 (1 + r_L \delta(L)) - \frac{1}{4} R_{\mu\nu}^2 (1 + r_R \delta(L)) - \frac{1}{4} X_{\mu\nu}^2 (1 + r_X \delta(L)) \\ \mathcal{L}_{mass} &= \frac{1}{2} \delta(L) \frac{\tilde{v}^2}{4} (g_{5,L} L_\mu^a - g_{5,R} R_\mu^a)^2 \end{aligned} \tag{64}$$

We have quite a few free parameters: the three bulk couplings $g_{5,L}$, $g_{5,R}$ and $g_{5,X}$. and the three IR brane kinetic terms $r_{L,R,X}$. We will always assume that at least one BKT is large. Depending on the relative values of the BKTs we will have an LKP sitting in a different 5D gauge field. We will concentrate on two cases,

- KK V: is a kind of KK Z occuring in the $SO(4)$ symmetric case $g_{5,L} = g_{5,R} \equiv g_5$, $r_L = r_R \equiv r \gg r_X$
- KK L: is a kind of KK Z occuring for $r_L \gg r_{R,X}$
- KK R: is a kind of KK photon occuring for $r_R \gg r_{L,X}$
- KK X: is a kind of KK photon occuring for $r_X \gg r_{R,L}$

We start generally however. The odd modes can sit in LR guage fields,

$$f_L^a(y) = \alpha_L^a S(y) \qquad f_R^a(y) = \alpha_R^a S(y) \tag{65}$$

or in X gauge fields,

$$f_X(y) = \alpha_X S(y) \tag{66}$$

Here $S(y)$ solves the equations of motion and vanishes on the IR branes. Approximately, $S(y) \sim \int_0^y a^{-2}$. For AdS $S(y) \sim a^{-2}(y) - 1$. The profile is thus sharply localized toward IR.

Up to overall normalization, the constants are determined by the boundary conditions which in the presence of EW breaking read

$$\begin{aligned}
a_L^2 \partial_5 f_X(L) - m^2 r_X f_X(L) &= 0 \\
a_L^2 \partial_5 f_L^a(L) - m^2 r_L f_L^a(L) &= -\frac{1}{4} g_{5,L} \tilde{v}^2 (g_{5,L} f_L^a(L) - g_{5,R} f_R^a(L)) \\
a_L^2 \partial_5 f_R^a(L) - m_n^2 r_R f_R^a(L) &= +\frac{1}{4} g_{5,R} L \tilde{v}^2 (g_L f_L^a(L) - g_{5,R} f_R^a(L))
\end{aligned} \tag{67}$$

From this BC it is obvious that the odd modes in LR and X do not mix (unlike the even one, where the mixing is introduced on the UV brane).

The X tower has the mass quantization condition,

$$a_L^2 S'(L) - m^2 r_X S(L) = 0 \tag{68}$$

which for large r_X would give the light eigenstate with $m_{X,1-} \approx 1/r_X \int_0^L a^{-2}$.

The left and right mix through EW breaking on the IR brane. In general, we find

$$\alpha_R = -\frac{g_{5,R}}{g_{5,L}} \alpha_L \frac{a_L^2 S'(L) - m^2 r_L S(L)}{a_L^2 S'(L) - m^2 r_R S(L)} \tag{69}$$

and the quantization condition

$$\begin{aligned}
&(a_L^2 S'(L) - m^2 r_L S(L))\,(a_L^2 S'(L) - m^2 r_R S(L)) \\
&= -\tfrac{\tilde{v}^2}{4} \left((g_{5,L}^2 + g_{5,R}^2) a_L^2 S'(L) - m^2 (g_{5,L}^2 r_R + g_{5,R}^2 r_L) S(L) \right)
\end{aligned} \tag{70}$$

In the limit of no EW breaking we have two eigenstates, one along L and one along R, with the mass given by the quantization condition:

$$a_L^2 S'(L) - m^2 r_{L,R} S(L) = 0 \tag{71}$$

In the generic case, when r_L and r_R are different enough, the masses of the two lightest eigenstates are different. In that case, EW breaking introduces only small mixing of order $v^2/M_{\rm KK}^2$. We will ignore the effects of this mixing, except for the special SO(4) symmetric case $r_L = r_R$. In the SO(4) case the eigenstates along L and R are degenerate before EW breaking. Then electroweak breaking decides who is the eigenstate, and it picks up the vector and axial combinations of L and R as eigenstates.

Now, we scan trough our special cases:

1. **KK L.** In this case the LKP sits almost fully in L. The profile (in AdS5) is approximately,

$$f_L^3(y) \approx \frac{1}{\sqrt{r_L}} \frac{e^{2ky} - 1}{e^{2kL}} \tag{72}$$

e.llkpp

Let's point out the properties of KK L.

- KK L has non-abelian gauge interactions coming from the L^2 kinetic term. In particular, it couples to the SM $W^{\pm}$ who also lives mostly in $L^{\pm}$. The coupling goes via the tri-linear vertex $W^{\pm}L^{\mp}L^3$ and a quartic one $W^{\pm}W^{\mp}L^3L^3$. The coupling is the same as the corresponding SM coupling. More precisely, we have the trilinear vertex:

$$\mathcal{L}_3 \approx -ig_L \left[(\partial_\mu L^3_\nu - \partial_\nu L^3_\mu)L^+_\mu W^-_\nu + L^3_\nu(\partial_\mu L^+_\nu - \partial_\nu L^+_\mu)W^-_\mu - L^3_\nu L^+_\mu(\partial_\mu W^-_\nu - \partial_\nu L^-_\mu)\right] + \text{h.c.} \tag{73}$$

where g_L is the SM SU(2) coupling.
We also have the quartic vertex:

$$\mathcal{L}_4 \approx -g_L^2 L^3_\mu L^3_\mu W^+_\nu W^-_\nu + \frac{g_L^2}{2} L^3_\mu L^3_\nu (W^+_\mu W^-_\nu + W^+_\nu W^-_\mu) \tag{74}$$

e.gqv

- KK L couples to the higgs boson (this is not so important for relic abundance, but important for direct detection). We note, that the wave function of the LKP at the IR brane is $1/\sqrt{r_L}$, the same as for the SM W boson. Therefore, the coupling of the LKP to the higgs is the SM one. More precisely, the coupling is:

$$\mathcal{L}_h \approx \frac{m_W^2}{v} h L^3_\mu L^3_\mu \tag{75}$$

- KK L may couple to the SM fermions that carry $SU(2)_L$ charge. However, by KKP conservation, the couplings must involve the odd fermionic mode. Recall that for $c > -1/2$ the odd modes are heavy, of order M_{KK} that we take of order 10 TeV. Annihilation diagrams will be then suppressed by the large mass of the odd mode exchanged in the t channel. On the other hand, for $c < -1/2$ the lightest KK odd fermion can be light, of order $e^{(c+1/2)kL} M_{\text{KK}}$ and the amplitude is not suppressed by a heavy fermion mass. The vertex is

$$\mathcal{L} = g_\psi L^3_\mu \overline{\psi}_0 \gamma^\mu \psi_{1-} \tag{76}$$

and the coupling is given by the overlap of the corresponding profiles: the LKP one given in eq. (72), the zero mode profile in eq. (59) and the odd mode profile in eq. (61),

$$g_\psi = T^3_L g_{5,L} \int_0^L f^3_L f_{L,1-} f_{L,0} \approx g_L T^3_L \frac{1-2c}{3-2c} \tag{77}$$

e.lkpfc

However, the annihilation into fermions is not very relevant for dark matter, unless the majority of the SM fermions has $c < -1/2$, something poorly motivated.

Typically (unless most of the SM fermions have light odd KK partners), the annihilation cross section is dominated by the annihilation into $W^{\pm}$: via t/u channel exchange of $L^{\pm}$ or directly through the quartic vertex. The authors of ref. [5] were so kind as to provide us with a suitably averaged cross section that already includes all the channels,

$$\sigma(L^3L^3 \to W^+W^-) = \frac{g_L^4}{18\pi m^2 s^3 \beta^2}\left(12m^2(s-2m^2)L + s\beta(12m^4 + 3sm^2 + 4s^2)\right), \tag{78}$$

where $\beta^2 = 1 - 4s^2/m^2_{LKP}$.

2. **KK V.** In this case it is convenient to introduce axial and vector combinations of the LR gauge fields

$$\begin{aligned} V_\mu^a &= \frac{1}{\sqrt{2}}\left(L_\mu^a + R_\mu^a\right) \\ A_\mu^a &= \frac{1}{\sqrt{2}}\left(L_\mu^a - R_\mu^a\right) \end{aligned} \tag{79}$$

Only the axial gets a mass from EW breaking. The vector mass is as usual, $m_{1-} \sim M_{\rm KK}/kr$ and unaffected by EW breaking, while the axial mass will be slightly (of order m_W^2) larger. **Thus, in the SO(4) symmetric case the neutral vector combination V_μ^3 is the LKP**. Let's point out the properties of KK V.

- KK V has non-abelian gauge interactions coming from the L^2 and R^2 kinetic terms. In particular, it couples to the SM $W^\pm$ who lives mostly in $L^\pm$. The coupling goes via the tri-linear vertex $W^\pm V^\mp V^3$ and a quartic one $W^\pm W^\mp V^3 V^3$. The coupling is **a half** of the corresponding SM coupling (because the couplings originatea from $L_{\mu\nu}^2$ and there is only $1/\sqrt{2}$ of L in V). More precisely, we have the trilinear vertex:

$$\mathcal{L}_{3v} = -i\frac{g_L}{2}\partial_\mu V_\nu^3 V_\mu^+ W_\nu^- + \dots \tag{80}$$

  where g_L is the SM SU(2) coupling, and dots are further terms with derivatives on W as in eq. (73). The analogous term in the SM has g_L instead of $g_L/2$. There is also the triple vertex with charged axials

$$\mathcal{L}_{3a} = -i\frac{g_L}{2}\partial_\mu V_\nu^3 A_\mu^+ W_\nu^- + \dots \tag{81}$$

  that has the same coupling.
  We also have the quartic vertex:

$$\mathcal{L}_4 = -\frac{g_L^2}{2} V_\mu^3 V_\mu^3 W_\nu^+ W_\nu^- + \dots \tag{82}$$

  where dots is another term with $\mu\nu$ differently placed as in eq. (74). The analogous term in the SM has g_L^2 instead of $g_L^2/2$.

- KK V does not feel the higgs vev, therefore it **does not couple to the higgs at all**. This will probably make it invisible to direct detection.
- KK V couplings to the fermions are irrelevant, for the same reason as for KK L.

If the fermions do not count, the annihilation cross section is dominated by the annihilation into $W^\pm$: via t/u channel exchange of $V^\pm$ and $A^\pm$ or directly through the quartic vertex. $V^\pm$ and $A^\pm$ are approximately degenerate, and they have the same couplings to $V^3 W$. Therefore the $V^\pm$ and $A^\pm$ exchange diagrams are approximately equal. Each is one quarter of the $L^\pm$ exchange diagram from the KK L case, so their sum is one half. The quartic vertex $V^2 W^2$ is also one half that of L^2W^2. We conclude that the annihilation cross section of V's is one quarter that of L's,

$$\sigma(V^3V^3 \to W^+W^-) = \frac{1}{4}\sigma(L^3L^3 \to W^+W^-) \tag{83}$$

so that KK V mass that gives the correct abundance would be approximately one half of the KK L mass that gives the correct abundance.

3. **KK R.** In this case the LKP sits almost fully in R, with a small component in L induced by EW breaking The profile (in AdS5) is approximately,

$$\begin{aligned} f_R^3(y) &\approx \frac{1}{\sqrt{r_R}}\frac{e^{2ky}-1}{e^{2kL}} \\ f_L^3(y) &\approx f_R^3(y)\cdot\mathcal{O}(m_W^2/m_{1-}^2) \end{aligned} \tag{84}$$

Some properties of KK R.

- KK R has non-abelian gauge interactions coming from the R^2 kinetic terms. However, there is very little of $W^\pm$ in R. So, the vertex $R^3R^\pm W^\mp$ is very much suppressed.
- KK R couples to the higgs boson. The coupling is:

$$\mathcal{L}_h \approx \frac{g_{5,R}^2}{g_{5,L}^2}\frac{m_W^2}{v}hR_\mu^3R_\mu^3 \tag{85}$$

  Almost the standard model one, up to the factor depending on the ratio of L and R gauge couplings, which could be anything but we expect it to be order one.
- Again, coupling to the SM fermions is small.

KK R seems a poor dark matter candidate: annihilation cross section is very small. I dont have an estimate, but certainly the mass needed to obtain a correct relic abundance must be ridiculously small.

4. **KK X.** In this case the LKP sits fully in X. The profile (in AdS5) is approximately,

$$f_X(y) \approx \frac{1}{\sqrt{r_X}}\frac{e^{2ky}-1}{e^{2kL}} \tag{86}$$

Some properties of KK X.

- KK X does not have non-abelian gauge interactions. So the vertex $X^3X^\pm W^\mp$ just does not exist.
- KK X does not couple to the higgs boson at all (the higgs field is not charged under $U(1)_X$).
- In this case, it is the only coupling this poor LKP would have. It is given by the formula analogous to eq. (77) with $g_LT_L^3$ replaced by g_Xq_X.

KK X seems a poor dark matter candidate: annihilation cross section is very small. I dont have an estimate, but certainly the mass needed to obtain a correct relic abundance must be ridiculously small.

Gauge-Higgs

I turn now to the gauge-higgs framework in which naturalness issues are really adressed. As an appetizer, I investigate a toy-model with $SU(2)$ gauge symmetry which is broken down to $U(1)$. At low energies we get a $U(1)$ theory plus a charged scalar coming from the A_5 components along the two broken. The latter plays a role of the higgs. Below we investigate contributions to the higgs potential from the gauge fields.

I will speak the language of refs. [2] (sorry). The higgs potential is derived from the so-called spectral function ρ, where $\rho = \det(-p^2 + m_n^2(\tilde{v}))$ is a function of momenta whose zeros encode the whole KK spectrum in the presence of EW breaking. With a spectral function at hand, we can compute the higgs potential from the Coleman-Weinberg formula,

$$V(\tilde{v}) = \frac{N}{(4\pi)^2} \int_0^\infty dp p^3 \log\left(\rho[-p^2]\right) \qquad (87) \quad \boxed{\texttt{e.cws}}$$

where $N = +3$ for the gauge fields. The spectral function can be computed by solving the equations of motion in the gauge-higgs background together with the boundary condtions. This is what i'm doing below.

The boundary condtions for the $SU(2)$ gauge fields are given by:

$$A_\mu^- = \begin{pmatrix} A_\mu^{1,2}[--] \\ A_\mu^3[++] \end{pmatrix} \qquad (88)$$

where $+$ stands for Neumann (or mixed in the presence of brane kinetic terms), while $-$ stands for Dirichlet BC on the UV and IR brane, respectively.

The gauge field is expanded as $A_\mu = A_{\mu,n}(x) f_n(y)$ (where the profile f is a matrix, $f = f^a \sigma^a$). The profiles satisfy the equations of motion:

$$D_y(a^2 D_y f) + p^2 f = 0 \qquad (89)$$

with $D_y f = \partial_y f - i g_5 [\langle A_5 \rangle, f]$, so that various f^a are mixed in the presence of the gauge-higgs vev. We can rotate away the higgs vev from the equations of motion by rewriting

$$f(y) = \Omega(y) \hat{f}(y) \Omega^\dagger(y) \qquad \Omega(y) = e^{i g_5 \int_0^y \langle A_5 \rangle} \qquad (90)$$

and we obtain the simple equation for the hat,

$$\partial_y(a^2 \partial_y \hat{f}) + p^2 \hat{f} = 0 \qquad (91) \quad \boxed{\texttt{e.eomgp}}$$

in which different f^a are no longer mixed. The higgs vev is now swept under IR boundary conditions. The UV boundary conditions are unchanged because i fixed $\Omega(0) = 1$.

We pick up the two independent solutions of eq. (91) and define two combinations $C(y)$ and $S(y)$ that satisfy $C(0) = 1$, $C'(0) = 0$, $S(0) = 0$, $S'(0) = m$. We can then write down the hatted profiles as

$$\hat{f}^{1,2}(y) = \alpha_{1,2} S(y) \qquad \hat{f}^3(y) = \alpha_3 C(y) \qquad (92) \quad \boxed{\texttt{e.lgp}}$$

The true profiles are found by rotating the hats with the matrix $\Omega(y)$. We choose the higgs vev to reside along σ^1,

$$\langle A_5\rangle = \frac{a^{-2}(y)}{\left[\int_0^L a^{-2}\right]^{1/2}}\frac{\sigma^1}{2}\tilde{v} \tag{93}$$

In such a case

$$\Omega(L) = \begin{pmatrix} \cos(\tilde{v}/2f) & i\sin(\tilde{v}/2f) \\ i\sin(\tilde{v}/2f) & \cos(\tilde{v}/2f) \end{pmatrix} \qquad f^2 = \frac{1}{g_5^2 \int_0^L a^{-2}} \tag{94}$$

The relation between hatted and unhatted turns out to be:

$$\begin{aligned} f^1(y) &= \hat{f}^1(y) \\ f^2(L) &= \cos(\tilde{v}/f)\hat{f}^2(L) + \sin(\tilde{v}/f)\hat{f}^3(L) \\ f^3(L) &= -\sin(\tilde{v}/f)\hat{f}^2(L) + \cos(\tilde{v}/f)\hat{f}^3(L) \end{aligned} \tag{95}$$

Pluggin in eq. (92) we find the solutions: $\alpha_{2,3} = 0$ and $S(L) = 0$ (this tower doesnt feel the higgs vev, therefore it does not contribute to the higgs potential) or $\alpha_1 = 0$ and

$$\begin{aligned} 0 &= \alpha_2\cos(\tilde{v}/f)S(L) + \alpha_3\sin(\tilde{v}/f)C(L) \\ 0 &= -\alpha_2\sin(\tilde{v}/f)\left(S'(L) - p^2 r_{IR}a^{-2}(L)S(L)\right) + \alpha_3\cos(\tilde{v}/f)\left(C'(L) - p^2 r_{IR}a^{-2}(L)C(L)\right) \end{aligned} \tag{96}$$

The determinant of this equation gives the quantization condtion, ergo the spectral function

$$\rho_+(p^2) = F(p^2) + \sin^2(\tilde{v}/f) \tag{97}$$

where ther form factor is

$$F(p^2) = p^{-1}a^2(L)C'(L)S(L) - p r_{IR}C(L)S(L) \tag{98}$$

For small momenta, below the mass of the lightest heavy mode, it behaves as $F(p^2) \approx -p^2/g_0^2 f^2$, where $g_0^2 = g_5^2/(r_{IR}+L)$ is the zero mode coupling. In fact, we can give the approximate expression that is valid all the way to the KK scale mode,

$$F(p^2) \approx -\frac{p^2}{g_0^2 f^2}\left(1 - \frac{p^2}{g_0^2 f^2}\frac{r_{IR}L}{(r_{IR}+L)^2}\right) \tag{99}$$

which can also be written as

$$F(p^2) \approx -\frac{p^2}{m_{1-}^2}\left(1 - \frac{p^2}{m_{1+}^2}\right) \tag{100}$$

where $m_{1-} \approx g_0 f$, $m_{1+}^2 \approx m_{1-}^2 r_{IR}/L$. One can see that $F(-p^2)$ grows as p^4 for $p > m_{1+}$, that is above the mass of the first excited KK state. For high momenta, when the whole KK tower sets in we have $F(p^2) \sim e^{2p/M_{\rm KK}}$, which ensures that the potential finite.

From the spectral function we can deduce that the higgs potential is of the form:

$$V_+(\tilde{v}) = a_0 + a_2\sin^2(\tilde{v}/f) + a_4\sin^4(\tilde{v}/f) + \ldots \tag{101}$$

(in fact a_4 is not a constant, it depends $\log\sin(\tilde{v}/f)$, but we can ignore this detail here) where the constants are given by

$$\begin{aligned} a_2 &= \frac{N}{(4\pi)^2}\int_0^\infty dpp^3 F^{-1}(p^2) \\ a_4 &\approx \frac{N}{(4\pi)^2}\int_{m_W}^\infty dpp^3 F^{-2}(p^2) \end{aligned} \tag{102}$$

The integrals are UV finite because of the exponential behaviour of the form factor, that effectively cuts them off at $M_{\rm KK}$. For large kinetic terms we can estimate $a_2 \sim a(L)\, M_{\rm KK}^3 r_{IR}^{-1}$, consequently, the higgs mass is of order $m^2 \sim a(L)\, M_{\rm KK}^3/r_{IR}f^2$. Since $f^2 \sim a(L)\, M_{\rm KK}/g_5^2 \sim a(L)\, M_{\rm KK}/r_{IR}g_{SM}^2$, we find the higgs mass is of order $M_{\rm KK}^2$. *From numerics it seems that the previous statement is not correct, and the quadratic divergence is cut off by the first even mode, at the scale m_{1+} and there is only log sensitivity to $M_{\rm KK}$.*

Now, what happens in the KK parity symmetric set-up? KK parity is equivalent to introducing another gauge field (that i call the shadow) whose UV boundary conditions are flipped with respect to the original one (that i call the light),

$$A_\mu^- = \begin{pmatrix} \tilde{A}_\mu^{1,2}[+-] \\ \tilde{A}_\mu^3[-+] \end{pmatrix} \tag{103}$$

We need to compute the spectral function for the shadow. Repeating the same steps as for the light we find

$$\rho_-(p^2) = F(p^2) + \cos^2(\tilde{v}/f) \tag{104}$$

where the form factor is the same one as for the light multiplet. Thus, the shadow differs from the light only by switching fromn a sine turns into a cosine.

Thus, the shadow contributes to the higgs potential as

$$V_-(\tilde{v}) = a_0 + a_2\cos^2(\tilde{v}/f) + a_4'\cos^4(\tilde{v}/f) + \dots \tag{105}$$

where a_2 is the same as for the light. Summing up the two contributions.

$$V_+(\tilde{v}) + V_-(\tilde{v}) = a_0 + a_2 + a_4(\cos^4(\tilde{v}/f) + \sin^4(\tilde{v}/f)) + \dots \tag{106}$$

so that the large, proportional to a_2 contribution to the mass exactly cancels between the light and the shadow, thanks to the KK parity. The only contribution to the higgs mass comes from the fake quartic, $a_4\cos^4(\tilde{v}/f)$ (and higher). Now $a_4 \sim a^2(L)\, M_{\rm KK}^2 r_{IR}^{-2}$, $m^2 \sim a^2(L)\, M_{\rm KK}^2/r_{IR}^2 f^2$ so we end up with $m^2 \sim a(L)\, M_{\rm KK}/r_{IR}$. This is much smaller than $M_{\rm KK}$, suppressed by the large IR brane kinetic term. Thus, at the end of the day the higgs mass generated by gauge loops is $m^2 \sim m_{1-}^2$, that is, proportional to the mass of the lightest **odd** mode!. Hallelujah.

Another, more straighforward way to estimate the contribution to the higgs mass: compute the second derivative of the potential at $\tilde{v} = 0$. We have

$$\begin{aligned} V_+''(0) &= \frac{N}{8\pi^2 f^2}\int_0^\infty dpp^3 \frac{1}{F(p^2)} \\ V_-''(0) &= -\frac{N}{8\pi^2 f^2}\int_0^\infty dpp^3 \frac{1}{(F(p^2)+1)} \end{aligned} \tag{107}$$

so that

$$V''(0) = \frac{N}{8\pi^2 f^2}\int_0^\infty dp p^3 \frac{1}{F(p^2)(F(p^2)+1)} \tag{108}$$

Now we can see clearly. For $p < m_{1-}$ the integrand $\sim p m_{1-}^2$ so that we are sensitive quadratically to the first cutoff at m_{1-}. But already at $p \sim m_{1-}$ we had $F(p^2) > 1$ so that the integrand goes as $\sim m_{1-}^4/p$ and we are only log sensitive to the next cutoff at m_{1+}. Thus we can estimate $m_h^2 \sim (1/4\pi^2) m_{1-}^2 \log(m_{1+}/m_{1-})$.

Appendix A The Art of Juggling

a.aoj

We show how to switch from four-fermion operators and vertex corrections to oblique parameters.

We start with the low energy effective lagrangian after electroweak breaking:

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{mass} + \mathcal{L}_{vertex} + \mathcal{L}_{4f} \tag{A.1}$$

e.sul

where

$$\begin{aligned}
\mathcal{L}_{kin} = \; & -\tfrac{1}{4}(L^a_{\mu\nu})^2\left(1 + 2\tfrac{g_L^2 v^2}{4}\delta_W\right) - \tfrac{1}{4}(B_{\mu\nu})^2\left(1 + 2\tfrac{g_Y^2 v^2}{4}\delta_B\right)\\
& -\alpha_S \tfrac{g_L g_Y v^2}{4}\tfrac{1}{2} W^3_{\mu\nu}B_{\mu\nu} - \tfrac{1}{4}\alpha_W(\partial_\rho L^a_{\mu\nu})^2 - \tfrac{1}{4}\alpha_Y(\partial_\rho B_{\mu\nu})^2
\end{aligned} \tag{A.2}$$

$$\mathcal{L}_{mass} = \frac{1}{2}\frac{g_L^2 v^2}{4} L^i_\mu L^i_\mu + \frac{1}{2}\frac{v^2}{4}\left(1 - \alpha_T \frac{g_Y^2 v^2}{2}\right)\left(g_L L^3_\mu - g_Y B_\mu\right)^2 \tag{A.3}$$

$$\begin{aligned}
\mathcal{L}_{vertex} = \; & \tfrac{g_L}{\sqrt{2}}\left(1 + m_W^2\beta_j\right)\overline{\psi}_j\gamma_\mu t^\pm \psi_j W^\pm_\mu + \left\{m_W^2 \rho_{ij}(\overline{\psi}_i\gamma_\mu\psi_j)W^+_\mu + \text{h.c.}\right\}\\
& + g_L\overline{\psi}_j\gamma_\mu t^3\psi_j\left(L^3_\mu + \tfrac{m_Z^2 g_L}{g_L^2+g_Y^2}\beta_j(g_L L^3_\mu - g_Y B_\mu)\right)\\
& + g_Y\overline{\psi}_j\gamma_\mu Y\psi_j\left(B_\mu - \tfrac{m_Z^2 g_Y}{g_L^2+g_Y^2}\gamma_j(g_L L^3_\mu - g_Y B_\mu)\right)
\end{aligned} \tag{A.4}$$

$$\mathcal{L}_{hdv} = \kappa_{L,j} g_L \overline{\psi}_j\gamma_\mu t^a\psi_j\partial_\nu L^a_{\nu\mu} + \kappa_{B,j} g_Y\overline{\psi}_j\gamma_\mu Y\psi_j\partial_\nu B_{\nu\mu} \tag{A.5}$$

$$\mathcal{L}_{4f} = -c_{ij} g_Y^2\overline{\psi}_i\gamma_\mu Y\psi_i\overline{\psi}_j\gamma_\mu Y\psi_j - d_{ij} g_L^2\overline{\psi}_i\gamma_\mu t^a\psi_i\overline{\psi}_j\gamma_\mu t^a\psi_j \tag{A.6}$$

The mass and vertex corrections can be readily understood as the effect of dimension six operators

$$\begin{aligned}
\mathcal{L}_{D6} = \; & -\tfrac{1}{4}\alpha_W(\partial_\rho L^a_{\mu\nu})^2 - \tfrac{1}{4}\alpha_Y(\partial_\rho B_{\mu\nu})^2\\
& -\alpha_T g_Y^2|H^\dagger D_\mu H|^2 + \alpha_S\tfrac{g_L g_Y}{2}(H^\dagger t^a H)W^a_{\mu\nu}B_{\mu\nu}\\
& -\left\{i\beta_j g_L^2(\overline{\psi}_j\gamma_\mu t^a\psi_j)(D_\mu H^\dagger t^a H) + i\tfrac{\gamma_j}{2}Y_j g_Y^2\overline{\psi_j}\gamma_\mu\psi_j(D_\mu H^\dagger H) + \text{h.c.}\right\}\\
& -\left\{i\tfrac{\rho_{ij}}{2}g_L^2(\overline{\psi}_i\gamma_\mu t^a\psi_j)(H^T\epsilon D_\mu H) + \text{h.c.}\right\}
\end{aligned} \tag{A.7}$$

after inserting the higgs vev. ρ_{ij} is non-zero only when $Y_i - Y_j = 1$, which leaves $i = u, j = d$ or $i = \nu, j = e$ (the latter if RH neutrino is present).

We now perform the following rescaling of the electroweak fields

$$\begin{aligned}
L^i_\mu &\rightarrow L^i_\mu + \hat{d}\frac{g_L^2 v^2}{4} L^i_\mu + a_j g_L \overline{\psi}_j \gamma_\mu t^i \psi_j - d\partial_\nu L^i_{\nu\mu} \\
L^3_\mu &\rightarrow L^3_\mu + \hat{d}\frac{g_L^2 v^2}{4} L^3_\mu + a_j g_L \overline{\psi}_j \gamma_\mu t^3 \psi_j - d\partial_\nu L^3_{\nu\mu} + x\frac{g_L g_Y v^2}{4} B_\mu \\
B_\mu &\rightarrow B_\mu + \hat{c}\frac{g_Y^2 v^2}{4} B_\mu + b_j g_Y \overline{\psi}_j \gamma_\mu Y \psi_j - c\partial_\nu B_{\nu\mu} + y\frac{g_L g_Y v^2}{4} L^3_\mu
\end{aligned} \tag{A.8}$$

We also rededine the couplings:

$$g_L \rightarrow g_L\left(1 + \frac{g_L^2 v^2}{4}(\delta_L - \hat{d})\right) \qquad g_Y \rightarrow g_Y\left(1 + \frac{g_Y^2 v^2}{4}(\delta_Y - \hat{c})\right) \tag{A.9}$$

Finally, we rescale the electroweak vev as

$$v \rightarrow v(1 - m_W^2 \delta_L) \tag{A.10}$$

so that the W mass remains $g_L v/2$. We assume all these a_j, b_j, c, d, x, y are small, so that we can retain only linear terms in these and the others. We get

$$\begin{aligned}
\mathcal{L}_{kin} = & -\tfrac{1}{4}(L^a_{\mu\nu})^2\left(1 + 2\tfrac{g_L^2 v^2}{4}(\delta_W + \hat{d} - d)\right) - \tfrac{1}{4}(B_{\mu\nu})^2\left(1 + 2\tfrac{g_Y^2 v^2}{4}(\delta_B + \hat{c} - c)\right) \\
& -\tfrac{g_L g_Y v^2}{4}\left(\alpha_S + c + d + x + y\right)\tfrac{1}{2} L^3_{\mu\nu} B_{\mu\nu} \\
& -\tfrac{1}{4}\left(\alpha_W + 2d\right)(\partial_\rho L^a_{\mu\nu})^2 - \tfrac{1}{4}\left(\alpha_B + 2c\right)(\partial_\rho B_{\mu\nu})^2
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\mathcal{L}_{mass} = & \tfrac{1}{2}\tfrac{g_L^2 v^2}{4} L^i_\mu L^i_\mu + \tfrac{1}{2}\tfrac{g_L^2 v^2}{4} L^3_\mu L^3_\mu\left(1 - 2(y + \alpha_T)\tfrac{g_Y^2 v^2}{4}\right) \\
& + \tfrac{1}{2}\tfrac{g_Y^2 v^2}{4} B_\mu B_\mu\left(1 - 2(x + \delta_L) m_W^2 + 2(\delta_Y - \alpha_T)\tfrac{g_Y^2 v^2}{4}\right) \\
& - \tfrac{g_L g_Y v^2}{4} B_\mu L^3_\mu\left(1 - (x + \delta_L) m_W^2 + (\delta_Y - y - 2\alpha_T)\tfrac{g_Y^2 v^2}{4}\right)
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
\mathcal{L}_{vertex} = & \tfrac{g_L}{\sqrt{2}}\overline{\psi}_j \gamma_\mu t^\pm \psi_j\left(L^\pm_\mu + m_W^2[\beta_j + a_j + \delta_L] L^\pm_\mu\right) \\
& + g_L \overline{\psi}_j \gamma_\mu t^3 \psi_j\left(L^3_\mu + \tfrac{g_L^2 v^2}{4}[\beta_j + a_j + \delta_L] L^3_\mu - \tfrac{g_L g_Y v^2}{4}[\beta_j + a_j - x] B_\mu\right) \\
& + g_Y \overline{\psi}_j \gamma_\mu Y \psi_j\left(B_\mu + \tfrac{g_Y^2 v^2}{4}[\gamma_j + b_j + \delta_Y] B_\mu - \tfrac{g_L g_Y v^2}{4}[\gamma_j + b_j - y] L^3_\mu\right)
\end{aligned} \tag{A.13}$$

$$\mathcal{L}_{hdv} = [\kappa_{L,j} + a_j - d] g_L \overline{\psi}_j \gamma_\mu t^a \psi_j \partial_\nu L^a_{\nu\mu} + [\kappa_{B,j} + b_j - c] g_Y \overline{\psi}_j \gamma_\mu Y \psi_j \partial_\nu B_{\nu\mu} \tag{A.14}$$

e.hdv

$$\mathcal{L}_{4f} = -[c_{ij} - (b_i + b_j)/2] g_Y^2 \overline{\psi}_i \gamma_\mu Y \psi_i \overline{\psi}_j \gamma_\mu Y \psi_j - [d_{ij} - (a_i + a_j)/2] g_L^2 \overline{\psi}_i \gamma_\mu t^a \psi_i \overline{\psi}_j \gamma_\mu t^a \psi_j \tag{A.15}$$

If we choose a transformation with $\delta_L = -x$ and $\delta_Y = -x$, then we are back to the start-up lagrangian (A.1) with just shifted coefficients:

$$\begin{aligned}
\delta_W &\rightarrow \delta_W + \hat{d} - d & \delta_B &\rightarrow \delta_B + \hat{c} - c \\
\alpha_S &\rightarrow \alpha_S + c + d + x + y & \alpha_T &\rightarrow \alpha_T + y \\
\alpha_W &\rightarrow \alpha_W + 2d & \alpha_B &\rightarrow \alpha_B + 2c \\
\beta_j &\rightarrow \beta_j + a_j - x & \gamma_j &\rightarrow \gamma_j + b_j - y \\
\kappa_{L,j} &\rightarrow \kappa_{L,j} + a_j - d & \kappa_{B,j} &\rightarrow \kappa_{B,j} + b_j - c \\
d_{ij} &\rightarrow d_{ij} - (a_i + a_j)/2 & c_{ij} &\rightarrow c_{ij} - (b_i + b_j)/2
\end{aligned} \tag{A.16}$$

This is the prescription for juggling with parameters of the dimension six operators in the SM. *(I don't quite understand why i need the constraint $\delta_L = -x$ etc.)*

Universal case

We assume that the coefficients in the start-up lagrangian have the following simple pattern

$$c_{ij} = \bar{c} \quad d_{ij} = \bar{d} \quad \beta_j = \beta \quad \gamma_j = \gamma \quad \kappa_{L,j} = \kappa_L \quad \kappa_{B,j} = \kappa_B \tag{A.17}$$

$$\alpha_S = \bar{\alpha}_S \quad \alpha_T = \bar{\alpha}_T \quad \alpha_W = \bar{\alpha}_W \quad \alpha_Y = \bar{\alpha}_Y \tag{A.18}$$

Then we can choose the rescaling coefficients as $a_j = \bar{d}$, $b_j = \bar{c}$, $c = \kappa_B + \bar{c}$, $d = \kappa_L + \bar{d}$, $x = \beta + \bar{d}$, $y = \gamma + \bar{c}$. This choice kill all four-fermion operators and vertex corrections. We are left only with four so-callled oblique (not involving fermions) operators with the coefficients:

$$\alpha_S = \bar{\alpha}_S + \beta + \gamma + 2\bar{d} + 2\bar{c} + \kappa_L + \kappa_B \qquad \alpha_T = \bar{\alpha}_T + \gamma + \bar{c} \tag{A.19}$$ e.vto

$$\alpha_W = \bar{\alpha}_W + 2\bar{c} \qquad \alpha_Y = \bar{\alpha}_Y + 2\bar{d} \tag{A.20}$$

The familiar S and T are connected to our parameters by:

$$S = 4\pi v^2 \alpha_S \qquad T = \frac{2\pi g_Y^2 v^2}{e^2} \alpha_T \tag{A.21}$$ e.tst

While hatted oblique parameters from ref. [4]

$$\hat{S} = m_W^2 \alpha_S \qquad \hat{T} = \frac{g_Y^2 v^2}{2} \alpha_T \qquad W = m_W^2 \alpha_W \qquad Y = m_W^2 \alpha_Y \tag{A.22}$$

Also worth remembering: $\hat{S} = \frac{g_L^2}{16\pi} S$

Appendix B Warped Trigonometry

a.wt

The gauge boson mode equation

$$\partial_5 (a^2(x_5) \partial_5 f(x_5)) + z^2 f(x_5) = 0 \tag{B.23}$$ e.gme

has two independent solutions. Denote them $C(x_5, z)$ and $S(x_5, z)$. We choose them such that they satisfy

$$C(0, z) = 1 \qquad C'(0, z) = 0 \qquad S(0, z) = 0 \qquad S'(0, z) = z \tag{B.24}$$

(so that they generalize cosines and sines). The Wronskian relation

$$S'(x_5, z) C(x_5, z) - C'(x_5, z) S(x_5, z) = z\, a^{-2}(x_5) \tag{B.25}$$ e.wr

is the warped analog of $\sin^2 + \cos^2 = 1$.

Integrating eq. (B.23) twice, we obtain the integral representation of the warped trigs

$$\begin{aligned} C(x_5, z) &= 1 - z^2 \int_0^{x_5} a^{-2}(y) \int_0^y C(y', z) \\ S(x_5, z) &= z \int_0^{x_5} a^{-2}(y) - z^2 \int_0^{x_5} a^{-2}(y) \int_0^y S(y', z) \end{aligned} \tag{B.26}$$

from which follows the expansion at small z:

$$
\begin{aligned}
C(x_5,z) &= 1 - z^2 \int_0^{x_5} y a^{-2}(y) + z^4 \int_0^{x_5} a^{-2}(y) \int_0^{y} \int_0^{y'} y'' a^{-2}(y'') + \dots \\
S(x_5,z) &= z \int_0^{x_5} a^{-2}(y) - z^3 \int_0^{x_5} a^{-2}(y) \int_0^{y} \int_0^{y'} a^{-2}(y'') + \dots
\end{aligned} \tag{B.27}
$$

$$
\begin{aligned}
C'(x_5,z) &= -z^2 a^{-2}(x_5) \left(x_5 - z^2 \int_0^{x_5} \int_0^{y} y' a^{-2}(y') + \dots \right) \\
S'(x_5,z) &= z a^{-2}(x_5) \left(1 - z^2 \int_0^{x_5} \int_0^{y} a^{-2}(y') + \dots \right)
\end{aligned} \tag{B.28}
$$

It is also useful to approximate the warped trigs in the region where the $z^2/a^2 \gg 1$ close to $x_5 \sim L$. Close to the IR brane, where z^2/a^2 is sizable we can write the solutions as

$$
\begin{aligned}
C(x_5,z) &\approx a^{-1/2}(x_5) |\alpha| \cos(z \int_L^{x_5} a^{-1}(y) + \phi_\alpha) \\
S(x_5,z) &\approx -a^{-1/2}(x_5) |\beta| \cos(z \int_L^{x_5} a^{-1}(y) + \phi_\beta)
\end{aligned} \tag{B.29}
$$

We introduced four unknown coefficients α, β, ϕ_α, ϕ_β In general, they are constrained by the Wronskian to satisfy the relation $\beta\alpha \sin(\phi_\beta - \phi_\alpha) = 1$. Moreover, for highly warped metrics we ahve the approximate relation $C'(L) \approx -zLS'(L)$ that follows from the perturbation expansion (B.27), with $y \to L$ substitution in the integrals. Then we have one less unknown:

$$
\begin{aligned}
C(x_5,z) &\approx a^{-1/2}(x_5) |\alpha| \cos(z \int_L^{x_5} a^{-1}(y) + \phi) \\
S(x_5,z) &\approx -a^{-1/2}(x_5) |\alpha| zL \cos(z \int_L^{x_5} a^{-1}(y) + \phi + \delta)
\end{aligned} \tag{B.30}
$$

where the phase shift δ is small. From the Wronskian $\alpha^2 zL\delta \approx 1$.

Let us see the warped sines and cosines in some particular, solvable backgrounds.

For **flat space**, $a(x_5) = 1$, they become the familiar trigs

$$
C(x_5,z) = \cos(z x_5) \qquad S(x_5,z) = \sin(z x_5) \tag{B.31}
$$

For AdS$_5$ we insert $a(x_5) = e^{-kx_5}$ and almost as easily solve eq. (B.23) in terms of Bessel functions. The solution is $a^{-1} Z_1(m/ka)$ (recall also that $[a^{-1} Z_1(m/ka)]' = z a^{-2} Z_0(m/ka)$) and we pick up the following combinations

$$
\begin{aligned}
C(x_5,z) &= \frac{\pi z}{2k} a^{-1}(x_5) \left[Y_0\left(\frac{z}{k}\right) J_1\left(\frac{z}{ka(x_5)}\right) - J_0\left(\frac{z}{k}\right) Y_1\left(\frac{z}{ka(x_5)}\right) \right] \\
S(x_5,z) &= \frac{\pi z}{2k} a^{-1}(x_5) \left[J_1\left(\frac{z}{k}\right) Y_1\left(\frac{z}{ka(x_5)}\right) - Y_1\left(\frac{z}{k}\right) J_1\left(\frac{z}{ka(x_5)}\right) \right]
\end{aligned} \tag{B.32}
$$

For $za_L^{-1} \gg 1$,

$$
\begin{aligned}
C(x_5, z) &\approx \sqrt{\frac{\pi z}{2k}} \frac{1}{\sin\delta} a^{-1/2} \cos\left(\frac{z}{ka} - \frac{3\pi}{4} + \delta\right) \\
S(x_5, z) &\approx \sqrt{\frac{2k}{\pi z}} a^{-1/2} \sin\left(\frac{z}{ka} - \frac{\pi}{4}\right)
\end{aligned}
\tag{B.33}
$$

where $\cot\delta = (2/\pi)\log(ze^{\gamma}/2k)$ This is in agreement with eq. (B.29).

Another solvable background is that with a power law warp factor $a(x_5) = \left(1 - \frac{kx_5}{\gamma-1}\right)^{\gamma}$ For $\gamma \to \infty$ we recover the exponential warp factor of the RS model. The solutions to eq. (B.23) can be written, simalarly as in the AdS$_5$ case, in terms of the Bessel functions,

$$
\begin{aligned}
C(x_5, z) &= \frac{\pi z}{2k} a^{-1+\frac{1}{2\gamma}} \left[Y_{\frac{1}{2\gamma-2}}\left(\frac{z}{k}\right) J_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{z}{ka^{1-\frac{1}{\gamma}}}\right) - J_{\frac{1}{2\gamma-2}}\left(\frac{z}{k}\right) Y_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{z}{ka^{1-\frac{1}{\gamma}}}\right)\right] \\
S(x_5, z) &= \frac{\pi z}{2k} a^{-1+\frac{1}{2\gamma}} \left[J_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{z}{k}\right) Y_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{z}{ka^{1-\frac{1}{\gamma}}}\right) - Y_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{z}{k}\right) J_{\frac{2\gamma-1}{2\gamma-2}}\left(\frac{z}{ka^{1-\frac{1}{\gamma}}}\right)\right]
\end{aligned}
\tag{B.34}
$$

Appendix C Fermions, once and for all

a.f

We discuss the KK decomposition of fermion, bulk equations of motion and its formal solutions. The quadratic fermionic lagrangian in the warped background has the following form

$$
\mathcal{L} = ia^3\overline{\psi}\gamma_\mu\partial_\mu\psi - a^4\overline{\psi}\left(\gamma_5\partial_5 + \gamma_5 2\frac{a'}{a} + M\right)\psi \tag{C.35}
$$

The possible boundary mass term will be discussed later. The factor $2a'/a$ comes from the spin connection (or is required by hermiticity, if you wish).

The bulk equations have the following structure:

$$
\begin{aligned}
\left(\partial_5 + 2\frac{a'}{a} + M\right) f_R &= za^{-1} f_L \\
\left(-\partial_5 - 2\frac{a'}{a} + M\right) f_L &= za^{-1} f_R
\end{aligned}
\tag{C.36}
$$

We define hat profiles as

$$
f_R = a^{-2}e^{-Mx_5}\hat{f}_R \qquad f_L = a^{-2}e^{Mx_5}\hat{f}_L \tag{C.37}
$$

that satisfy simpler equations

$$
\begin{aligned}
\partial_5\hat{f}_R &= za^{-1}e^{2Mx_5}\hat{f}_L \\
\partial_5\hat{f}_L &= -za^{-1}e^{-2Mx_5}\hat{f}_R
\end{aligned}
\tag{C.38}
$$

Disentangling, the hatted profiles satisfy the second order differential equations

$$
\begin{aligned}
\left[\partial_5^2 + \left(\frac{a'}{a} + 2M\right)\partial_5 + \frac{z^2}{a^2}\right]\hat{f}_L &= 0 \\
\left[\partial_5^2 + \left(\frac{a'}{a} - 2M\right)\partial_5 + \frac{z^2}{a^2}\right]\hat{f}_R &= 0
\end{aligned}
\tag{C.39}
$$

We denote the two independent solutions of the *first* of these equations by $C_M(x_5, z)$, $S_M(x_5, z)$ (consequently, the *second* is solved by $C_{-M}(x_5, z)$, $S_{-M}(x_5, z)$). We define them such that they satisfy the following boundary conditions at $x_5 = 0$,

$$C_M(0, z) = 1 \qquad C'_M(0, z) = 0 \qquad S_M(0, z) = 0 \qquad S'_M(0, z) = z \tag{C.40}$$

Thus, we can write the general fermion profiles as

$$\begin{aligned} \hat{f}_L &= \alpha S_M(x_5, z) + \beta C_M(x_5, z) \\ \hat{f}_R &= -\alpha C_{-M}(x_5, z) + \beta S_{-M}(x_5, z) \end{aligned} \tag{C.41}$$

The integration constant in $\hat{f}_R$ has to much those in $\hat{f}_L$, which can be proven by use of eq. (C.38) and the boundary conditions. For completeness, the unhatted profiles

$$\begin{aligned} f_L &= a^{-2}(x_5) e^{M x_5} \left(\alpha S_M(x_5, z) + \beta C_M(x_5, z) \right) \\ f_R &= a^{-2}(x_5) e^{-M x_5} \left(-\alpha C_{-M}(x_5, z) + \beta S_{-M}(x_5, z) \right) \end{aligned} \tag{C.42}$$

In the case where f_R satisfies Dirichlet BC on the UV brane we have (below $\alpha \neq \alpha$)

$$\begin{aligned} f_L &= \alpha a^{-2}(x_5) e^{M x_5} C_M(x_5, z) \\ f_R &= \alpha a^{-2}(x_5) e^{-M x_5} S_{-M}(x_5, z) \end{aligned} \tag{C.43}$$

while in the case where f_L satisfies Dirichlet BC on the UV brane we have

$$\begin{aligned} f_L &= -\alpha a^{-2}(x_5) e^{M x_5} S_M(x_5, z) \\ f_R &= \alpha a^{-2}(x_5) e^{-M x_5} C_{-M}(x_5, z) \end{aligned} \tag{C.44}$$

Comments:

- The funny notation I introduced serves to stress the analogy to cosines and sines. We will call C_M the warped (fermionic) cosine and S_M the warped (fermionic) sine. There is a relationship between the warped sine and cosine:

  $$S'_M(x_5, z) = z a^{-1}(x_5) e^{-2M x_5} C_{-M}(x_5, z) \qquad C'_M(x_5, z) = -z a^{-1}(x_5) e^{-2M x_5} S_{-M}(x_5, z) \tag{C.45}$$

  e.wscd

  which vaguely resemble something.... For $a(x_5) = 1$, and $M = 0$ we indeed recover the familiar trig functions: $C_M = \cos(x_5 z)$, $S_M = \sin(x_5 z)$

- There exists a generalization of $\cos^2 + \sin^2 = 1$ to warped trigs. It is called the Wronskian formula:

  $$S'_M(x_5, z) C_M(x_5, z) - C'_M(x_5, z) S_M(x_5, z) = z e^{-2M x_5} a^{-1}(x_5) \tag{C.46}$$

  Using eq. (C.45) we can also derive the Crowian formula

  $$C_M(x_5, z) C_{-M}(x_5, z) + S_M(x_5, z) S_{-M}(x_5, z) = 1 \tag{C.47}$$

  which clearly resembles the familiar primary school formula...

- For the fermions with left-handed zero modes we flip the sign of the bulk mass term. Then in all the above formulae we have to flip $M \to -M$. This trick, apart from introducing some mess, also has some advantage. The zero mode profile is always $f \sim a^{-2} e^{-M x_5}$ for both left- and right-handed fermions. In particular, in AdS$_5$ the flat profile always correspond to $c = 1/2$.

For small z we can integrate eq. (C.39) perturbatively. This yields the small mass asymptotic formulae:

$$\begin{aligned} C_M(x_5, z) &= 1 - z^2 \int_0^{x_5} e^{-2My} a^{-1}(y) \int_0^y e^{2My'} a^{-1}(y') + \mathcal{O}(z^4) \\ S_M(x_5, z) &= z \int_0^{x_5} a^{-1}(y) e^{-2My} \left[1 - z^2 \int_0^y a^{-1}(y') e^{2My'} \int_0^{y'} a^{-1}(y'') e^{-2My''} + \mathcal{O}(z^4) \right] \end{aligned} \tag{C.48}$$

Now, we consider the boundary conditions for the profiles and how they depend on the presence of boundary terms.

- No boundary terms.
  In this basic case one of the chiralities has to vanish at the boundary.

$$f_R(0) = 0 \quad \text{xor} \quad f_L(0) = 0 \tag{C.49}$$

  The profiles satisfying the UV conditions are given by

$$\begin{aligned} f_{L,n} &= \alpha_n a^{-2}(x_5) e^{M x_5} C_M(x_5, m_n) & \text{xor} \qquad f_{L,n} &= -\alpha_n a^{-2}(x_5) e^{M x_5} S_M(x_5, m_n) \\ f_{R,n} &= \alpha_n a^{-2}(x_5) e^{-M x_5} S_{-M}(x_5, m_n) & f_{R,n} &= \alpha_n a^{-2}(x_5) e^{-M x_5} C_{-M}(x_5, m_n) \end{aligned} \tag{C.50}$$

  The IR boundary conditions

$$f_R(L) = 0 \quad \text{xor} \quad f_L(L) = 0 \tag{C.51}$$

  determine allowed values of m_n. The constants α_n are fixed by the normalization conditions

$$\int_0^L a^3 |f_{L,n}|^2 = 1 = \int_0^L a^3 |f_{R,n}|^2 \tag{C.52}$$

  Thanks to the equations of motion connecting f_L and f_R both left and right profile normalization yield the same value of α.

- Brane kinetic terms.
  We consider a brane kinetic term for the chirality that would be non-vanishing in the absence thereof (can i say it more simply?)

$$\mathcal{L}_{\rm UV} = i l_0 a^3 \delta(0) \overline{\psi}_L \gamma^\mu \partial_\mu \psi_L \quad \text{xor} \quad \mathcal{L}\text{UV} = i l_0 a^3 \delta(0) \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R \tag{C.53}$$

$$\mathcal{L}_{\rm UV} = i l_L a^3 \delta(L) \overline{\psi}_L \gamma^\mu \partial_\mu \psi_L \quad \text{xor} \quad \mathcal{L}\text{UV} = i l_L a^3 \delta(L) \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R \tag{C.54}$$

The UV boundary conditions now read

$$f_R(0) = m_n l_0 a_0^{-1} f_L(0) \quad \text{xor} \quad f_L(0) = -m_n l_0 a_0^{-1} f_R(0) \tag{C.55}$$

The profiles

$$\begin{aligned} f_{L,n} &= \alpha_n a^{-2}(x_5) e^{Mx_5} \left(C_M(x_5, m_n) - m_n l_0 S_M(x_5, m_n) \right) \\ f_{R,n} &= \alpha_n a^{-2}(x_5) e^{-Mx_5} \left(S_{-M}(x_5, m_n) + m_n l_0 C_{-M}(x_5, m_n) \right) \end{aligned} \tag{C.56}$$

xor

$$\begin{aligned} f_{L,n} &= -\alpha_n a^{-2}(x_5) e^{Mx_5} \left(S_M(x_5, m_n) + m_n l_0 C_M(x_5, m_n) \right) \\ f_{R,n} &= \alpha_n a^{-2}(x_5) e^{-Mx_5} \left(C_{-M}(x_5, m_n) - m_n l_0 S_{-M}(x_5, m_n) \right) \end{aligned} \tag{C.57}$$

The IR boundary conditions

$$f_{R,n}(L) = -m_n l_L a_L^{-1} f_{L,n}(L) \quad \text{xor} \quad f_{L,n}(L) = m_n l_L a_L^{-1} f_{R,n}(L) \tag{C.58}$$

determine allowed values of m_n. The constants α_n are fixed by the normalization conditions

$$\int_0^L a^3 \left(1 + l_0 \delta(0) + l_L \delta(0)\right) |f_{L,n}|^2 = 1 = \int_0^L a^3 |f_{R,n}|^2 \tag{C.59}$$

xor the 3 permutations thereof.

- Mass mixing with a UV boundary fermion

Consider now a UV boundary fermion $\tilde{\psi}_R$ mixing via a Dirac mass term with a bulk fermion

$$\mathcal{L}_{\rm UV} = i a^3 \delta(0) \overline{\tilde{\psi}_R} \gamma^\mu \partial_\mu \tilde{\psi}_R + -(a^4 \delta(0) \sqrt{\tilde{M}} \overline{\psi_L} \psi_R + \text{h.c.}) \tag{C.60}$$

The new fermion also has KK expansion $\tilde{\psi}_R(x) = \tilde{f}_{R,n} \psi_R(x)$ but its profiles are not profiles but a collection of numbers. Diagonalization in the KK basis fixes $\tilde{f}_{R,n} = a_0 \sqrt{\tilde{M}}/m_n f_{L,n}(0)$. The UV boundary condition for the bulk fermion happens to be

$$f_{R,n}(0) = -a_0 \frac{\tilde{M}}{m_n} f_{L,n}(0) \tag{C.61}$$

- Mass mixing via a UV boundary fermion

Consider now a UV boundary fermion $\tilde{\psi}_R$ mixing via a Dirac mass term with two bulk fermions

$$\mathcal{L}_{\rm UV} = i a^3 \delta(0) \overline{\tilde{\psi}_R} \gamma^\mu \partial_\mu \tilde{\psi}_R + -a^4 \delta(0) (\sqrt{\tilde{M}_1} \overline{\psi_L^1} \psi_R + \sqrt{\tilde{M}_2} \overline{\psi_L^2} \psi_R + \text{h.c.}) \tag{C.62}$$

Diagonalization in the KK basis fixes the brane fermion "profile",

$$\tilde{f}_{R,n} = a_0 \sqrt{\tilde{M}_1}/m_n f_{L,n}^1(0) + a_0 \sqrt{\tilde{M}_2}/m_n f_{L,n}^2(0) \tag{C.63}$$

The UV boundary condition for the bulk fermion happens to be

$$
\begin{aligned}
f^1_{R,n}(0) &= -a_0\frac{\tilde{M}_1}{m_n}f^1_{L,n}(0) - a_0\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}f^2_{L,n}(0) \\
f^2_{R,n}(0) &= -a_0\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}f^1_{L,n}(0) - a_0\frac{\tilde{M}_2}{m_n}f^2_{L,n}(0)
\end{aligned}
\tag{C.64}
$$

The bulk fermion profiles can be written as

$$
\begin{aligned}
f^1_{L,n} &= a^{-2}(x_5)e^{M_1x_5}\left(\alpha^1_n[C_{M_1}(x_5,m_n) + \frac{\tilde{M}_1}{m_n}S_{M_1}(x_5,m_n)] + \alpha^2_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}S_{M_1}(x_5,m_n)\right) \\
f^1_{R,n} &= a^{-2}(x_5)e^{-M_1x_5}\left(\alpha^1_n[S_{-M_1}(x_5,m_n) - \frac{\tilde{M}_1}{m_n}C_{-M_1}(x_5,m_n)] - \alpha^2_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}C_{-M_1}(x_5,m_n)\right) \\
f^2_{L,n} &= a^{-2}(x_5)e^{M_2x_5}\left(\alpha^1_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}S_{M_2}(x_5,m_n) + \alpha^2_n[C_{M_2}(x_5,m_n) + \frac{\tilde{M}_2}{m_n}S_{M_2}(x_5,m_n)]\right) \\
f^2_{R,n} &= a^{-2}(x_5)e^{-M_2x_5}\left(-\alpha^1_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}C_{-M_2}(x_5,m_n) + \alpha^2_n[S_{-M_2}(x_5,m_n) - \frac{\tilde{M}_2}{m_n}C_{-M_2}(x_5,m_n)]\right)
\end{aligned}
\tag{C.65}
$$

- Boundary Dirac mass terms.
  Now, consider IR brane localized Dirac mass terms

$$
\mathcal{L} = -a^4\delta(L)\tilde{m}\overline{\psi}_L\chi_R + \text{h.c.} \quad \text{xor} \quad \mathcal{L} = -a^4\delta(L)\tilde{m}\overline{\psi}_R\chi_L + \text{h.c.}
\tag{C.66}
$$

where χ is some other bulk fermion. The equation of motion are modfied as

$$
\begin{aligned}
\left(\partial_5 + 2\frac{a'}{a} + M\right)f^\psi_R &= za^{-1}f^\psi_L - \tilde{m}\delta(L)f^\chi_R \\
\left(-\partial_5 - 2\frac{a'}{a} + M\right)f^\psi_L &= za^{-1}f^\psi_R - \tilde{m}\delta(L)f^\chi_R
\end{aligned}
\tag{C.67}
$$

This results in IR boundary conditions

$$
f^\psi_R(L) = \tilde{m}f^\chi_R(L) \quad \text{xor} \quad f^\psi_L(L) = -\tilde{m}f^\chi_L(L)
\tag{C.68}
$$

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