Note on Precision Observables in Warped Models

(last compiled February 22, 2007)

1 Precision observables

We collect facts and formalism to parametrize new physics beyond the Standard Model (SM) in a model-independent way.

s.d6

1.1 Dimension Six Six Six

We write the low energy theory as

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \mathcal{L}_{D6} \tag{1.1}$$

where $\mathcal{L}$SM is the SM lagrangian with the gauge couplings g, g_Y (numerically they are not the PDG couplings, because dimension six will modify the observable-couplings relations).

$$\begin{aligned} \mathcal{L}_{\rm SM} = -\tfrac{1}{2}{\rm Tr}\{L_{\mu\nu}L_{\mu\nu}\} - \tfrac{1}{4}B_{\mu\nu}B_{\mu\nu} + i\textstyle\sum_j \overline{\psi_j}\gamma_\mu D_\mu \psi_j + |D_\mu H|^2 - V(H) + \text{Yukawa} \\ D_\mu\psi_j = (\partial_\mu - ig_L L_\mu^a T^a - ig_Y Y_j B_\mu)\psi_j \\ D_\mu H = (\partial_\mu - ig_L L_\mu^a T^a - ig_Y \tfrac{1}{2}B_\mu)H \end{aligned} \tag{1.2}$$

($T^a = \sigma^a/2$) and $\mathcal{L}_{D6}$ are the higher-dimensional operators

$$\begin{aligned} \mathcal{L}_{D6} = \alpha_T |H^\dagger D_\mu H|^2 + \alpha_S (H^\dagger T^a H) W^a_{\mu\nu} B_{\mu\nu} \\ - \left\{ i\beta_j g_L^2 (\overline{\psi}_j \gamma_\mu t^a \psi_j)(D_\mu H^\dagger t^a H) + i\tfrac{\gamma_j}{2} Y_j g_Y^2 \overline{\psi_j}\gamma_\mu\psi_j (D_\mu H^\dagger H) + \text{h.c.} \right\} \\ + \text{fermion}^4 \end{aligned} \tag{1.3}$$

We assume that the coefficients of in $\mathcal{L}_{D6}$ are much smaller than $1/v^2$.

After inserting the Higgs vev $H = (0, v/\sqrt{2})^T$, the SM part yields, in particular, neutral and charged current interactions

$$\mathcal{L}_{\rm SM} \to \frac{g_L}{\sqrt{2}} \overline{\psi}_j \gamma_\mu T^\pm \psi_j W^\pm_\mu + \frac{g_L^2 T^3 - g_Y^2 Y_j}{\sqrt{g_L^2 + g_Y^2}} \overline{\psi}_j \gamma_\mu \psi_j Z_\mu \tag{1.4}$$

and gauge boson masses:

$$\mathcal{L}_{\rm SM} \to \frac{g_L^2 v^2}{4} W^+_\mu W^-_\mu + \frac{(g_L^2 + g_Y^2)v^2}{4}\frac{1}{2} Z_\mu^2 \tag{1.5}$$

The dimensionsixes contribute additional interaction terms:

$$\mathcal{L}_{D6} \to \frac{g_L}{\sqrt{2}} m_W^2 \beta_j \overline{\psi}_j \gamma_\mu T^\pm \psi_j W^\pm_\mu + \frac{1}{\sqrt{g_L^2 + g_Y^2}} m_Z^2 (\beta_j g_L^2 T^3 - \gamma_j g_Y^2 Y_j) \overline{\psi}_j \gamma_\mu \psi_j Z_\mu \tag{1.6}$$

They also contribute to the Z boson mass, so that the masses now become

$$m_W^2 = \frac{g_L^2 v^2}{4} \qquad m_Z^2 = \frac{(g_L^2 + g_Y^2) v^2}{4} \left(1 + \frac{v^2 \alpha_T}{2} \right) \tag{1.7}$$

Finally, α_S yields kinetic mixing between W_μ^3 and B_μ:

$$\mathcal{L}_{D6} \to -\frac{1}{4} \alpha_S v^2 W_{\mu\nu}^3 B_{\mu\nu} \tag{1.8}$$

1.2 Equivalence of S and universal vertex corrections

s.vto

We show that for S is equivalent to universal vertex corrections.

Consider the SM supplemented by the S dimensions six operator:

$$\mathcal{L}_{D6} = -\alpha_S (H^\dagger T^a H) W_{\mu\nu}^a B_{\mu\nu} \tag{1.9}$$

For $\alpha_S \neq 0$ the gauge boson kinetic terms include a term mixing W_μ^3 and B_μ,

$$\mathcal{L}_{\rm kin} = -\frac{1}{2} |W_{\mu\nu}^+| - \frac{1}{4} W_{\mu\nu}^3 W_{\mu\nu}^3 - \frac{1}{4} B_{\mu\nu} B_{\mu\nu} - \frac{1}{4} \alpha_S v^2 W_{\mu\nu}^3 B_{\mu\nu} \tag{1.10}$$

We can remove the ugly mixing term by redefining:

$$\begin{aligned} W_\mu^3 &\to \cos\theta_1 W_\mu^3 + \sin\theta_2 B_\mu \\ B_\mu &\to \cos\theta_2 B_\mu + \sin\theta_1 W_\mu^3 \end{aligned} \tag{1.11}$$

where the rotation angles should satisfy

$$\sin(\theta_1 + \theta_2) \approx \theta_1 + \theta_2 = -\frac{1}{2} \alpha_S v^2 \tag{1.12}$$

However, due to this redefinition α_S will pop out in thousand different places. First, the relation between W_μ^3, B_μ and the mass eigenstates Z_μ, A_μ gets modified:

$$\begin{aligned} W_\mu^3 &= \frac{(g_L - g_Y \theta_1) Z_\mu + (g_Y - g_L \theta_2) A_\mu}{g_L^2 + g_Y^2 - 2 g_L g_Y (\theta_1 + \theta_2)} \\ B_\mu &= \frac{(g_L - g_Y \theta_1) A_\mu - (g_Y - g_L \theta_2) Z_\mu}{g_L^2 + g_Y^2 - 2 g_L g_Y (\theta_1 + \theta_2)} \end{aligned} \tag{1.13}$$

The gauge boson masses are

$$m_W^2 = \frac{g_L^2 v^2}{4} \qquad m_Z^2 = \frac{(g_L^2 + g_Y^2) v^2}{4} \left(1 + \frac{g_L \, g_Y}{g_L^2 + g_Y^2} \alpha_S v^2 \right) \tag{1.14}$$

e.sim

The interactions with fermions become:

$$\begin{aligned} \mathcal{L} = \tfrac{g_L}{\sqrt{2}} \overline{\psi} \gamma_\mu T^\pm \psi W_\mu^\pm + \tfrac{g_L g_Y}{\sqrt{g_L^2 + g_Y^2}} (T^3 + Y) \left(1 - \tfrac{g_L \, g_Y}{g_L^2 + g_Y^2} \tfrac{\alpha_S v^2}{2} \right) \overline{\psi} \gamma_\mu \psi A_\mu \\ \left[\tfrac{g_L^2}{\sqrt{g_L^2 + g_Y^2}} T^3 \left(1 + \tfrac{g_Y^3}{g_L (g_L^2 + g_Y^2)} \tfrac{\alpha_S v^2}{2} \right) - \tfrac{g_Y^2}{\sqrt{g_L^2 + g_Y^2}} Y_j \left(1 + \tfrac{g_L^3}{g_Y (g_L^2 + g_Y^2)} \tfrac{\alpha_S v^2}{2} \right) \right] \overline{\psi} \gamma_\mu \psi Z_\mu \end{aligned} \tag{1.15}$$

We now rescale the couplings to put the interactions into the form compatible with our dimension six parametrization,

$$
\begin{aligned}
g_L &\to g_L(1 + m_W^2 \beta) \\
g_Y &\to g_Y(1 - m_W^2 \frac{g_Y^2}{g_L^2}\beta + \frac{g_Y}{2g_L} v^2 \alpha_S) \\
v &\to v(1 - m_W^2 \beta)
\end{aligned} \tag{1.16}
$$

The gauge interactions and masses turn into

$$
\begin{aligned}
\mathcal{L} = \tfrac{g_L}{\sqrt{2}}(1 + m_W^2\beta)(\overline{\psi}\gamma_\mu T^{\pm}\psi W_\mu^{\pm} + \tfrac{g_L g_Y}{\sqrt{g_L^2+g_Y^2}}(T^3 + Y)\overline{\psi}\gamma_\mu\psi A_\mu \\
\left[\tfrac{g_L^2}{\sqrt{g_L^2+g_Y^2}} T^3 \left(1 + m_Z^2\beta\right) - \tfrac{g_Y^2}{\sqrt{g_L^2+g_Y^2}} Y_j \left(1 + m_Z^2 (\tfrac{2}{g_L g_Y}\alpha_S - \beta\right)\right] \overline{\psi}\gamma_\mu\psi Z_\mu
\end{aligned} \tag{1.17}
$$

$$
m_W^2 = \frac{g_L^2 v^2}{4} \qquad m_Z^2 = \frac{(g_L^2 + g_Y^2)v^2}{4}\left(1 + \frac{g_Y}{g_L}\alpha_S v^2 - \frac{g_Y^2}{2}\beta v^2\right) \tag{1.18}
$$

e.sim2

This is the SM+D6 lagrangian with $\alpha_S = 0$, $\beta_j = \beta$, $\gamma_j = \frac{2}{g_L g_Y}\alpha_S - \beta$, $\alpha_T = \frac{2g_Y}{g_L}\alpha_S - g_Y^2\beta$

Summarizing, we have shown that we are always allowed to shift the coefficient of the dimension six operators in the following way:

$$
\begin{aligned}
\alpha_S &\to \alpha_S + \Delta\alpha_S \\
\beta_j &\to \beta_j + \Delta\beta \\
\gamma_j &\to \gamma_j - \frac{2}{g_L g_Y}\Delta\alpha_S - \Delta\beta \\
\alpha_T &\to \alpha_T - \frac{2g_Y}{g_L}\Delta\alpha_S - g_Y^2\Delta\beta
\end{aligned} \tag{1.19}
$$

where $\Delta\alpha_S$ and $\Delta\beta$ are arbitrary (except that they are much smaller than $1/m_W^2$). This is better expressed by a universal shift of the vertex corrections

$$
\begin{aligned}
\beta_j &\to \beta_j + \Delta\beta \\
\gamma_j &\to \gamma_j + \Delta\gamma \\
\alpha_S &\to \alpha_S - \frac{g_L g_Y}{2}\left(\Delta\beta + \Delta\gamma\right) \\
\alpha_T &\to \alpha_T + g_Y^2\Delta\gamma
\end{aligned} \tag{1.20}
$$

One particular application of this result concerns the situation where the vertex corrections are universal: $\beta_j = \beta$ and $\gamma_j = \gamma$. Then, choosing $\Delta\beta = -\beta$, $\Delta\gamma = -\gamma$ we can set $\beta_j \to 0$, $\gamma_j \to 0$. In this case the beyond SM corrections are parametrized by the two oblique operators only. Those get shifted in the process by the amount

$$
\Delta\alpha_S = \frac{g_Y g_L}{2}\left(\beta + \gamma\right) \qquad \Delta\alpha_T = -g_Y^2\gamma \tag{1.21}
$$

e.vto

1.3 Various Parametrizations

Our parameters are simply related to the *styx* used in [1]

$$
\begin{aligned}
\alpha_S &= \frac{g_L g_Y}{16\pi^2 v^2} s \\
\alpha_T &= -\frac{1}{16\pi^2 v^2} t \\
\beta_j &= \frac{1}{16\pi^2 g_L^2 v^2} x_j \\
\gamma_j &= \frac{1}{16\pi^2 Y_H Y g_Y^2 v^2} y_j
\end{aligned}
\tag{1.22}
$$

In the oblique case, the familiar S and T can be read from

$$
S = \frac{8\pi v^2}{g_L g_Y}\alpha_S \qquad T = -\frac{2\pi v^2}{e^2}\alpha_T \tag{1.23}
$$

e.tst

Barbieri's epsilons:

$$
\begin{aligned}
\epsilon_1 &= -\frac{v^2}{2}\left(\alpha_T - g_Y^2\gamma\right) \\
\epsilon_3 &= \frac{v^2}{2}\frac{g_L}{g_Y}\left(\alpha_S + \frac{g_L g_Y}{2}(\beta+\gamma)\right)
\end{aligned}
\tag{1.24}
$$

In fact, what is certain above is the relation between ϵ's and α's. The dependence on β and γ should be such that ϵ are invariant under the vertex rescaling (1.20). I believie that's the idea of this parametrization. However, when i try to derive it from Barbieri's lectures i don't seem to get the signs right.

2 Warped Standard Model (3-2-1)

We consider a 5D $SU(3)_c \times SU(2)_W \times U(1)_Y$ gauge theory with the 5th dimension being a finite interval, $x_5 \in [0, L]$. The gravitational background is warped, with the line element

$$
ds^2 = a^2(x_5)\eta_{\mu\nu}dx^\mu dx^\nu - dx_5^2 \tag{2.1}
$$

e.wb

We don't specify the warp factor, and we will perform computations in a background independent way. The bulk theory contains the SM gauge fields coupled to Dirac fermions: SU(2) doublets denoted as D and $SU(2)$ singlets denoted as S. The Higgs field and Yukawa interactions are localized at the IR boundary.

The 5D action (we ignore the gluons henceforth)

$$
\begin{aligned}
S = \int d^4x \int_0^L dx_5 \sqrt{g}\,\big\{(&-\tfrac{1}{2}\mathrm{Tr}\{W_{MN}W_{MN}\} - \tfrac{1}{4}B_{MN}B_{MN} \\
&+\overline{D}(i\Gamma_N D_N + M_d)D + \overline{S}(i\Gamma_N D_N - M_s)S\big\} \\
\int d^4x dx_5 \sqrt{g_4}\,\delta(L)\,&\big(|D_\mu H|^2 - V(H) - \tilde{y}H\overline{D}_L S_R - \tilde{y}H^\dagger \overline{S}_R D_L\big)
\end{aligned}
\tag{2.2}
$$

where

$$\begin{aligned}
D_M D &= (\partial_M - i\tilde{g}_5 W_M - i\tilde{g}'_5 Y_d B_M) D \\
D_M S &= (\partial_M - i\tilde{g}'_5 Y_s B_M) S \\
D_\mu H &= (\partial_\mu - i\tilde{g}_5 W_\mu - i\tilde{g}'_5 Y_H B_\mu) H \\
W_M &= W_M^a \frac{\sigma^a}{2} \qquad W_{MN} = \partial_M W_N - \partial_N W_M - i[W_M, W_N] \\
\Gamma^\mu &= \gamma^\mu \qquad \Gamma^5 = i\gamma_5 \qquad \gamma_5 \psi_{L,R} = \mp \psi_{L,R} \\
D &= \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} D_{L,1} \\ D_{R,1} \\ D_{L,2} \\ D_{R,2} \end{pmatrix} \\
S &= \begin{pmatrix} S_L \\ S_R \end{pmatrix}
\end{aligned} \tag{2.3}$$

The doublet stands for $D = (L_i, Q_i)$, $i = 1 \ldots 3$, the singlets stands for $S = (u_i, d_i, e_i)$. The sign of the doublet's bulk mass has been flipped for convenience.

The boundary conditions in the absence of the Higgs vev read:

$$\partial_5 W_\mu^a|_{0,L} = \partial_5 B_\mu|_{0,L} = W_5^a|_{0,L} = B_5|_{0,L} = 0 \tag{2.4}$$ e.wsmbc

$$D_R|_{0,L} = S_L|_{0,L} = 0 \tag{2.5}$$

These boundary conditions signify that, in the absence of electroweak breaking W_μ^a, B_μ, S_R and D_L have a massless mode. To lowest order in m_W^2/M_{KK}^2 these can be indentified with the SM gauge and fermion fields.

2.1 KK expansion in the presence of electroweak breaking

The quadratic 5D action for the gauge field, in the background $\langle H \rangle = \tilde{v}$ can be written as

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}|\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+|^2 + a^2(x_5)|\partial_5 W_\mu^+|^2 \\
&-\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 + \frac{1}{2}a^2(x_5)(\partial_5 Z_\mu)^2 \\
&-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}a^2(x_5)(\partial_5 A_\mu)^2 \\
&+a^2\delta(L)\left(\frac{1}{2}L\tilde{m}_W^2|W_\mu^+|^2 + \frac{1}{2}L\tilde{m}_Z^2|Z_\mu|^2\right)
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2) \\
A_\mu &= \sin\tilde{\theta}_W W_\mu^3 + \cos\tilde{\theta}_W B_\mu \\
Z_\mu &= \cos\tilde{\theta}_W W_\mu^3 - \sin\tilde{\theta}_W B_\mu \\
\sin\tilde{\theta}_W &= \frac{\tilde{g}'_5}{\sqrt{\tilde{g}_5^2 + \tilde{g}_5'^2}} \qquad \cos\tilde{\theta}_W = \frac{\tilde{g}_5}{\sqrt{\tilde{g}_5^2 + \tilde{g}_5'^2}} \\
L\tilde{m}_W^2 &= \frac{\tilde{g}_5^2 \tilde{v}^2}{2} \qquad L\tilde{m}_Z^2 = \frac{(\tilde{g}_5^2 + \tilde{g}_5'^2)\tilde{v}^2}{2}
\end{aligned} \tag{2.7}$$

We rewrite the 5D gauge field in the KK basis:

$$\begin{aligned} W_\mu^\pm(x,x_5) &= \sum_n f_n^W(x_5) W_{\mu,n}^\pm(x) \\ Z_\mu(x,x_5) &= \sum_n f_n^Z(x_5) Z_{\mu,n}(x) \\ A_\mu(x,x_5) &= \sum_n f_n^\gamma(x_5) A_{\mu,n}(x) \end{aligned} \tag{2.8}$$

We wish to arrive at the quadratic action that is diagonal in n. To this end we choose the gauge KK profiles f_n to be solutions of the mode equation:

$$\partial_5(a^2\partial_5 f_n) + m_n^2 f_n = 0 \tag{2.9}$$ e.gfme

(the same equation for W, Z and γ). The profiles satisfy the UV boundary conditions

$$\partial_5 f_n(0) = 0 \tag{2.10}$$ e.wsmgpbc

(also the same equation for W, Z and γ). On the IR brane we impose

$$\partial_5 f_n^\gamma(L) = 0 \qquad \partial_5 f_n^W(L) = -\tilde{m}_W^2 f_n^W(L) \qquad \partial_5 f_n^Z(L) = -\tilde{m}_Z^2 f_n^Z(L) \tag{2.11}$$

These conditions are chosen such that after integrating ∂_5 by parts in eq. (2.6) the boundary integration terms cancel the boundary mass terms (this is needed to really diagonalize the action in n). Finally, there's normalization/orthogonality condition

$$\int_0^L f_n f_m = \delta_{nm} \tag{2.12}$$

Inserting the solutions back into the 5D action and integrating over the fifth dimension we get a quadratic 4D action that is diagonal in the KK basis:

$$\begin{aligned} \mathcal{L} = \textstyle\sum_n \big\{ &-\tfrac{1}{2}|\partial_\mu W_{\nu,n}^+ - \partial_\nu W_{\mu,n}^+|^2 + m_{W,n}^2 |W_{\mu,n}^+| \\ &-\tfrac{1}{4}(\partial_\mu Z_{\nu,n} - \partial_\nu Z_{\mu,n})^2 + \tfrac{1}{2} m_{Z,n}^2 (Z_{\mu,n})^2 \\ &-\tfrac{1}{4}(\partial_\mu A_{\nu,n} - \partial_\nu A_{\mu,n})^2 + \tfrac{1}{2} m_{\gamma,n}^2 (A_{\mu,n})^2 \big\} \end{aligned} \tag{2.13}$$

The zero (not massless, in general) modes are the SM gauge bosons: $W_{\mu,0}^\pm \equiv W_\mu^\pm$, $Z_{\mu,0} \equiv Z_\mu$, $A_{\mu,0} \equiv A_\mu$. The masses $m_{\gamma,0}^2 = 0$, $m_{W,0}^2 \equiv m_W^2$, $m_{Z,0}^2 \equiv m_Z^2$ are the physical gauge boson masses (while twiddle masses $\tilde{m}_W^2$, $\tilde{m}_Z^2$ are merely parameters of the 5D lagrangian).

Denote the two independent solutions of eq. (2.9) as $C(x_5, m_n)$ and $S(x_5, m_n)$ (of course they depend also on the warp factor). Furthermore, we can normalize the solutions such that

$$C(0,m_n) = 1 \qquad C'(0,m_n) = 0 \qquad S(0,m_n) = 0 \qquad S'(0,m_n) = m_n \tag{2.14}$$

(so that they generalize cosines and sines). Then we can formally write down the solutions to the gauge profiles equation as

$$\begin{aligned} f_n^\pm(x_5) &= c_\pm C(x_5, m_{W,n}) \\ f_n^Z(x_5) &= c_Z C(x_5, m_{Z,n}) \\ f_n^\gamma(x_5) &= c_\gamma C(x_5, m_{\gamma,n}) \end{aligned} \tag{2.15}$$

The constant are fixed by the normalization condtion, while the mass towers are obtained by solving the quantization condition (the IR boundary condition)

$$
\begin{aligned}
C'(L, m_{\gamma,n}) &= 0 \\
C'(L, m_{W,n}) + L\tilde{m}_W^2 C(L, m_{W,n}) &= 0 \\
C'(L, m_{Z,n}) + L\tilde{m}_Z^2 C(L, m_{Z,n}) &= 0
\end{aligned}
\qquad (2.16)
$$

Let us concentrate on the zero modes. The photon zero mode is easy:

$$
f_\gamma(x_5) = \frac{1}{\sqrt{L}} \qquad m_\gamma = 0 \qquad (2.17)
$$

e.zmpp

In order to determine the W and Z profiles we need to know the small mass asymptotics of the solution $C(x_5, z)$. For $z \ll M_{KK} \sim \pi \int_0^L a^{-1}$ we can approximate the solutions to eq. (2.9) as

$$
\begin{aligned}
C(x_5, z) &= 1 - z^2 \int_0^{x_5} (y a^{-2}(y)) + z^4 \int_0^{x_5} a^{-2}(y) \int_0^y \int_0^{y'} (y'' a^{-2}(y'')) + \mathcal{O}(z^6) \\
S(x_5, z) &= z \left\{ \int_0^{x_5} a^{-2}(y) - z^2 \int_0^{x_5} a^{-2}(y) \int^y \int^{y'} a^{-2}(y'') + \mathcal{O}(z^4) \right\}
\end{aligned}
\qquad (2.18)
$$

(the S asymptotics is not needed, but included here just for completeness). Inserting this into the quantization condition we get, to order m_W^4/M_{KK}^2:

$$
\begin{aligned}
m_W^2 L a_L^{-2} - m_W^4 a_L^{-2} \textstyle\int_0^L \int_0^y y' a^{-2}(y') &= L\tilde{m}_W^2 [1 - m_W^2 \textstyle\int_0^L y a^{-2}(y)] \\
m_Z^2 L a_L^{-2} - m_Z^4 a_L^{-2} \textstyle\int_0^L \int_0^y y' a^{-2}(y') &= L\tilde{m}_Z^2 [1 - m_Z^2 \textstyle\int_0^L y a^{-2}(y)]
\end{aligned}
\qquad (2.19)
$$

or

$$
\begin{aligned}
m_W^2 &= a_L^2 \tilde{m}_W^2 \left[1 + a_L^2 \tilde{m}_W^2 \left(L^{-1} \int_0^L \int_0^y y' a^{-2}(y') - \int_0^L y a^{-2}(y) \right) \right] \\
m_Z^2 &= a_L^2 \tilde{m}_Z^2 \left[1 + a_L^2 \tilde{m}_Z^2 \left(L^{-1} \int_0^L \int_0^y y' a^{-2}(y') - \int_0^L y a^{-2}(y) \right) \right]
\end{aligned}
\qquad (2.20)
$$

The true gauge boson masses m differ from the boundary masses $a_L \tilde{m}$ by the factor of order $(a_L \tilde{m})^2/M_{KK}^2$ (the factor a_L appears together with $\tilde{m}_W$ because we used a vev of a non-canonically normalized Higgs in the definition of the boundary masses).

The normalized gauge boson profiles, to order m^2/M_{KK}^2 can be written as:

$$
\begin{aligned}
f^W(x_5) &= \frac{1}{\sqrt{L}} \left(1 + m_W^2 L^{-1} \int_0^L \int_0^y y' a^{-2}(y') - m_W^2 \int_0^{x_5} y' a^{-2}(y') + \mathcal{O}(m_W^4/M_{KK}^4) \right) \\
f^Z(x_5) &= \frac{1}{\sqrt{L}} \left(1 + m_Z^2 L^{-1} \int_0^L \int_0^y y' a^{-2}(y') - m_Z^2 \int_0^{x_5} y' a^{-2}(y') + \mathcal{O}(m_Z^4/M_{KK}^4) \right)
\end{aligned}
\qquad (2.21)
$$

Unlike the photon profile, the massive gauge boson profiles are not flat; instead they are tilted $\mathcal{O}(m_W^2/M_{KK}^2)$ by terms. Note that the tilt is orthogonal to the zeroth order constant profile:

$$
\int_0^L dx_5 \, 1 \cdot \left(m_W^2 L^{-1} \int_0^L \int_0^y y' a^{-2}(y') - m_W^2 \int_0^{x_5} y' a^{-2}(y') \right) = 0 \qquad (2.22)
$$

This is the usual story that the first-order perturbation is orthogonal to the zeroth order function.

Now we play the same game with the fermions. The quadratic fermionic lagrangian reads

$$\begin{aligned}\mathcal{L} = ia^3\overline{D}\gamma_\mu\partial_\mu D + ia^3\overline{S}\gamma_\mu\partial_\mu S \\ -a^4\overline{D}\left(\gamma_5\partial_5 + \gamma_5 2\tfrac{a'}{a} - M_d\right)D - a^4\overline{S}\left(\gamma_5\partial_5 + \gamma_5 2\tfrac{a'}{a} + M_S\right)S \\ -\delta(L)a^4\tilde{m}(\overline{D}_L S_R + \overline{S}_R D_L)\end{aligned} \tag{2.23}$$

The factor $2a'/a$ comes from the spin connection (or is required by hermiticity, if you wish). The boundary mass is $\tilde{m} = \tilde{y}\tilde{v}$ (unnormalized vev, again). We rewrite the fermion fields in the KK basis:

$$\begin{aligned}D_L(x, x_5) &= \sum_n f^D_{L,n}(x_5)\psi_{L,n}(x) \\ D_R(x, x_5) &= \sum_n f^D_{R,n}(x_5)\psi_{R,n}(x) \\ S_L(x, x_5) &= \sum_n f^S_{L,n}(x_5)\psi_{L,n}(x) \\ S_R(x, x_5) &= \sum_n f^S_{R,n}(x_5)\psi_{R,n}(x)\end{aligned} \tag{2.24}$$

In order to digonalize the quadratic fermionic action we need the following. The KK profiles should satisfy the equations of motion

$$\begin{aligned}(\partial_5 + 2\frac{a'}{a} - M_d)f^D_{R,n} &= m_n a^{-1} f^D_{L,n} \\ (-\partial_5 - 2\frac{a'}{a} - M_d)f^D_{L,n} &= m_n a^{-1} f^D_{R,n} \\ (\partial_5 + 2\frac{a'}{a} + M_s)f^S_{R,n} &= m_n a^{-1} f^S_{L,n} \\ (-\partial_5 - 2\frac{a'}{a} + M_s)f^S_{L,n} &= m_n a^{-1} f^S_{R,n}\end{aligned} \tag{2.25}$$

the UV boundary conditions

$$f^D_{R,n}(0) = f^S_{R,n}(0) = 0\,, \tag{2.26}$$

the IR boundary conditions

$$f^D_{R,n}(L) = \tilde{m} f^S_{R,n}(L) \qquad f^S_{L,n}(L) = -\tilde{m} f^D_{L,n}(L) \tag{2.27}$$

and the normalization condition

$$\int_0^L a^3(|f^D_{L,n}|^2 + |f^S_{L,n}|^2) = 1 \qquad \int_0^L a^3(|f^D_{R,n}|^2 + |f^S_{R,n}|^2) = 1 \tag{2.28}$$

e.fn

(in fact the second follows from the first and the equations of motion).

Having fulfilled all this the quadratic action for fermions takes the diagonal form

$$\sum_n \left\{ i\overline{\psi}_{L,n}\gamma_\mu\partial_\mu\psi_{L,n} + i\overline{\psi}_{R,n}\gamma_\mu\partial_\mu\psi_{R,n} - m_n\overline{\psi}_{L,n}\psi_{R,n} - m_n\overline{\psi}_{R,n}\psi_{L,n} \right\} \tag{2.29}$$

The solution to the equations of motion can be formally written as:

$$\begin{aligned} f^D_{L,n} &= c_{d,n}a^{-2}e^{-M_d x_5}C_{-M_d}(x_5, m_n) \\ f^D_{R,n} &= c_{d,n}a^{-2}e^{M_d x_5}S_{M_d}(x_5, m_n) \\ f^S_{L,n} &= -c_{s,n}a^{-2}e^{M_s x_5}S_{M_s}(x_5, m_n) \\ f^S_{R,n} &= c_{s,n}a^{-2}e^{-M_s x_5}C_{-M_s}(x_5, m_n) \end{aligned} \tag{2.30}$$

where we define the $C_M(x_5, z)$ and $S_M(x_5, z)$ as the two independent solutions of the differential equation:

$$\left[\partial_5^2 + \left(\frac{a'}{a} + 2M\right)\partial_5 + \frac{z^2}{a^2}\right] f = 0 \tag{2.31}$$

satisfying the UV boundary conditions

$$C_M(0, m_n) = 1 \qquad C'_M(0, m_n) = 0 \qquad S_M(0, m_n) = 0 \qquad S'_M(0, m_n) = m_n \tag{2.32}$$

With this choice the fermionic profiles obviously satisfy the UV boundary conditions. The two IR boundary conditions determine the quantization condition:

$$S_{M_d}(L, m_n)S_{M_s}(L, m_n) = \tilde{m}^2 C_{-M_d}(L, m_n)C_{-M_s}(L, m_n) \tag{2.33}$$

and also relate two normalization constants:

$$\frac{c_{s,n}}{c_{d,n}} = \sqrt{\frac{C_{-M_d}(L, m_n)S_{M_d}(L, m_n)}{C_{-M_s}(L, m_n)S_{M_s}(L, m_n)}} \tag{2.34}$$

Let us concentrate on the zero mode. To this end we need the small z asymptotics of $C(x_5, z)$.

$$\begin{aligned} C_M(x_5, z) &= 1 - z^2 \int_0^{x_5} e^{-2My}a^{-1}(y)\int_0^y e^{2My'}a^{-1}(y') + \mathcal{O}(z^4) \\ S_M(x_5, z) &= z\left\{\int_0^{x_5} a^{-1}(y)e^{-2My} - z^2\int_0^{x_5} a^{-1}(y)e^{-2My}\int_0^y a^{-1}(y')e^{2My'}\int_0^{y'} a^{-1}(y'')e^{-2My''} + \mathcal{O}(z^4)\right\} \end{aligned} \tag{2.35}$$

Using this we easily find the zero mode mass:

$$m = y a_L \tilde{v} \qquad y = \tilde{y}\sqrt{\frac{a_L^{-1}e^{-2M_d L}}{\int_0^L a^{-1}e^{-2M_d y}}}\sqrt{\frac{a_L^{-1}e^{-2M_s L}}{\int_0^L a^{-1}e^{-2M_s y}}} \tag{2.36}$$

The normalized profiles are slightly more complicated this time:

$$f_L^D(x_5) = \frac{a^{-2}e^{-M_d x_5}}{(\int_0^L a^{-1}e^{-2M_d y})^{1/2}}\left[1 - m^2\int_0^{x_5} a^{-1}(y)e^{2M_d y}\int_0^y a^{-1}(y')e^{-2M_d y'}\right.$$
$$\left.+m^2\frac{\int_0^L a^{-1}e^{-2M_d y}\int_0^y a^{-1}(y')e^{2M_d y'}\int_0^{y'} a^{-1}(y'')e^{-2M_d y''}}{\int_0^L a^{-1}e^{-2M_d y}} - \frac{m^2}{2}\frac{\int_0^L a^{-1}e^{2M_s y}(\int_0^y a^{-1}e^{-2M_s y'})^2}{\int_0^L a^{-1}e^{-2M_s y}} + \mathcal{O}(m^4)\right]$$
$$f_L^S(x_5) = m\frac{a^{-2}e^{M_s x_5}}{(\int_0^L a^{-1}e^{-2M_s y})^{1/2}}\int_0^{x_5} a^{-1}e^{-2M_s y} + \mathcal{O}(m^3)$$
$$f_R^S(x_5) = -\frac{a^{-2}e^{-M_s x_5}}{(\int_0^L a^{-1}e^{-2M_s y})^{1/2}}\left[1 - m^2\int_0^{x_5} a^{-1}(y)e^{2M_s y}\int_0^y a^{-1}(y')e^{-2M_s y'}\right.$$
$$\left.+m^2\frac{\int_0^L a^{-1}e^{-2M_s y}\int_0^y a^{-1}(y')e^{2M_s y'}\int_0^{y'} a^{-1}(y'')e^{-2M_s y''}}{\int_0^L a^{-1}e^{-2M_s y}} - \frac{m^2}{2}\frac{\int_0^L a^{-1}e^{2M_d y}(\int_0^y a^{-1}e^{-2M_d y'})^2}{\int_0^L a^{-1}e^{-2M_d y}} + \mathcal{O}(m^4)\right]$$
$$f_R^D = m\frac{a^{-2}e^{M_d x_5}}{(\int_0^L a^{-1}e^{-2M_d y})^{1/2}}\int_0^{x_5} a^{-1}e^{-2M_d y} + \mathcal{O}(m^3) \tag{2.37}$$

Looks nasty but note that the tilt is non-neglible only for 3rd generation fermions.

2.2 Zero mode gauge interactions

We move to study the interactions between the light (means zero mode) fermions and the light gauge bosons. The 5D interaction terms are given by:

$$\mathcal{L} = a^3\left\{\frac{\tilde{g}_5\tilde{g}_5'}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}Q(\overline{D}\gamma_\mu D + \overline{S}\gamma_\mu S)A_\mu\right.$$
$$+\frac{\tilde{g}_5}{\sqrt{2}}\overline{D}_1\gamma_\mu D_2 W_\mu^+ + \text{h.c.}$$
$$\left.+\frac{\tilde{g}_5^2 T_3 - \tilde{g}_5'^2 Y_d}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}\overline{D}\gamma_\mu D Z_\mu - \frac{\tilde{g}_5'^2 Y_s}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}\overline{S}\gamma_\mu S Z_\mu\right\} \tag{2.38}$$

The zero mode interaction lagrangian can be found by inserting

$$\begin{array}{rcl}
D_L(x,x_5) & \to & f_L^D(x_5)\psi_L(x)\\
D_R(x,x_5) & \to & f_R^D(x_5)\psi_R(x)\\
S_L(x,x_5) & \to & f_L^S(x_5)\psi_L(x)\\
S_R(x,x_5) & \to & f_R^S(x_5)\psi_R(x)\\
W_\mu^\pm(x,x_5) & \to & f^\pm(x_5)W_\mu^\pm(x)\\
Z_\mu(x,x_5) & \to & f^Z(x_5)Z_\mu(x)\\
A_\mu(x,x_5) & \to & f^\gamma(x_5)A_\mu(x)
\end{array} \tag{2.39}$$

with the zero mode profiles given in eqs. (2.17), (2.21) and (2.37), and integrating over the extra dimensions.

1. Electromagnetic current.

$$\mathcal{L} = \frac{\tilde{g}_5\tilde{g}_5'}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}Q \quad \Big[\overline{\psi}_R\gamma_\mu\psi_R A_\mu \int_0^L a^3 f^\gamma(|f_R^S|^2+|f_R^D|^2) \\ +\overline{\psi}_L\gamma_\mu\psi_L A_\mu \int_0^L a^3 f^\gamma(|f_L^S|^2+|f_L^D|^2)\Big] \tag{2.40}$$

Since the photon profile is constant, the integral reduces to the normalization (2.28) and we obtain

$$\mathcal{L}_{em} = eQ\overline{\psi}\gamma_\mu\psi A_\mu \qquad e = \frac{\tilde{g}\tilde{g}'}{\sqrt{\tilde{g}^2+\tilde{g}'^2}} \tag{2.41}$$

where $\tilde{g} = L^{-1/2}\tilde{g}_5$, $\tilde{g}' = L^{-1/2}\tilde{g}_5'$.

2. Charged current.

$$\mathcal{L} = \frac{\tilde{g}_5}{\sqrt{2}}\left[\overline{\psi}_L^1\gamma_\mu\psi_L^2 W_\mu^+ \int_0^L a^3 f^W f_L^{D_1} f_L^{D_2} + \text{h.c.} + \overline{\psi}_R^1\gamma_\mu\psi_R^2 W_\mu^+ \int_0^L a^3 f^W f_R^{D_1} f_R^{D_2} + \text{h.c.}\right] \tag{2.42}$$

We obtain

$$\mathcal{L}_{cc} = \frac{\tilde{g}}{\sqrt{2}}[1+m_W^2 F_G(M_d)][1-\frac{1}{2}m_{\psi_1}^2 F_F(M_{S_1})][1-\frac{1}{2}m_{\psi_2}^2 F_F(M_{S_2})]\overline{\psi}_L^1\gamma_\mu\psi_L^2 W_\mu^+ + \text{h.c.} \tag{2.43}$$

e.cc

and

$$\mathcal{L}_{ccR} = \frac{\tilde{g}}{\sqrt{2}} m_{\psi_1} m_{\psi_2} F_F(M_d)\overline{\psi}_R^1\gamma_\mu\psi_R^2 W_\mu^+ + \text{h.c.} \tag{2.44}$$

where

$$\begin{aligned} F_G(M) &= \frac{\int_0^L\int_0^y y' a^{-2}(y')}{L} - \frac{\int_0^L a^{-1}(y)e^{-2My}\int_0^y y' a^{-2}(y')}{\int_0^L a^{-1}(y)e^{-2My}} \\ F_F(M) &= \frac{\int_0^L a^{-1}(y)e^{2My}(\int_0^y a^{-1}(y')e^{-2My'})^2}{\int_0^L a^{-1}(y)e^{-2My}} \end{aligned} \tag{2.45}$$

3. Neutral current.

$$\mathcal{L}_{ncL} = \left(\frac{\tilde{g}_5^2 T_3 - \tilde{g}_5'^2 Y_d}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}\int_0^L a^3 f^Z |f_L^D|^2 - \frac{\tilde{g}_5'^2 Y_s}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}\int_0^L a^3 f^Z |f_L^S|^2\right)\overline{\psi}_L\gamma_\mu\psi_L Z_\mu \tag{2.46}$$

$$\mathcal{L}_{ncR} = \left(\frac{\tilde{g}_5^2 T_3 - \tilde{g}_5'^2 Y_d}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}\int_0^L a^3 f^Z |f_R^D|^2 - \frac{\tilde{g}_5'^2 Y_s}{\sqrt{\tilde{g}_5^2+\tilde{g}_5'^2}}\int_0^L a^3 f^Z |f_R^S|^2\right)\overline{\psi}_R\gamma_\mu\psi_R Z_\mu \tag{2.47}$$

We obtain

$$\mathcal{L}_{ncL} = \left(\tfrac{\tilde{g}^2 T_3 - \tilde{g}'^2 Y_d}{\sqrt{\tilde{g}^2+\tilde{g}'^2}}[1+m_Z^2 F_G(M_d)][1-m_\psi^2 F_F(M_s)] - m_\psi^2 F_F(M_s)\tfrac{\tilde{g}'^2 Y_s}{\sqrt{\tilde{g}^2+\tilde{g}'^2}}\right) \\ \cdot\overline{\psi}_L\gamma_\mu\psi_L Z_\mu \tag{2.48}$$

$$\mathcal{L}_{ncR} = \left(m_\psi^2 F_F(M_d) \tfrac{\tilde{g}^2 T_3 - \tilde{g}'^2 Y_d}{\sqrt{\tilde{g}^2+\tilde{g}'^2}} - [1 + m_Z^2 F_G(M_s)][1 - m_\psi^2 F_F(M_d)] \tfrac{\tilde{g}'^2 Y_s}{\sqrt{\tilde{g}^2+\tilde{g}'^2}} \right) \cdot \overline{\psi}_R \gamma_\mu \psi_R Z_\mu \tag{2.49}$$

4. Gauge boson masses.
Recall, from eq. (2.20) that the masses of W and Z, including the first order corrections, are given by

$$\begin{aligned} m_W^2 &= \tilde{m}_W^2 \left[1 + \tilde{m}_W^2 F\right] \\ m_Z^2 &= \tilde{m}_Z^2 \left[1 + \tilde{m}_Z^2 F\right], \end{aligned} \tag{2.50}$$

where

$$F = L^{-1} \int_0^L \int_0^y y' a^{-2}(y') - \int_0^L y\, a^{-2}(y) \tag{2.51}$$

The twiddle masses are related to the brane Higgs vev, and there is the relation: $\tilde{m}_Z^2 = \tilde{m}_W^2(\tilde{g}^2 + \tilde{g}'^2)/\tilde{g}^2$. We can rewrite the mass formulae in a slightly different form that will prove convenient in a moment,

$$m_W^2 = \frac{\tilde{g}^2 \tilde{v}^2}{4} \qquad m_Z^2 = \frac{(\tilde{g}^2 + g'^2)\tilde{v}^2}{4} \left(1 + \frac{\tilde{g}'^2 v^2}{4} F\right) \tag{2.52}$$ e.wzm

where $\tilde{g}^2 \tilde{v}^2 = 4\tilde{m}_W^2[1 + \tilde{m}_W^2 F]$.

We now reproduce these interactions in the effective theory consisting of the SM lagrangian supplemented by dimensions six operators, as in Section 1.1. Before we kick off let us define two geometrical integrals

$$I_1(x_5) = L^{-1} \int_0^{x_5} \int_0^y y' a^{-2}(y') \qquad I_2(x_5) = \int_0^{x_5} y a^{-2}(y) \tag{2.53}$$

We will see that the oblique corrections can be expressed in terms of these two integrals. The integrals have dimension $[mass]^{-2}$ and are expected to be of order $1/M_{KK}^2$. We note that I_1 is safe while I_2 is *vicious*, by which we mean that in AdS_5 $I_2(L)$ is enhanced by $\log M_{\rm Pl}/M_{\rm EW}$ wrt to the KK scale.

From eqs. (2.52), (2.43), (2.48) we can read out the coefficients of dimension six operators defined in Section 1.1:

$$\begin{aligned} \alpha_T &= \frac{1}{2} g'^2 \left(I_1(L) - I_2(L)\right) \\ \alpha_S &= 0 \\ \beta_j &= I_1(L) - \frac{\int_0^L a^{-1}(y) e^{-2M_j y} I_2(y)}{\int_0^L a^{-1}(y) e^{-2M_j y}} \\ \gamma_j &= I_1(L) - \frac{\int_0^L a^{-1}(y) e^{-2M_j y} I_2(y)}{\int_0^L a^{-1}(y) e^{-2M_j y}} \end{aligned} \tag{2.54}$$

There is a universal piece in the vertex corrections. As explained in Section 1.2 we can sweep it into S and T. After this is done we wake up with

$$\begin{aligned}
\alpha_T &= -\frac{1}{2}g'^2\left(I_1(L)+I_2(L)\right)\\
\alpha_S &= \tilde{g}\tilde{g}' I_1(L)\\
\beta_j &= -\frac{\int_0^L a^{-1}(y)e^{-2M_j y}I_2(y)}{\int_0^L a^{-1}(y)e^{-2M_j y}}\\
\gamma_j &= -\frac{\int_0^L a^{-1}(y)e^{-2M_j y}I_2(y)}{\int_0^L a^{-1}(y)e^{-2M_j y}}
\end{aligned} \tag{2.55}$$

The vertex corrections are in general non-universal. They could be made universal by setting all bulk masses equal, in which case we can parametrize all corrections by the two oblique parameters, but in then we lose the nice mechanism of generating the fermion mass hierarchy. Note however that if the fermions are strictly localized on the UV brane, which corresponds to $M_j \to \infty$, the vertex corrections all vanish. In such case the corrections are purely oblique and can be parametrized by S and T. Thus, parametrizing deviations from the SM by just oblique correction is justified as long as the light fermions are localized sufficiently close to the UV brane. In such circumstances the tree-level S and T are given by

$$S = 8\pi v^2 I_1(L) \qquad\qquad T = \frac{4\pi m_Z^2}{g_L^2}\left(I_1(L)+I_2(L)\right) \tag{2.56}$$ e.sst

Note that T contains the vicious integral I_2, therefore it comes out huge in $\mathrm{AdS}_\mathbf{5}$.

Let us now investigate the oblique corrections in fashionable backgrounds

Flat: $a(x_5)=1$

We find the integrals:

$$I_1(L) = \frac{1}{6}L^2 \qquad I_2(L) = \frac{1}{2}L^2 \tag{2.57}$$

In terms of the KK scale, $M_{\mathrm{KK}} = \pi/L$,

$$I_1(L) = \frac{\pi^2}{6M_{\mathrm{KK}}^2} \qquad I_2(L) = \frac{\pi^2}{2M_{\mathrm{KK}}^2} \tag{2.58}$$

In the oblique case we find S and T,

$$S = \frac{4\pi^3 v^2}{3M_{\mathrm{KK}}^2} \qquad\qquad T = \frac{8\pi^3 m_Z^2}{3g_L^2 M_{\mathrm{KK}}^2} \tag{2.59}$$ e.sstf

$\mathrm{AdS}_\mathbf{5}$: $a(x_5) = \exp(-kx_5)$

We find the integrals:

$$I_1(L) = \frac{1}{4k^2a_L^2}\left[1-\frac{1}{kL}+a_L^2\left(1+\frac{1}{kL}\right)\right] \qquad I_2(L) = \frac{1}{4k^2a_L^2}\left[2kL-1+a_L^2\right] \tag{2.60}$$

For $a_L \ll 1$ and inserting $M_{\rm KK} = \pi k a_L$

$$I_1(L) \approx \frac{\pi^2}{4M_{\rm KK}^2}\left[1 - \frac{1}{kL}\right] \qquad I_2(L) \approx \frac{\pi^2}{4M_{\rm KK}^2}\left[2kL - 1\right] \tag{2.61}$$

In the oblique case we find S and T,

$$S \approx \frac{2\pi^3 v^2}{M_{\rm KK}^2} \qquad T \approx \frac{2\pi^3 kL m_Z^2}{g_L^2 M_{\rm KK}^2} \tag{2.62}$$ e.sstrs

Here T ends up being very large; it is enhanced the large log $kL = \log(M_{\rm Pl}/M_{\rm EW})$. Apparently, we need custodial symmetry to tame T.

$$\text{Power: } a(x_5) = \left(1 - \tfrac{kx_5}{\gamma-1}\right)^\gamma$$

The integrals become too complicated to display them in their full glory. For $a_L \ll 1$ and inserting $M_{\rm KK} = \pi k a_L^{1-1/\gamma}$ we obtain

$$I_1(L) \approx \frac{(\gamma-1)\pi^2}{2(2\gamma-1)M_{\rm KK}^2}\frac{(2\gamma-3-(2\gamma-1)a_L^{1/\gamma}}{(1-a_L^{1/\gamma})(2\gamma-3)} \qquad I_2(L) \approx \frac{(\gamma-1)\pi^2}{2(2\gamma-1)M_{\rm KK}^2}(2\gamma-2-(2\gamma-1)a_L^{1/\gamma})a_L^{-1/\gamma} \tag{2.63}$$

For $\gamma \to \infty$ (the conformal limit) we recover the AdS$_\mathbf{5}$results. Away from the conformal limit we have a power enhancment of the vicious integral I_2, in consequence T gets really huuuge.

3 Warped Custodial Model

We consider a 5D $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X$ gauge theory in the background (2.1). The bulk theory contains the gauge fields coupled to Dirac fermions. $SU(2)_L$ doublets will be denoted by l and $SU(2)_R$ doublets will be denoted by r. The Higgs field Φ and Yukawa interactions are localized at the IR boundary.

3.1 Gauge sector

The 5D action (we ignore the gluons henceforth)

$$\begin{aligned} S = \int d^4x \int_0^L dx_5 \sqrt{g}\left\{-\tfrac{1}{2}{\rm Tr}\{L_{MN}L_{MN}\} - \tfrac{1}{2}{\rm Tr}\{R_{MN}R_{MN}\} - \tfrac{1}{4}X_{MN}X_{MN}\right\} \\ \int d^4x dx_5 \sqrt{g_4}\delta(L)\left({\rm Tr}|D_\mu\Phi|^2 - V(\Phi)\right) \end{aligned} \tag{3.1}$$

A bunch of definitions:

$$\begin{aligned} D_\mu\Phi &= \partial_\mu\Phi - ig_L L_\mu \Phi + ig_R \Phi R_\mu \\ L_M &= L_M^a t^a \qquad L_{MN} = \partial_M L_N - \partial_N L_M - i[L_M, L_N] \\ R_M &= R_M^a t^a \qquad R_{MN} = \partial_M R_N - \partial_N R_M - i[R_M, R_N] \\ \Phi &= \Phi^0 \frac{I}{2} + \Phi^a t^a \qquad t^a = \frac{\sigma^a}{2} \end{aligned} \tag{3.2}$$

$\Phi^{0,a}$ are real and so the Higgs is just a usual SM doublet written in a different notation. The Higgs acquires vev $\langle\Phi\rangle = \frac{\tilde{v}}{2} a_L^{-1} I$. This results in the IR brane mass terms:

$$\mathcal{L}_{mass} = \frac{\tilde{v}^2}{4}\delta(L)(g_L L_\mu^a - g_R R_\mu^a)^2 \tag{3.3}$$

We assume that, without a Higgs vev, the IR brane breaks no gauge symmetries – all gauge bosons BC's are Neumann. Then, in the presence of the Higgs vev, the IR boundary conditions read

$$\begin{aligned} \partial_5 X_\mu|_L &= 0 \\ \partial_5(g_R L_\mu^a + g_L R_\mu^a)|_L &= 0 \\ \partial_5(g_L L_\mu^a - g_R R_\mu^a)|_L &= -(g_L^2 + g_R^2) a_L^{-2}\frac{\tilde{v}^2}{4}(g_L L_\mu^a - g_R R_\mu^a)|_L \end{aligned} \tag{3.4}$$

The UV brane is assumed to break $SU(2)_R \times U(1)_X$ down to $U(1)_Y$. Thus we set up the boundary conditions as follows:

$$\begin{aligned} \partial_5 L_\mu^a|_0 &= 0 \\ R_\mu^i|_0 &= 0 \qquad i = 1,2 \\ \partial_5(s_x R_\mu^3 + c_x X_\mu)|_0 &= 0 \qquad s_x = \frac{g_X}{\sqrt{g_X^2 + g_R^2}} \\ (-c_x R_\mu^3 + s_x X_\mu)|_0 &= 0 \end{aligned} \tag{3.5}$$

Thus L_μ^a and the combination $B_\mu = s_x R_\mu^3 + c_x X_\mu$ have Neumann BC on the UV brane. They have a chance to contain a light zero-mode excitation which, in the first approximation, can be identified with the SM gauge fields $W_\mu^a(x)$ and $B_\mu(x)$. Consider a bulk matter field in represenation t_R of $SU(2)_R$ and with charge q_X under $U(1)_X$. The field B_μ couples to it with the coupling g_Y and charge Y, where

$$g_Y = \frac{g_X g_R}{\sqrt{g_X^2 + g_R^2}} \qquad Y = t_R^3 + q_X \tag{3.6}$$

The mass eigenstate will be, in general, combinations of L, R and X. We rewrite the 5D gauge field in the KK basis:

$$\begin{aligned} L_\mu^a(x, x_5) &= \sum_n f_{L,n}^a(x_5) A_{\mu,n}(x) \\ R_\mu^a(x, x_5) &= \sum_n f_{R,n}^a(x_5) A_{\mu,n}(x) \\ X_\mu(x, x_5) &= \sum_n f_{X,n}(x_5) A_{\mu,n}(x) \end{aligned} \tag{3.7}$$

We choose the profiles such that they satisfy BC on the UV brane:

$$\begin{aligned} f_{L,n}^a(x_5) &= \alpha_L^a C(x_5, m_n) \\ f_{R,n}^i(x_5) &= \alpha_R^i S(x_5, m_n) \\ f_{R,n}^3(x_5) &= \alpha_N s_x C(x_5, m_n) - \alpha_D\, c_x S(x_5, m_n) \\ f_{X,n}(x_5) &= \alpha_N c_x C(x_5, m_n) + \alpha_D\, s_x S(x_5, m_n) \end{aligned} \tag{3.8}$$

The masses m_n and the ratios of α's (overall is fixed by normalization) are fixed by IR boundary conditions.

$$\left[\begin{array}{cc} g_R C'(L) & g_L S'(L) \\ g_L(C'(L)+(g_L^2+g_R^2)a_L^{-2}\frac{\tilde{v}^2}{4}C(L)) & -g_R(S'(L)+(g_L^2+g_R^2)a_L^{-2}\frac{\tilde{v}^2}{4}S(L)) \end{array}\right]\left(\begin{array}{c}\alpha_L^i\\ \alpha_R^i\end{array}\right)=0 \tag{3.9}$$

$$\left[\begin{array}{ccc} 0 & c_x C'(L) & s_x S'(L) \\ g_R C'(L) & g_L s_x C'(L) & -g_L c_x S'(L) \\ g_L \hat{C}'(L) & -g_R s_x \hat{C}'(L) & g_R c_x(S'(L)+(g_L^2+g_R^2)a_L^{-2}\frac{\tilde{v}^2}{4}S(L)) \end{array}\right]\left(\begin{array}{c}\alpha_L^3\\ \alpha_N \\ \alpha_D\end{array}\right)=0 \tag{3.10}$$

where we shortanded $C(x_5, m_n) \to C(x_5)$ etc.

In the **charged sector** we find the quantization condition:

$$S'(L)C'(L)+a_L^{-2}\frac{\tilde{v}^2}{4}\left(g_L^2 S'(L)C(L)+g_R^2 S(L)C'(L)\right)=0 \tag{3.11}$$ e.cwm

From the IR BC and normalization we find:

$$\begin{aligned}(\alpha_{L,n}^i)^{-2} &= \int_0^L dy\left\{C^2(y,m_n)+\frac{g_R^2}{g_L^2}\left(\frac{C'(L,m_n)}{S'(L,m_n)}\right)^2 S^2(y,m_n)\right\}\\ \alpha_{R,n}^i &= -\frac{g_R}{g_L}\frac{C'(L,m_n)}{S'(L,m_n)}\alpha_{L,n}^i\end{aligned} \tag{3.12}$$

The eigenvector with a smallest mass is identified with the W boson.

In the **neutral sector** the quantization condition can be separated into two. One possibility has the simple quantization condition:

$$C'(L,m_n)=0 \tag{3.13}$$

The eigenvectors have the form:

$$\begin{aligned}(\alpha_{L,n}^3)^{-2} &= \frac{1}{\sin^2\tilde{\theta}_W}\int_0^L dy C^2(y,m_n) \qquad \sin\tilde{\theta}_W=\frac{g_Y}{g_Y^2+g_L^2}\\ \alpha_N &= \frac{\cos\tilde{\theta}_W}{\sin\tilde{\theta}_W}\alpha_L^3 \qquad \alpha_D=0\end{aligned} \tag{3.14}$$

This is the photon KK tower, as it includes the solution with $m_\gamma=0$, $C(x_5,0)=1$, $\alpha_L^3=L^{-1/2}\sin\tilde{\theta}_W$, $\alpha_N=L^{-1/2}\cos\tilde{\theta}_W$. The other possibility is the quantization condition:

$$S'(L)C'(L)+a_L^{-2}\frac{\tilde{v}^2}{4}\left(g_L^2 S'(L)C(L)+g_R^2 S(L)C'(L)+m_n a_L^{-2}g_Y^2\right)=0 \tag{3.15}$$ e.czm

The lightest solution is identified with the Z boson. The eigenvectors:

$$\begin{aligned}(\alpha_{L,n}^3)^{-2} &= \int_0^L dy\left\{\frac{1}{\cos^2\tilde{\theta}_W}C^2(x_5,m_n)+\frac{c_x^2}{s_x^2}\frac{g_Y^2}{g_L^2}\left(\frac{C'(L,m_n)}{S'(L,m_n)}\right)^2 S^2(x_5,m_n)\right\}\\ \alpha_{N,n} &= -\frac{g_Y}{g_L}\alpha_{L,n}^3\\ \alpha_{D,n} &= \frac{c_x}{s_x}\frac{g_Y}{g_L}\frac{C'(L,m_n)}{S'(L,m_n)}\alpha_{L,n}^3\end{aligned} \tag{3.16}$$

We will be interested in the zero modes: the SM fields. From the above result we read out the "zero-mode content" of the 5D fields:

$$
\begin{aligned}
L^i_\mu(x,x_5) &\to \frac{1}{\sqrt{L}}\alpha_W C(x_5,m_W)W^i_\mu(x) \\
R^i_\mu(x,x_5) &\to -\frac{1}{\sqrt{L}}\alpha_W \frac{g_R}{g_L}\frac{C'(L,m_W)}{S'(L,m_W)}S(x_5,m_W)W^i_\mu(x) \\
\alpha_W &= \sqrt{L}\left(\int_0^L dy\left\{C^2(y,m_W)+\frac{g_R^2}{g_L^2}\left(\frac{C'(L,m_W)}{S'(L,m_W)}\right)^2 S^2(y,m_W)\right\}\right)^{-1/2}
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
L^3_\mu(x,x_5) &\to \frac{1}{\sqrt{L}}\sin\tilde\theta_W A_\mu(x)+\frac{1}{\sqrt{L}}\alpha_Z\cos\tilde\theta_W C(x_5,m_Z)Z_\mu(x) \\
R^3_\mu(x,x_5) &\to \frac{1}{\sqrt{L}}\cos\tilde\theta_W s_x A_\mu(x)-\frac{1}{\sqrt{L}}\alpha_Z s_x\sin\tilde\theta_W\left(C(x_5,m_Z)+\frac{c_x^2}{s_x^2}\frac{C'(L,m_Z)}{S'(L,m_Z)}S(x_5,m_Z)\right)Z_\mu(x) \\
X_\mu(x,x_5) &\to \frac{1}{\sqrt{L}}\cos\tilde\theta_W c_x A_\mu(x)-\frac{1}{\sqrt{L}}\alpha_Z c_x\sin\tilde\theta_W\left(C(x_5,m_Z)-\frac{C'(L,m_Z)}{S'(L,m_Z)}S(x_5,m_Z)\right)Z_\mu(x) \\
\alpha_Z &= \sqrt{L}\left(\int_0^L dy\left\{C^2(y,m_Z)+\frac{c_x^2}{s_x^2}\sin^2\tilde\theta_W\left(\frac{C'(L,m_Z)}{S'(L,m_Z)}\right)^2 S^2(y,m_Z)\right\}\right)^{-1/2}
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
0 &= S'(L,m_W)C'(L,m_W)+a_L^{-2}\frac{\tilde v^2}{4}\left(g_L^2 S'(L,m_W)C(L,m_W)+g_R^2 S(L,m_W)C'(L,m_W)\right) \\
0 &= S'(L,m_Z)C'(L,m_Z)+a_L^{-2}\frac{\tilde v^2}{4}\left(g_L^2 S'(L,m_Z)C(L,m_Z)+g_R^2 S(L,m_Z)C'(L,m_Z)+m_n a_L^{-2}g_Y^2\right)
\end{aligned}
\tag{3.19}
$$

3.2 Fermionic sector

We describe various ways the SM fermions can be embedded in the custodial model. We will explicitly discuss the 3rd generation quarks; leptons can be realized in an analogous way.

v1: Minimal fermions

We review the minimal version introduced in [1]. The SM left doublets are simply promoted to $(2,1)_X$ representation of the bulk gauge group, $X=Y$. Then the right fermions have to belong to $SU(2)_R$ doublets (so that we can write brane Yukawas with our $(2,\overline{2})$ Higgs), that is to $(1,2)_X$ with $Y=t^3_R+X$. But right top and bottom have to originate from different $SU(2)_R$ multiplets, otherwise they would have the same yukawas and the top and bottom would be degenerate in mass. Thus the third generation looks like:

$$
\underset{(2,1)_{1/6}}{Q=\begin{pmatrix} t \\ b\end{pmatrix}} \quad \underset{(1,2)_{1/6}}{T=\begin{pmatrix} t^c \\ \tilde b^c\end{pmatrix}} \quad \underset{(1,2)_{1/6}}{B=\begin{pmatrix} \tilde t^c \\ b^c\end{pmatrix}}
\tag{3.20}
$$

The fermionic lagrangian

$$\mathcal{L} = \overline{Q}(i\Gamma_N D_N + M_Q)Q + \overline{T}(i\Gamma_N D_N - M_T)T + \overline{B}(i\Gamma_N D_N - M_B)B \tag{3.21}$$

We impose the IR boundary conditions (in the absence of the Higgs vev)

$$Q_R(L) = T_L(L) = B_L(L) = 0 \tag{3.22}$$

These bc's do preserve $SU(2)_R$ as the right group is only spontaneously broken there. We can now write the yukawas

$$\mathcal{L} = -2\delta(L)\sqrt{-g}\left(y_t \overline{Q}_L \Phi T_R + y_b \overline{Q}_L \Phi B_R + \text{h.c.}\right) \tag{3.23}$$

which spurns the mass boundary terms

$$\mathcal{L}_{mass} = -\delta(L) a_L^3 \tilde{v}\left(y_t \bar{t}_L t_R^c + y_t \bar{b}_L \tilde{b}_R^c + y_b \bar{t}_L \tilde{t}_R^c + y_b \bar{b}_L b_R^c + \text{h.c.}\right) \tag{3.24}$$

The UV boundary condition may break $SU(2)_R$ and we readily make use of this possibility. We impose:

$$Q_R(0) = t_L^c(0) = \tilde{b}_R^c(0) = \tilde{t}_R^c(0) = b_L^c(0) = 0 \tag{3.25}$$

In this way we kill the would-be zero modes of $\tilde{b}$ and $\tilde{t}$. These fields, which have SM quantum numbers of bottom and top appear only at the KK scale.

The fermionic profiles satisfying the UV boundary conditions can be written as (see Appendix B for derivation)

$$\begin{aligned}
f_{R,n}^{t}(x_5) &= \alpha_n^t a^{-2}(x_5) e^{M_Q x_5} S_{M_Q}(x_5, m_n) \\
f_{L,n}^{t}(x_5) &= \alpha_n^t a^{-2}(x_5) e^{-M_Q x_5} C_{-M_Q}(x_5, m_n) \\
f_{R,n}^{b}(x_5) &= \alpha_n^b a^{-2}(x_5) e^{M_Q x_5} S_{M_Q}(x_5, m_n) \\
f_{L,n}^{b}(x_5) &= \alpha_n^b a^{-2}(x_5) e^{-M_Q x_5} C_{-M_Q}(x_5, m_n) \\
f_{L,n}^{t^c}(x_5) &= -\alpha_n^{t^c} a^{-2}(x_5) e^{M_T x_5} S_{M_T}(x_5, m_n) \\
f_{R,n}^{t^c}(x_5) &= \alpha_n^{t^c} a^{-2}(x_5) e^{-M_T x_5} C_{-M_T}(x_5, m_n) \\
f_{L,n}^{\tilde{b}^c}(x_5) &= \alpha_n^{\tilde{b}^c} a^{-2}(x_5) e^{M_T x_5} C_{M_T}(x_5, m_n) \\
f_{R,n}^{\tilde{b}^c}(x_5) &= \alpha_n^{\tilde{b}^c} a^{-2}(x_5) e^{-M_T x_5} S_{-M_T}(x_5, m_n) \\
f_{L,n}^{\tilde{t}^c}(x_5) &= \alpha_n^{\tilde{t}^c} a^{-2}(x_5) e^{M_B x_5} C_{M_B}(x_5, m_n) \\
f_{R,n}^{\tilde{t}^c}(x_5) &= \alpha_n^{\tilde{t}^c} a^{-2}(x_5) e^{-M_B x_5} S_{-M_B}(x_5, m_n) \\
f_{L,n}^{b^c}(x_5) &= -\alpha_n^{b^c} a^{-2}(x_5) e^{M_B x_5} S_{M_B}(x_5, m_n) \\
f_{R,n}^{b^c}(x_5) &= \alpha_n^{b^c} a^{-2}(x_5) e^{-M_B x_5} C_{-M_B}(x_5, m_n)
\end{aligned} \tag{3.26}$$

In the presence of the the Higgs vev we get mass terms

$$\mathcal{L} = -\delta(L) a_L^3 \tilde{v}\left(y_t \bar{t}_L t_R^c + y_b \bar{t}_L \tilde{t}_R^c + y_t \bar{b}_L \tilde{b}_R^c + y_b \bar{b}_L b_R^c + \text{h.c.}\right) \tag{3.27}$$

that lead to IR boundary conditions for the profiles

$$
\begin{aligned}
f^{t}_{R,n}(L) &= y_t\tilde{v}a_L^{-1}f^{t^c}_{R,n}(L) + y_b\tilde{v}a_L^{-1}f^{\tilde{t}^c}_{R,n}(L) \\
f^{t^c}_{L,n}(L) &= -y_t\tilde{v}a_L^{-1}f^{t}_{L,n}(L) \\
f^{\tilde{t}^c}_{L,n}(L) &= -y_b\tilde{v}a_L^{-1}f^{t}_{L,n}(L) \\
f^{b}_{R,n}(L) &= y_b\tilde{v}a_L^{-1}f^{b^c}_{R,n}(L) + y_t\tilde{v}a_L^{-1}f^{\tilde{b}^c}_{R,n}(L) \\
f^{b^c}_{L,n}(L) &= -y_b\tilde{v}a_L^{-1}f^{b}_{L,n}(L) \\
f^{\tilde{b}^c}_{L,n}(L) &= -y_t\tilde{v}a_L^{-1}f^{b}_{L,n}(L)
\end{aligned}
\qquad (3.28)
$$

In the top sector

$$
\begin{bmatrix}
e^{M_Q L}S_{M_Q}(L) & -y_t\tilde{v}a_L^{-1}e^{-M_T L}C_{-M_T}(L) & -y_b\tilde{v}a_L^{-1}e^{-M_B L}S_{-M_B}(L) \\
y_t\tilde{v}a_L^{-1}e^{-M_Q L}C_{-M_Q}(L) & -e^{M_T L}S_{M_T}(L) & 0 \\
y_b\tilde{v}a_L^{-1}e^{-M_Q L}C_{-M_Q}(L) & 0 & e^{M_B L}C_{M_B}(L)
\end{bmatrix}
\begin{pmatrix} \alpha^t_n \\ \alpha^{t^c}_n \\ \alpha^{\tilde{t}^c}_n \end{pmatrix} = 0
\qquad (3.29)
$$

yields the quantization condition:

$$
\begin{gathered}
S_{M_Q}(L)S_{M_T}(L)C_{M_B}(L) = \\
\tilde{v}^2 a_L^{-2}\left(y_t^2 e^{-2(M_Q+M_T)L}C_{M_B}(L)C_{-M_T}(L)C_{-M_Q}(L) - y_b^2 e^{-2(M_Q+M_B)L}S_{M_T}(L)S_{-M_B}(L)C_{-M_Q}(L)\right)
\end{gathered}
\qquad (3.30)
$$

In the bottom sector

$$
\begin{bmatrix}
e^{M_Q L}S_{M_Q}(L) & -y_b\tilde{v}a_L^{-1}e^{-M_B L}C_{-M_B}(L) & -y_t\tilde{v}a_L^{-1}e^{-M_T L}S_{-M_T}(L) \\
y_b\tilde{v}a_L^{-1}e^{-M_Q L}C_{-M_Q}(L) & -e^{M_B L}S_{M_B}(L) & 0 \\
y_t\tilde{v}a_L^{-1}e^{-M_Q L}C_{-M_Q}(L) & 0 & e^{M_T L}C_{M_T}(L)
\end{bmatrix}
\begin{pmatrix} \alpha^b_n \\ \alpha^{b^c}_n \\ \alpha^{\tilde{b}^c}_n \end{pmatrix} = 0
\qquad (3.31)
$$

e.cv1be

yields the quantization condition:

$$
\begin{gathered}
S_{M_Q}(L)S_{M_B}(L)C_{M_T}(L) = \\
\tilde{v}^2 a_L^{-2}C_{-M_Q}(L)\left(y_b^2 e^{-2(M_Q+M_B)L}C_{M_T}(L)C_{-M_B}(L) - y_t^2 e^{-2(M_Q+M_T)L}S_{M_B}(L)S_{-M_T}(L)\right)
\end{gathered}
\qquad (3.32)
$$

v2: Custodial fermions

The label 'custodial' this time refers to Zbb rather than T. It was pointed out in [3] that dangerous corrections to the Zb_Lb_L vertex can be reduced if the there exist an approximate LR parity symmetry and if the left b quark has $t^3_L = t^3_R$. Thus, the left b quark should originates from a bifundamental $(2,\overline{2})$ wrt to $SU(2)_L \times SU(2)_R$. Below we review a model with desired properties (this one was studied in [4])

- Left doublet from a $SU(2)_L \times SU(2)_R$ bifundamental

$$
Q = \begin{bmatrix} t & \chi \\ b & \tilde{t} \end{bmatrix} \qquad (2,\overline{2})_{2/3} \quad Q \to LQR^\dagger \qquad (3.33)
$$

Under SM, $(t,b) \to 2_{1/6}$ is the usual 3rd generation doublet, while $(\chi,\tilde{t}) \to 2_{7/6}$ is an exotic double containing a charge 5/3 and a charge 2/3 quark.

- Singlet top from a $SU(2)_L \times SU(2)_R$ singlet

$$t^c \qquad (1,1)_{2/3} \tag{3.34}$$

- Singlet bottom from a $SU(2)_R$ triplet

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}}\tilde{t}^c & \chi^c \\ b^c & -\frac{1}{\sqrt{2}}\tilde{t}^c \end{bmatrix} \qquad (1,3)_{2/3} \quad B \to RBR^\dagger \tag{3.35}$$

Here $b^c \to 1_{-1/3}$ is the bottom singlet. $\tilde{t}^c \to 1_{2/3}$ and $\chi^c \to 1_{5/3}$ are exotic quarks.

- Junk from a $SU(2)_L$ triplet (to keep LR symmetry)

$$B' = \begin{bmatrix} \frac{1}{\sqrt{2}}\tilde{t}'^c & \chi'^c \\ b'^c & -\frac{1}{\sqrt{2}}\tilde{t}'^c \end{bmatrix} \qquad (3,1)_{2/3} \quad B \to LBL^\dagger \tag{3.36}$$

This is a SM triplet of exotic quarks.

The model is... well... late baroque.[1] In a generation, there are 5 top-like quarks, 3 bottom-like quarks and three 5/3 charge exotic quarks. To describe it, we will need 22 different profiles, 11 boundary conditions, 3 quantization conditions...

The fermionic lagrangian

$$\mathcal{L} = \mathrm{Tr}[\overline{Q}(i\Gamma_N D_N + M_Q)Q] + \overline{t}^c(i\Gamma_N D_N - M_T)t^c + \mathrm{Tr}[\overline{B}(i\Gamma_N D_N - M_B)B] + \mathrm{Tr}[\overline{B}'(i\Gamma_N D_N - M_B)B'] \tag{3.37}$$

The covariant derivatives are

$$\begin{aligned} D_N Q &= \partial_N Q - ig_L L_N Q + ig_L Q R_N - i(2/3)g_X X_N Q \\ D_N t^c &= \partial_N t^c - i(2/3)g_X X_N t^c \\ D_N B &= \partial_N B - ig_L R_N B + ig_L B R_N - i(2/3)g_X X_N B \\ D_N B' &= \partial_N B' - ig_L L_N B' + ig_L B' L_N - i(2/3)X_N B' \end{aligned} \tag{3.38}$$

Note that by LR symmetry we set $g_R = g_L$.

In the absence of the Higgs vev, we impose the IR boundary conditions that preserve $SU(2)_R$ and the LR parity

$$Q_R(L) = t^c_L(L) = B_L(L) = B'_L(L) = 0 \tag{3.39}$$

The yukawas

$$\mathcal{L} = -2\delta(L)\sqrt{-g}\left(y_t \mathrm{Tr}[\overline{Q}_L\Phi]t^c_R + y_b \mathrm{Tr}[\overline{Q}_L \Phi B_R] + y_b \mathrm{Tr}[\overline{Q}_L B'_R \Phi] + \mathrm{h.c.}\right) \tag{3.40}$$

yield the mass boundary terms

$$\begin{aligned} \mathcal{L}_{mass} = -\delta(L)\tilde{v}a_L^3 \Big(& y_t \overline{t}_L t^c_R + y_t \overline{\tilde{t}}_L t^c_R \\ & + \tfrac{1}{\sqrt{2}} y_b \overline{t}_L \tilde{t}^c_R - \tfrac{1}{\sqrt{2}} y_b \overline{\tilde{t}}_L \tilde{t}^c_R + y_b \overline{b}_L b^c_R + y_b \overline{\chi}_L \chi^c_R \\ & + \tfrac{1}{\sqrt{2}} y_b \overline{t}_L \tilde{t}'^c_R - \tfrac{1}{\sqrt{2}} y_b \overline{\tilde{t}}_L \tilde{t}'^c_R + y_b \overline{b}_L b'^c_R + y_b \overline{\chi}_L \chi'^c_R \Big) + \mathrm{h.c.} \end{aligned} \tag{3.41}$$

[1] It could be less artificial if $SU(2)_L \times SU(2)_R$ were unified into $SO(5)$. We have the decomposition $5 = (2,2) \oplus (1,1)$ and $10 = (2,2) \oplus (1,3) \oplus (3,1)$, thus e.g. Q, B and B' fit into just one $SO(5)$ representation.

The UV boundary we impose $SU(2)_R$ breaking (also LR breaking) conditions so as to remove all junk from the low energy spectrum

$$\tilde{t}_L(0) = \tilde{\chi}^c_L(0) = \tilde{\chi}^c_R(0) = \tilde{t}^c_R(0) = \tilde{\chi}'^c_R(0) = \tilde{t}'^c_R(0) = b'^c_R(0) \tag{3.42}$$

and keep the zero modes for the SM fields

$$t_R(0) = b_R(0) = t^c_L(0) = b^c_L(0) = 0 \tag{3.43}$$

We can start discussing the fermion profiles. In the following we restrict to the bottom sector only.

The bottom profiles satisfying the UV boundary conditions can be written as (see Appendix B for a derivation)

$$\begin{aligned}
f^b_{L,n}(x_5) &= \alpha^b_n a^{-2}(x_5) e^{-M_Q x_5} C_{-M_Q}(x_5, m_n) \\
f^b_{R,n}(x_5) &= \alpha^b_n a^{-2}(x_5) e^{M_Q x_5} S_{M_Q}(x_5, m_n) \\
f^{b^c}_{L,n}(x_5) &= -\alpha^{b^c}_n a^{-2}(x_5) e^{M_B x_5} S_{M_B}(x_5, m_n) \\
f^{b^c}_{R,n}(x_5) &= \alpha^{b^c}_n a^{-2}(x_5) e^{-M_B x_5} C_{-M_B}(x_5, m_n) \\
f^{b'^c}_{L,n}(x_5) &= \alpha^{b'^c}_n a^{-2}(x_5) e^{M_B x_5} C_{M_B}(x_5, m_n) \\
f^{b'^c}_{R,n}(x_5) &= \alpha^{b'^c}_n a^{-2}(x_5) e^{-M_B x_5} S_{-M_B}(x_5, m_n)
\end{aligned} \tag{3.44}$$

In the presence of the boundary mass term we find the following IR boundary conditions for the profiles

$$\begin{aligned}
f^b_{R,n}(L) &= y_b \tilde{v} a_L^{-1} f^{b^c}_{R,n}(L) + y_b \tilde{v} a_L^{-1} f^{b'^c}_{R,n}(L) \\
f^{b^c}_{L,n}(L) &= -y_b \tilde{v} a_L^{-1} f^b_{L,n}(L) \\
f^{b'^c}_{L,n}(L) &= -y_b \tilde{v} a_L^{-1} f^b_{L,n}(L)
\end{aligned} \tag{3.45}$$

This leads to the eigenvalue equations

$$\begin{bmatrix}
e^{M_Q L} S_{M_Q}(L) & -y_b \tilde{v} a_L^{-1} e^{-M_B L} C_{-M_B}(L) & -y_b \tilde{v} a_L^{-1} e^{-M_B L} S_{-M_B}(L) \\
y_b \tilde{v} a_L^{-1} e^{-M_Q L} C_{-M_Q}(L) & -e^{M_B L} S_{M_B}(L) & 0 \\
y_b \tilde{v} a_L^{-1} e^{-M_Q L} C_{-M_Q}(L) & 0 & e^{M_B L} C_{M_B}(L)
\end{bmatrix}
\begin{pmatrix} \alpha^b_n \\ \alpha^{b^c}_n \\ \alpha^{b'^c}_n \end{pmatrix} = 0 \tag{3.46}$$

e.cv2be

and the quantization condition:

$$\begin{aligned}
& S_{M_Q}(L) S_{M_B}(L) C_{M_B}(L) = \\
& y_b^2 \tilde{v}^2 a_L^{-2} e^{-2(M_Q + M_B)L} C_{-M_Q}(L) \left(C_{M_B}(L) C_{-M_B}(L) - S_{M_B}(L) S_{-M_B}(L) \right)
\end{aligned} \tag{3.47}$$

Note that all the formulae in bottom sector happen to be the same as in the minimal model with $y_t = y_b$, $M_T = M_B$.

3.3 Perturbative expansion

We add another assumption now, namely that the brane mass $\tilde{v}^2$ is much smaller than the resonance scale. In consequence, m_W and m_Z are always much lighter than the resonances.[2] Their masses are of order $\tilde{v}^2$, plus corrections of order $\tilde{v}^4/M_{KK}^2$. Similarly, the W and Z profiles can written as a constant + corrections of order $m_W^2/M_{\rm KK}^2$.

More precisely, using eq. (2.18), the quantization conditions (3.19) can be expanded in $m_W^2/M_{\rm KK}^2$. Using the integrals defined in Appendix A, the gauge boson masses to order m_W^4 can be written as

$$
\begin{aligned}
m_W^2 &\approx \tilde{g}_L^2 \frac{\tilde{v}^2}{4}\left(1+\frac{\tilde{v}^2}{4}\left[\tilde{g}_L^2(I_1(L)-I_2(L))-\tilde{g}_R^2 I_3(L)\right]\right) \\
m_Z^2 &\approx (\tilde{g}_L^2+\tilde{g}_Y^2)\frac{\tilde{v}^2}{4}\left(1+\frac{\tilde{v}^2}{4}\left[\tilde{g}_L^2(I_1(L)-I_2(L))-\tilde{g}_R^2 I_3(L)+\tilde{g}_Y^2(I_1(L)+I_4(L))\right]\right)
\end{aligned}
\tag{3.48}
$$

Next, the profiles of the SM gauge fields. We find

$$
\begin{aligned}
L_\mu^i(x,x_5) &\to \frac{1}{\sqrt{L}}\left(1+m_W^2[I_1(L)-I_2(x_5)]\right)W_\mu^i(x) \\
R_\mu^i(x,x_5) &\to \frac{1}{\sqrt{L}} m_W^2 \frac{\tilde{g}_R}{\tilde{g}_L} I_3(x_5) W_\mu^i(x)
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
L_\mu^3(x,x_5) &\to \frac{1}{\sqrt{L}}\left(\sin\tilde{\theta}_W A_\mu(x)+\cos\tilde{\theta}_W(1+m_Z^2[I_1(L)-I_2(x_5)])Z_\mu(x)\right) \\
R_\mu^3(x,x_5) &\to \frac{s_x}{\sqrt{L}}\left(\cos\tilde{\theta}_W A_\mu(x)-\sin\tilde{\theta}_W\left(1+m_Z^2[I_1(L)-I_2(x_5)+I_3(x_5)-\frac{1}{s_x^2}I_3(x_5)]\right)Z_\mu(x)\right) \\
X_\mu(x,x_5) &\to \frac{c_x}{\sqrt{L}}\left(\cos\tilde{\theta}_W A_\mu(x)-\sin\tilde{\theta}_W\left(1+m_Z^2[I_1(L)-I_2(x_5)+I_3(x_5)]\right)Z_\mu(x)\right)
\end{aligned}
\tag{3.50}
$$

We can now read out SM gauge interactions with light fermions by convoluting the above profiles with the massless fermion profile:

$$
f_F(x_5)=\frac{e^{-Mx_5}a^{-2}(x_5)}{\left(\int_0^L e^{-2My}a^{-1}(y)\right)^{1/2}}
\tag{3.51}
$$

This way we find:

1. Electromagnetic current.

$$
\mathcal{L}_{em}=e(t_L^3+t_R^3+q_X)\overline{\psi}\gamma_\mu\psi A_\mu \qquad e=\frac{g_L g_Y}{\sqrt{g_L^2+g_Y^2}}
\tag{3.52}
$$

2. Charged current.

$$
\mathcal{L}_{cc}=\frac{g_L}{\sqrt{2}}\left(1+m_W^2[I_1(L)-J_2(L,M_j)]\right)\overline{\psi}_j\gamma_\mu T^{\pm}\psi_j W_\mu^{\pm}
\tag{3.53}
$$

e.ccc

3. Neutral current.
   We obtain

$$
\begin{aligned}
\mathcal{L}_{ncL} = \frac{1}{\sqrt{\tilde{g}_L^2+\tilde{g}_Y^2}}&\{g_L^2 t_L^3\left(1+m_Z^2[I_1(L)-J_2(L,M_j)]\right) \\
-g_Y^2 Y\left(1+m_Z^2[I_1(L)-J_2(L,M_j)+J_3(L,M_j)]\right)&+g_R^2 t_R^3 m_Z^2 J_3(L,M_j)\}\,\overline{\psi}_j\gamma_\mu\psi_j Z_\mu
\end{aligned}
\tag{3.54}
$$

[2] W and Z can sometimes be parametrically lighter than the resonances even if $\tilde{v}$ is large. This is e.g. the case in AdS$_5$, which observation is the starting point of the higgsless business.

We now match these results to the SM lagrangian supplemented by dimension six operators, see Section 1. Defining:

$$v^2 = \tilde{v}^2 \left(1 + \frac{\tilde{v}^2}{4}\left[\tilde{g}_L^2[I_1(L) - I_2(L)] - g_R^2 I_3(L)\right]\right) \tag{3.55}$$

we can rewrite the gauge boson masses as

$$m_W^2 = \frac{g_L^2 v^2}{4} \qquad m_Z^2 = \frac{(g_L^2 + g_Y^2)v^2}{4}\left(1 + \frac{g_Y^2 v^2}{4}\left[I_1(L) + I_4(L)\right]\right) + \mathcal{O}(v^6) \tag{3.56}$$

e.cwzm

From eqs. (3.56), (3.53), (3.54) we can read out the coefficients of dimension six operators defined in Section 1.1:

$$\begin{aligned}
\alpha_T &= \frac{1}{2} g_Y^2 \left(I_1(L) + I_4(L)\right) \\
\alpha_S &= 0 \\
\beta_j &= I_1(L) - J_2(L, M_j) \\
\gamma_j &= I_1(L) - J_2(L, M_j) + J_3(L, M_j) - \frac{\tilde{g}_R^2 t_R^3}{\tilde{g}_Y^2 Y} J_3(L, M_j)
\end{aligned} \tag{3.57}$$

There is a universal piece in the vertex corrections. As explained in Section 1.2 we can sweep it into S and T. After this is done we wake up with

$$\begin{aligned}
\alpha_T &= \frac{1}{2} g_Y^2 \left(-I_1(L) + I_4(L)\right) \\
\alpha_S &= g_L g_Y I_1(L) \\
\beta_j &= -J_2(L, M_j) \\
\gamma_j &= -J_2(L, M_j) + J_3(L, M_j) - \frac{g_R^2 t_R^3}{g_Y^2 Y} J_3(L, M_j)
\end{aligned} \tag{3.58}$$

The vertex corrections are in general non-universal. What is funny is that they persist being non-universal even if the bulk mass of all the fermions is universal. This is due to the last piece in γ_j that depends on the bulk representations via t_R^3/Y.

Note however that if the fermions are strictly localized on the UV brane, which corresponds to $M_j \to \infty$, the vertex corrections all vanish. In such case the tree level corrections are purely oblique and can be parametrized by S and T. We find

$$S = 8\pi v^2 I_1(L) \qquad\qquad T = \frac{4\pi m_Z^2}{g_L^2}\left(I_1(L) - I_4(L)\right) \tag{3.59}$$

e.cst

In our setup fermions cannot be strictly localized on the UV brane (except in the Higgsless limit) as they need to have some overlap with the Higgs on the IR brane. However typically light fermions are localized towards the UV and the vertex corrections turn out to be neglibly small. In such case one can safely use S and T of eq. (3.59) to constrain the scale of new physics. Note that the vicious integrals I_2 and I_3 do not occur in the final result; the custodial symmetry is at work.

For the third generation we should not assume that the fermions are localized towards UV (we would run into problem with perturbativity of top yukawa) and we expect sizable vertex correction. Below we discuss the corrections to the neutral current interactions, which are the most interesting since they are pretty well measured in the case of the b quark. The vertex has the structure

$$\frac{g_L^2 t_L^3 - g_Y^2 Y}{\sqrt{g_L^2 + g_Y^2}} \left(1 + \delta g_{Zjj}\right) \overline{\psi}_j \gamma_\mu \psi_j Z_\mu \tag{3.60}$$

Here δg_{Zjj} describes corrections to from new physics to the familiar SM coupling. It is simply related to β and γ:

$$\delta g_{Zjj} = m_Z^2 \frac{\beta_j g_L^2 t_L^3 - \gamma_j g_Y^2 Y}{g_L^2 t_L^3 - g_Y^2 Y} \tag{3.61}$$

We find a simple expression

$$\delta g_{Zjj} = m_Z^2 (J_3(L, M_j) - J_2(L, M_j)) - m_Z^2 J_3(L, M_j) \frac{g_L^2 t_L^3 - g_R^2 t_R^3}{g_L^2 t_L^3 - g_Y^2 Y} \tag{3.62}$$

e.cvc

This second term depends on the potentially vicious integral J_3 (in the first viciousness cancels between J_3 and J_2). If the integral is not damped by the bulk mass terms, the correction is volume enhanced, as compared to the naive estimate $1/M_{\rm KK}^2$. This can be dangerous for the Zb_Lb_L vertex, which is measured to accuracy $\sim 0.25\%$.

One way to keep this vertex under control is to put the left b quark into a $(2, \overline{2})$ representation of $SU(2)_L \times SU(2)_R$ and ensure the parity symmetry $g_L = g_R$. This is the custodial symmetry for Zb_Lb_L advocated in [3]. As can be seen from eq. (3.62), the potentially vicious term is cancelled. What remains are safe corrections of order $m_Z^2/M_{\rm KK}^2$.

3.3.1 Perturbative expansion of fermionic profiles

The corrections to the zero mode constant gauge boson profiles are of order $m_Z^2/M_{\rm KK}^2$. Similarly the corrections to the zero mode fermionic profile $a^{-2}e^{-Mx_5}$ are expected to be of order $m_f^2/M_{\rm KK}^2$. The only ones worth looking at are those of order $m_t^2/M_{\rm KK}^2$. This will happen for the top quark of course, but it may also happen for the b quark, if it mixes with an exotic quark from a top a multiplet. Below we study one such effect in the v1 minimal fermion embedding.

Perturbatively, the top and bottom masses are given by

$$m_b^2 \approx y_b^2 \tilde{v}^2 \frac{a_L^{-2} e^{-2(M_Q+M_B)L}}{\int_0^L a^{-1}(y) e^{-2M_Q y} \int_0^L a^{-1}(y) e^{-2M_B y}} \qquad m_t^2 \approx y_t^2 \tilde{v}^2 \frac{a_L^{-2} e^{-2(M_Q+M_B)L}}{\int_0^L a^{-1}(y) e^{-2M_Q y} \int_0^L a^{-1}(y) e^{-2M_T y}} \tag{3.63}$$

From the eigenvalue equations (3.31) we get

$$\begin{aligned} \alpha^{b^c} &= y_b \tilde{v} a_L^{-1} e^{-(M_Q+M_B)L} \frac{C_{-M_Q}(L, m_b)}{S_{M_B}(L, m_b)} \alpha^b \\ \alpha^{\tilde{b}^c} &= -y_t \tilde{v} a_L^{-1} e^{-(M_Q+M_T)L} \frac{C_{-M_Q}(L, m_b)}{C_{M_T}(L, m_b)} \alpha^b \end{aligned} \tag{3.64}$$

The left handed b quark is distributed in the bulk fields as follows

$$\begin{aligned}
b_L(x,x_5) &\rightarrow \alpha^b a^{-2}(x_5) e^{-M_Q x_5} C_{-M_Q}(x_5, m_b) b_L(x) \\
b_L^c(x,x_5) &\rightarrow -y_b \tilde{v} a_L^{-1} e^{-(M_Q+M_B)L} \frac{C_{-M_Q}(L,m_b)}{S_{M_B}(L,m_b)} \alpha^b a^{-2}(x_5) e^{M_B x_5} S_{M_B}(x_5, m_b) b_L(x) \\
\tilde{b}_L^c(x,x_5) &\rightarrow -y_t \tilde{v} a_L^{-1} e^{-(M_Q+M_T)L} \frac{C_{-M_Q}(L,m_b)}{C_{M_T}(L,m_b)} \alpha^b a^{-2}(x_5) e^{M_T x_5} C_{M_T}(x_5, m_b) b_L(x)
\end{aligned} \tag{3.65}$$

Perturbatively, keeping only corrections of order m_t, and ignoring those of order m_b,

$$\begin{aligned}
b_L(x,x_5) &\rightarrow \alpha^b a^{-2}(x_5) e^{-M_Q x_5} b_L(x) \\
b_L^c(x,x_5) &\rightarrow 0 \\
\tilde{b}_L^c(x,x_5) &\rightarrow -\alpha^b m_t \left(\int_0^L a^{-1}(y) e^{-2M_Q y} \int_0^L a^{-1}(y) e^{-2M_T y} a^{-2}(x_5) e^{M_T x_5} \right)^{1/2} b_L(x)
\end{aligned} \tag{3.66}$$

where, from normalization

$$\alpha^b \approx \frac{1}{\left(\int_0^L a^{-1}(y) e^{-2M_Q y}\right)^{1/2}} \left[1 - \frac{1}{2} m_t^2 \int_0^L a^{-1}(y) e^{-2M_T y} \int_0^L a^{-1}(y) e^{2M_T y} \right] \tag{3.67}$$

We see there can be a sizable, order $m_t^2/M_{\rm KK}^2$ mixing of the SM left b quark with the exotic b quark. This mixing shifts the Zbb vertex by

$$\delta g_{Z b_L b_L}(F) = m_t^2 \int_0^L a^{-1}(y) e^{-2M_T y} \int_0^L a^{-1}(y) e^{2M_T y} \left(-1 + \frac{\tilde{g}_L^2 t_L^3(\tilde{b}^c) - \tilde{g}_Y^2 Y(\tilde{b}^c)}{\tilde{g}_L^2 t_L^3(b) - \tilde{g}_Y^2 Y(b)} \right) \tag{3.68}$$

that adds up to the gauge induced shift. Here $t_L^3(\tilde{b}^c) = 0$, $Y(\tilde{b}^c) = -1/3$, $t_L^3(b) = -1/2$, $Y(b) = 1/6$. Note that there are no volume enhanced integrals occuring here.

In the v2 fermion embedding, all the formulae in the bottom sector are the same as above with $y_t = y_b$, $M_T = M_B$. Which means there is no order m_t^2 tilt of the b-quark profile; only m_b^2 effects are present.

3.4 Results for concrete and steel backgrounds

The values of the integrals in specific backgrounds are given in Appendix A. We use those results to compute electroweak parameters

Flat: $a(x_5) = 1$

In the oblique case we find S and T,

$$S = \frac{4\pi^3 v^2}{3M_{\rm KK}^2} \qquad T = -\frac{4\pi^3 m_Z^2}{3M_{\rm KK}^2} \tag{3.69}$$

e.cstf

where $M_{\rm KK} = \pi/L$ (this is the first KK photon mass). Not much difference from the non-custodial case…except that T is now negative. There is no much use for the custodial in the flat case: as the cutoff is low there is no log corrections from running between $M_{\rm Pl}$ and $M_{\rm EW}$.

The bound $S < 0.2$ constrains the KK scale:

$$M_{\rm KK} > 3.5\,{\rm TeV} \tag{3.70}$$

$$\text{AdS}_{\mathbf{5}}\textbf{:}\ a(x_5) = \exp(-kx_5)$$

For $a_L \ll 1$ and defining $M_{\rm KK} = \pi k a_L$ (the first KK photon mass is slighly less than $M_{\rm KK}$) we get S and T,

$$S \approx \frac{2\pi^3 v^2}{M_{\rm KK}^2} \qquad\qquad T \approx -\frac{\pi^3 m_Z^2}{kLM_{\rm KK}^2} \tag{3.71}$$ e.cstrs

Here the custodian is at work: compared to S, T is suppressed by additional $\log M_{\rm Pl}/M_{\rm EW}$.

The bound $S < 0.2$ constrains the KK scale:

$$M_{\rm KK} > 4.3\,{\rm TeV} \tag{3.72}$$

In the v1 fermion embedding we find the correction to the Zb_Lb_L vertex

$$\delta g_{Zb_Lb_L} \approx m_Z^2(J_3(L,M_j) - J_2(L,M_j)) - m_Z^2 J_3(L,M_j)\frac{\tilde{g}_L^2}{\tilde{g}_L^2 + 1/3\tilde{g}_Y^2} \tag{3.73}$$

For $c_Q \ll 1/2$ we find

$$\delta g_{Zb_Lb_L} \approx -\frac{\pi^2 kLm_Z^2}{4M_{\rm KK}^2}\frac{2(1-2c)}{3-2c} \tag{3.74}$$

while for $c_Q = 1/2$

$$\delta g_{Zb_Lb_L} \approx -\frac{\pi^2 m_Z^2}{4M_{\rm KK}^2} \tag{3.75}$$

The condition $|\delta g_{Zb_Lb_L}| < 0.0025$ constrains the KK scale

$$\begin{array}{lc} M_{\rm KK} > 18\,{\rm TeV} & \qquad c_Q = -1/2 \\ M_{\rm KK} > 3\,{\rm TeV} & \qquad c_Q = 1/2 \end{array} \tag{3.76}$$

Unless the left doublet profile is flat, Zbb yields the most stringent bound in this model.

In the v2 fermion embedding we find the correction to the Zb_Lb_L vertex

$$\delta g_{Zb_Lb_L} \approx m_Z^2(J_3(L,M_j) - J_2(L,M_j)) \tag{3.77}$$

For $c_Q \ll 1/2$ we find

$$\delta g_{Zb_Lb_L} \approx -\frac{\pi^2}{4M_{\rm KK}^2}\frac{(1-2c)(5-2c)}{(3-2c)^2} \tag{3.78}$$

while for $c_Q = 1/2$

$$\delta g_{Zb_Lb_L} \approx \frac{\pi^2}{4M_{\rm KK}^2}\frac{1}{kL} \tag{3.79}$$

The condition $|\delta g_{Zb_Lb_L}| < 0.0025$ constrains the KK scale

$$\begin{array}{lc} M_{\rm KK} > 2.5\,{\rm TeV} & \qquad c_Q = -1/2 \\ M_{\rm KK} > 0.5\,{\rm TeV} & \qquad c_Q = 1/2 \end{array} \tag{3.80}$$

Thus, in the v2 the most stringent bound always comes from S, whatever the shape of the doublet profile.

Power: $a(x_5) = \left(1 - \frac{kx_5}{\gamma-1}\right)^{\gamma}$

For $a_L \ll 1$ and defining $M_{\rm KK} = \pi k a_L^{1-1/\gamma}$ we obtain In the oblique situation

$$S \approx \frac{4(\gamma-1)\pi^3 v^2}{(2\gamma-1)M_{\rm KK}^2} \frac{2\gamma - 3 - (2\gamma-1)a_L^{1/\gamma}}{(1-a_L^{1/\gamma})(2\gamma-3)} \tag{3.81}$$

$$T \approx \frac{4(\gamma-1)\pi^3 m_Z^2}{(2\gamma-1)(2\gamma-3)M_{\rm KK}^2} \frac{a_L^{1/\gamma}}{(1-a_L^{1/\gamma})} \tag{3.82}$$

For $\gamma \to \infty$ (the conformal limit) we recover the AdS$_{\mathbf{5}}$results. Away from the conformal limit T is power law suppressed. But since it was small before there is no big gain. On the other hand, with $M_{\rm KK}$ kept constant, S decreases slightly as we move away from the conformal limit, but the change can hardly be larger than by a factor of one half (we cannot get too close to 1 as then the solution to the hierarchy problem goes away).

4 The higgsless limit

We consider the higgsless limit of the warped custodial model (the warped standard model does not make much sense in this limit, as it predicts $m_W = m_Z$), which amounts to taking the limit $\tilde{v} \to \infty$. In which case we can no longer expand in powers of $\tilde{v}$. However we can still expand in powers of SM masses, as long as there is a large enough gap between the electroweak scale and the KK scale. This is true in AdS$_{\mathbf{5}}$ where the gap is $m_Z^2/M_{\rm KK}^2 \sim 1/kL$. It seems to be the gap exists whenever the volume is large.

Note that in the previous

As for the fermions, the IR brane Yukawa interactions in the higgsless limit become simply IR brane Dirac mass terms. We could consider sending the brane masses $\tilde{m} \equiv y\tilde{v}$ to infinity, but it does not seem to lead to anything sensible. Therefore, in the higgsless limit we keep $\tilde{m}$ finite and, why not, much smaller than the KK scale. Under these assumptions, the fermionic sector is exactly the same as in the higgsyes case and all the mass formulae, profile etc. carry over without changes.

Let us now discuss the gauge boson masses in the higgless limit. Of course the approximate expressions for the gauge boson masses, eq. (3.48), are no longer valid. Taking the $\tilde{v} \to \infty$ limit of eq. (3.11) and eq. (3.15) we get the quantization conditions

$$g_L^2 S'(L, m_{W,n}) C(L, m_{W,n}) + g_R^2 S(L, m_{W,n}) C'(L, m_{W,n}) = 0 \tag{4.83}$$ `e.hwm`

$$g_L^2 S'(L, m_{Z,n}) C(L, m_{Z,n}) + g_R^2 S(L, m_{Z,n}) C'(L, m_{Z,n}) + m_{Z,n} a_L^{-2} g_Y^2 = 0 \tag{4.84}$$ `e.hzm`

and the lightest solutions in the two towers are identified with W and Z. Using Wronski, w more convenient form of these equations can be derived

$$(g_L^2 + g_R^2) S(L, m_{W,n}) C'(L, m_{W,n}) + m_{Z,n} a_L^{-2} g_L^2 = 0 \tag{4.85}$$ `e.hwmc`

$$(g_L^2 + g_R^2) S(L, m_{Z,n}) C'(L, m_{Z,n}) + m_{Z,n} a_L^{-2} (g_L^2 + g_Y^2) = 0 \tag{4.86}$$ `e.hzmc`

For the lowest solution, we assume $m_W, m_Z \ll M_{\rm KK}$. The we can expand the warped sines and cosines in powers of $m/M_{\rm KK}$ and solve the above equations perturbatively. Doing so to quartic order in m we find

$$\begin{aligned}
m_W^2 &= g_L^2 \frac{1}{(g_L^2+g_R^2)L\int_0^L a^{-2}(y)}\left[1+g_L^2\frac{1}{(g_L^2+g_R^2)L\int_0^L a^{-2}(y)}\left(I_1(L)+I_5(L)\right)\right]\\
m_Z^2 &= (g_L^2+g_Y^2)\frac{1}{(g_L^2+g_R^2)L\int_0^L a^{-2}(y)}\left[1+(g_L^2+g_Y^2)\frac{1}{(g_L^2+g_R^2)L\int_0^L a^{-2}(y)}\left(I_1(L)+I_5(L)\right)\right]
\end{aligned} \tag{4.87}$$

$$\begin{aligned}
m_W^2 &= \frac{g_L^2 v^2}{4}\\
m_Z^2 &= \frac{(g_L^2+g_Y^2)v^2}{4}\left[1+g_Y^2\frac{v^2}{4}\left(I_4(L)+I_5(L)\right)\right]+\mathcal{O}(v^6)
\end{aligned} \tag{4.88}$$

where we defined

$$v^2 = \frac{4}{(g_L^2+g_R^2)L\int_0^L a^{-2}(y)}\left[1+g_L^2\frac{v^2}{4}\left(I_4(L)+I_5(L)\right)\right] \tag{4.89}$$

e.hfs

In the last step we replaced I_1 with I_4 as the two are the same up to terms suppressed by the volume factor.

We see there is one qualitative change in the higgsless limit. Before, $m_W/M_{\rm KK}$ was a tunable parameter controlled by $\tilde{v}$. In the higgsless limit m_W is intimately tied to $M_{\rm KK}$. The precise relation depends on the background geometry and it turns out to be of the form $m_W^2 \sim M_{\rm KK}^2/\mathcal{V}$.

Let us take a look at the expansion of the profiles in the higgsless limit. The normalization constant for the W profiles is given, as before, by

$$(\alpha_{L,W})^{-2} = \int_0^L dy\left\{C^2(y,m_W)+\frac{g_R^2}{g_L^2}\left(\frac{C'(L,m_W)}{S'(L,m_W)}\right)^2 S^2(y,m_W)\right\} \tag{4.90}$$

Before we expanded till $m_W^2/M_{\rm KK}^2$. Now we expand till $m_W^4/M_{\rm KK}^4$ to see what happens. We get

$$\begin{aligned}
(\alpha_{L,W})^{-2} = L\Big\{1 &- 2m_W^2 L^{-1}\int^L\!\!\int y a^{-2}\\
&+m_W^4 L^{-1}\int^L\!(\textstyle\int y a^{-2})^2 + m_W^4\frac{g_R^2}{g_L^2}L\int^L\!(\int a^{-2})^2 + 2m_W^4 L^{-1}\int^L a^{-2}\int\!\!\int y a^{-2}\Big\}
\end{aligned} \tag{4.91}$$

In the higgsless limit we drop all volume suppressed terms. Thus, we can trade the y under the integrals for L before integrals (this can be shown by integration by parts).

$$\begin{aligned}
(\alpha_{L,W})^{-2} \approx L\Big\{1 &- 2m_W^2\int^L\!\!\int a^{-2}\\
&+m_W^4 L\int^L\!(\textstyle\int a^{-2})^2 + m_W^4\frac{g_R^2}{g_L^2}L\int^L\!(\int a^{-2})^2 + 2m_W^4\int^L a^{-2}\int\!\!\int a^{-2}\Big\}
\end{aligned} \tag{4.92}$$

The last term in the second line is indeed of order $m_W^4/M_{\rm KK}^4$, but the first two, because of the L in front can be rewritten as $m_W^2/M_{\rm KK}^2$ terms. After some integration by parts and using the explicit formula for the W mass we get, at the leading order:

$$(\alpha_{L,W})^{-2} \approx L\left\{1 - m_W^2(I_4(L) + I_5(L))\right\} \tag{4.93}$$

The same things happen for the Z normalization constant. The bottom line is that The SM gauge boson profiles are those of eq. (3.17) and eq. (3.18) with $I_1(L) \to (I_4(L) + I_5(L))/2$. The normalization factor changes for two reasons. One is that in the approximations we are using now $I_1 = I_4$, the difference being suppressed by the volume factor. Moreover there are $\mathcal{V} m_W^4/M_{\rm KK}^4$ terms which are of the same order as $m_W^2/M_{\rm KK}^2$ terms in the higgsless case and they contribute $-I_4/2 + I_5/2$ to the normalization.

The full set of corrections to the SM lagrangian is given by

$$\begin{aligned}
\alpha_T &= \frac{1}{2}\tilde{g}_Y^2\left(I_4(L) + I_5(L)\right) \\
\alpha_S &= 0 \\
\beta_j &= \frac{1}{2}\left(I_4(L) + I_5(L)\right) - J_2(L, M_j) \\
\gamma_j &= \frac{1}{2}\left(I_4(L) + I_5(L)\right) - J_2(L, M_j) + J_3(L, M_j) - \frac{\tilde{g}_R^2 t_R^3}{\tilde{g}_Y^2 Y} J_3(L, M_j)
\end{aligned} \tag{4.94}$$

or, swiping the universal part of the vertex correction under S and T

$$\begin{aligned}
\alpha_T &= 0 \\
\alpha_S &= \frac{g_L g_Y}{2}\left(I_4(L) + I_5(L)\right) \\
\beta_j &= -J_2(L, M_j) \\
\gamma_j &= -J_2(L, M_j) + J_3(L, M_j) - \frac{g_R^2 t_{R,j}^3}{g_Y^2 Y_j} J_3(L, M_j)
\end{aligned} \tag{4.95}$$

We can write the S parameter as[3]

$$S = 4\pi v^2\left(I_4(L) + I_5(L)\right) \sim \frac{12\pi}{\mathcal{V}(g_L^2 + g_R^2)} \tag{4.96}$$

e.hs

We would need a large volume factor to suppress the gauge contribution to S. Recall that in AdS$_{\mathbf{5}}$ $\mathcal{V} \sim 30$, which is not enough. Note that a volume factor large enough to make $S < 0.3$ would also make the KK scale heavier than 1 TeV. It seems unlikely that the longitudinal WW scattering amplitude could be unitarized by such heavy resonances. Therefore the only way to bring S down to acceptable level is by a suitable choice of fermionic profiles. More precisely, one assumes that the fermion bulk masses corresponding to the light SM generations are such that the integrals J_2 and J_3 are not suppressed. Moreover, the bulk masses for the different light SM fields should be almost equal, $M_j \approx M_{\rm ref}$. By eq. (1.20), choosing $\Delta\beta = J_2$, $\Delta\gamma = J_2 - J_3$, the universal part of the vertex corrections can be traded for a shift in S:

$$\Delta S = -8\pi v^2\left(J_2(L, M_{\rm ref}) - \frac{1}{2}J_3(L, M_{\rm ref})\right) \tag{4.97}$$

[3] This expression is consistent with the one derived for general metrics in [5].

Since the shift is negative, we can cancel the positive gauge contribution in eq. (4.96) (in AdS$_5$, the cancellation occurs for $M \sim 0.47k$). Note however that there will still remain sizable non-universal vertex corrections, $\gamma_j = -(g_R^2 t^3_{R,j}/g_Y^2 Y_j) J_3(L, M_j)$, which depend on the embedding of the SM fields into $SU(2)_R$. Thus constraints based just on the oblique parameters can be misleading and one would need a more refined fit.

Appendix A Background integrals

a.i It is convenient to define integral functions of the geometrical background:

$$
\begin{aligned}
I_1(x_5) &= L^{-1} \int_0^{x_5} \int_0^{y} y' a^{-2}(y') \\
I_2(x_5) &= \int_0^{x_5} y a^{-2}(y) \\
I_3(x_5) &= L \int_0^{x_5} a^{-2}(y) \\
I_4(x_5) &= \int_0^{x_5} \int_0^{y} a^{-2}(y') \\
I_5(x_5) &= \frac{\int_0^{x_5} a^{-2}(y) \int_0^{y} \int_0^{y'} a^{-2}(y'')}{\int_0^L a^{-2}(y)}
\end{aligned}
\tag{A.1}
$$

Note that $I_4(L) = I_3(L) - I_2(L)$. All the integrals have dimension $[mass]^{-2}$ and are expected to be of order $1/M_{KK}^2$. Moreover, I_1 and I_4 are *safe* while I_2 and I_3 are volume enhanced: because of the y factor in the integral the final result comes out proportional to L/M_{KK}^2 rather than $1/M_{KK}^2$. For example, in AdS$_5$ $I_2(L)$ and $I_3(L)$ are enhanced by $\log M_{\rm Pl}/M_{\rm EW} \sim 30$ wrt to the KK scale. In general, in backgrounds solving the hierarchy problem we expect the volume to be numerically large, say between 5 and 50. Therefore, $I_2(L)$ and $I_3(L)$ are *vicious*, and we better avoid them in sensitive observables.

We also define yet another class of integrals

$$
J_n(L, M) = \frac{\int_0^L a^{-1}(y) e^{-2My} I_n(y)}{\int_0^L a^{-1}(y) e^{-2My}}
\tag{A.2}
$$

that depend on the bulk fermion masses.

Let as regard the values of these integrals in specific backgrounds

Flat: $a(x_5) = 1$

$$
I_1(L) = \frac{1}{6} L^2 \quad I_2(L) = \frac{1}{2} L^2 \quad I_3(L) = L^2 \quad I_4(L) = \frac{1}{2} L^2
\tag{A.3}
$$

In terms of the KK scale, $M_{\rm KK} = \pi/L$,

$$
I_1(L) = \frac{\pi^2}{6M_{\rm KK}^2} \quad I_2(L) = \frac{\pi^2}{2M_{\rm KK}^2} \quad I_3(L) = \frac{\pi^2}{M_{\rm KK}^2} \quad I_4(L) = \frac{\pi^2}{2M_{\rm KK}^2}
\tag{A.4}
$$

AdS$_5$: $a(x_5) = \exp(-kx_5)$

We find the integrals:

$$
\begin{aligned}
I_1(L) &= \frac{1}{4k^2a_L^2}\left[1-\frac{1}{kL}+a_L^2\left(1+\frac{1}{kL}\right)\right] \\
I_2(L) &= \frac{1}{4k^2a_L^2}\left[2kL-1+a_L^2\right] \\
I_3(L) &= \frac{1}{4k^2a_L^2}\left[2kL-a_L^2 2kL\right] \\
I_4(L) &= \frac{1}{4k^2a_L^2}\left[1-a_L^2\left(2kL+1\right)\right] \\
J_2(L,ck) &= \frac{1}{4k^2a_L^2}\frac{(2c-1)(2kL(2c-3)-2c+5)+(2c-3)^2a_L^2-4a_L^{3-2c}}{(2c-3)^2(1-a_L^{1-2c})} \\
J_3(L,ck) &= \frac{1}{4k^2a_L^2}2kL\frac{(2c-1)-(2c-3)a_L^2-2a_L^{3-2c}}{(2c-3)(1-a_L^{1-2c})}
\end{aligned}
\tag{A.5}
$$

For $a_L \ll 1$ and inserting $M_{\rm KK} = \pi k a_L$

$$
I_1(L)\approx\frac{\pi^2}{4M_{\rm KK}^2}\left[1-\frac{1}{kL}\right]\quad I_4(L)\approx\frac{\pi^2}{4M_{\rm KK}^2}\quad I_2(L)\approx\frac{\pi^2}{4M_{\rm KK}^2}\left[2kL-1\right]\quad I_3(L)\approx\frac{\pi^2}{4M_{\rm KK}^2}2kL
\tag{A.6}
$$

For the fermionic integral the approximation depends on c. For $c \gg 3/2$

$$
J_2(L,ck)\approx\frac{\pi^2}{(2c-3)M_{\rm KK}^2}a_L^2 \qquad J_3(L,ck)\approx\frac{\pi^2}{(2c-3)M_{\rm KK}^2}kLa_L^2
\tag{A.7}
$$

they are Planck scale suppressed. For $1/2 \ll c \ll 3/2$

$$
J_2(L,ck)\approx\frac{\pi^2}{4M_{\rm KK}^2}a_L^{2c-1}\frac{(2c-1)(2kL(3-2c)+2c-5)}{(3-2c)^2} \qquad J_3(L,ck)\approx\frac{\pi^2}{4M_{\rm KK}^2}2kLa_L^{2c-1}\frac{2c-1}{3-2c}
\tag{A.8}
$$

they are still suppressed by a_L^{2c-1}. For $c \ll 1/2$

$$
J_2(L,ck)\approx\frac{\pi^2}{4M_{\rm KK}^2}\frac{(1-2c)(2kL(3-2c)-2c+5)}{(3-2c)^2} \qquad J_3(L,ck)\approx\frac{\pi^2}{4M_{\rm KK}^2}2kL\frac{1-2c}{3-2c}
\tag{A.9}
$$

they turn vicious: they are log enhanced. But note that $J_3 - J_2$ is safe. Note also that for $c = 1/2$

$$
J_2(L,k/2)\approx\frac{\pi^2}{4M_{\rm KK}^2}\left(1-\frac{1}{kL}\right) \qquad J_3(L,k/2)\approx\frac{\pi^2}{4M_{\rm KK}^2}
\tag{A.10}
$$

the integrals are not vicious yet.

Power: $a(x_5) = \left(1-\frac{kx_5}{\gamma-1}\right)^\gamma$

The integrals become too complicated to display them in their full glory. For $a_L \ll 1$ and inserting $M_{\rm KK} = \pi k a_L^{1-1/\gamma}$ we obtain

$$
\begin{aligned}
I_1(L) &\approx \frac{\pi^2}{2M_{\rm KK}^2}\frac{\gamma-1}{2\gamma-3}\left[1-\frac{2(\gamma-1)}{kL(2\gamma-1)}\right] \\
I_4(L) &\approx \frac{\pi^2}{2M_{\rm KK}^2}\frac{\gamma-1}{2\gamma-1} \\
I_5(L) &\approx \frac{\pi^2}{2M_{\rm KK}^2}\frac{\gamma-1}{4\gamma-3}
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
I_2(L) &\approx \frac{\pi^2}{2M_{\rm KK}^2}kLa_L^{-1/\gamma}\left[1-\frac{\gamma-1}{kL(2\gamma-1)}\right] \\
I_3(L) &\approx \frac{\pi^2}{2M_{\rm KK}^2}kLa_L^{-1/\gamma}\frac{2(\gamma-1)}{2\gamma-1}
\end{aligned} \tag{A.12}
$$

As we move γ away from the conformal limit the effect of volume enhancement becomes more dramatic: $I_2(L)$ and $I_3(L)$ grow as a power of the large number $1/a_L$ (rather than a logarithm as in AdS$_\mathbf{5}$). The remaining integrals, those that are not volume enhanced, are of order $1/M_{\rm KK}^2$ and weakly depend on the shape of the warp factor. In consequence, we cannot significantly reduce the S parameter by varying γ (γ too close to 1 is not attractive as we need to fine-tune L to generate large hierarchy).

Appendix B Fermions, once and for all

a.f

We discuss the KK decomposition of fermion, bulk equations of motion and its formal solutions. The quadratic fermionic lagrangian in the warped background has the following form

$$
\mathcal{L} = ia^3\overline{\psi}\gamma_\mu\partial_\mu\psi - a^4\overline{\psi}\left(\gamma_5\partial_5 + \gamma_5 2\frac{a'}{a} + M\right)\psi \tag{B.13}
$$

The possible boundary mass term will be discussed later. The factor $2a'/a$ comes from the spin connection (or is required by hermiticity, if you wish).

The bulk equations have the following structure:

$$
\begin{aligned}
\left(\partial_5 + 2\frac{a'}{a} + M\right) f_R &= za^{-1}f_L \\
\left(-\partial_5 - 2\frac{a'}{a} + M\right) f_L &= za^{-1}f_R
\end{aligned} \tag{B.14}
$$

We define hat profiles as

$$
f_R = a^{-2}e^{-Mx_5}\hat{f}_R \qquad f_L = a^{-2}e^{Mx_5}\hat{f}_L \tag{B.15}
$$

that satisfy simpler equations

$$
\begin{aligned}
\partial_5\hat{f}_R &= za^{-1}e^{2Mx_5}\hat{f}_L \\
\partial_5\hat{f}_L &= -za^{-1}e^{-2Mx_5}\hat{f}_R
\end{aligned} \tag{B.16}
$$

Disentangling, the hatted profiles satisfy the second order differential equations

$$
\begin{aligned}
\left[\partial_5^2 + \left(\frac{a'}{a} + 2M\right)\partial_5 + \frac{z^2}{a^2}\right]\hat{f}_L &= 0 \\
\left[\partial_5^2 + \left(\frac{a'}{a} - 2M\right)\partial_5 + \frac{z^2}{a^2}\right]\hat{f}_R &= 0
\end{aligned}
\tag{B.17}
$$

We denote the two independent solutions of the *first* of these equations by $C_M(x_5, z)$, $S_M(x_5, z)$ (consequently, the *second* is solved by $C_{-M}(x_5, z)$, $S_{-M}(x_5, z)$). We define them such that they satisfy the following boundary conditions at $x_5 = 0$,

$$
C_M(0, z) = 1 \qquad C'_M(0, z) = 0 \qquad S_M(0, z) = 0 \qquad S'_M(0, z) = z \tag{B.18}
$$

Thus, we can write the general fermion profiles as

$$
\begin{aligned}
\hat{f}_L &= \alpha S_M(x_5, z) + \beta C_M(x_5, z) \\
\hat{f}_R &= -\alpha C_{-M}(x_5, z) + \beta S_{-M}(x_5, z)
\end{aligned}
\tag{B.19}
$$

The integration constant in $\hat{f}_R$ has to much those in $\hat{f}_L$, which can be proven by use of eq. (B.16) and the boundary conditions. For completeness, the unhatted profiles

$$
\begin{aligned}
f_L &= a^{-2}(x_5)e^{Mx_5}\left(\alpha S_M(x_5, z) + \beta C_M(x_5, z)\right) \\
f_R &= a^{-2}(x_5)e^{-Mx_5}\left(-\alpha C_{-M}(x_5, z) + \beta S_{-M}(x_5, z)\right)
\end{aligned}
\tag{B.20}
$$

In the case where f_R satisfies Dirichlet BC on the UV brane we have (below $\alpha \neq \alpha$)

$$
\begin{aligned}
f_L &= \alpha a^{-2}(x_5)e^{Mx_5}C_M(x_5, z) \\
f_R &= \alpha a^{-2}(x_5)e^{-Mx_5}S_{-M}(x_5, z)
\end{aligned}
\tag{B.21}
$$

while in the case where f_L satisfies Dirichlet BC on the UV brane we have

$$
\begin{aligned}
f_L &= -\alpha a^{-2}(x_5)e^{Mx_5}S_M(x_5, z) \\
f_R &= \alpha a^{-2}(x_5)e^{-Mx_5}C_{-M}(x_5, z)
\end{aligned}
\tag{B.22}
$$

Comments:

- The funny notation I introduced serves to stress the analogy to cosines and sines. We will call C_M the warped (fermionic) cosine and S_M the warped (fermionic) sine. There is a relationship between the warped sine and cosine:

  $$
  S'_M(x_5, z) = z a^{-1}(x_5)e^{-2Mx_5}C_{-M}(x_5, z) \qquad C'_M(x_5, z) = -z a^{-1}(x_5)e^{-2Mx_5}S_{-M}(x_5, z) \tag{B.23}
  $$

  e.wscd

  which vaguely resemble something.... For $a(x_5) = 1$, and $M = 0$ we indeed recover the familiar trig functions: $C_M = \cos(x_5 z)$, $S_M = \sin(x_5 z)$

- There exists a generalization of $\cos^2 + \sin^2 = 1$ to warped trigs. It is called the Wronskian formula:

  $$
  S'_M(x_5, z)C_M(x_5, z) - C'_M(x_5, z)S_M(x_5, z) = z e^{-2Mx_5}a^{-1}(x_5) \tag{B.24}
  $$

  Using eq. (B.23) we can also derive the Crowian formula

  $$
  C_M(x_5, z)C_{-M}(x_5, z) + S_M(x_5, z)S_{-M}(x_5, z) = 1 \tag{B.25}
  $$

  which clearly resembles the familiar primary school formula...

- For the fermions with left-handed zero modes we flip the sign of the bulk mass term. Then in all the above formulae we have to flip $M \to -M$. This trick, apart from introducing some mess, also has some advantage. The zero mode profile is always $f \sim a^{-2} e^{-Mx_5}$ for both left- and right-handed fermions. In particular, in $\mathrm{AdS}_\mathbf{5}$ the flat profile always correspond to $c = 1/2$.

For small z we can integrate eq. (B.17) perturbatively. This yields the small mass asymptotic formulae:

$$\begin{aligned} C_M(x_5, z) &= 1 - z^2 \int_0^{x_5} e^{-2My} a^{-1}(y) \int_0^y e^{2My'} a^{-1}(y') + \mathcal{O}(z^4) \\ S_M(x_5, z) &= z \int_0^{x_5} a^{-1}(y) e^{-2My} \left[1 - z^2 \int_0^y a^{-1}(y') e^{2My'} \int_0^{y'} a^{-1}(y'') e^{-2My''} + \mathcal{O}(z^4) \right] \end{aligned} \tag{B.26}$$

Now, we consider the boundary conditions for the profiles and how they depend on the presence of boundary terms.

- No boundary terms.
  In this basic case one of the chiralities has to vanish at the boundary.

  $$f_R(0) = 0 \quad \text{xor} \quad f_L(0) = 0 \tag{B.27}$$

  The profiles satisfying the UV conditions are given by

  $$\begin{array}{lcl} f_{L,n} = \alpha_n a^{-2}(x_5) e^{Mx_5} C_M(x_5, m_n) & \text{xor} & f_{L,n} = -\alpha_n a^{-2}(x_5) e^{Mx_5} S_M(x_5, m_n) \\ f_{R,n} = \alpha_n a^{-2}(x_5) e^{-Mx_5} S_{-M}(x_5, m_n) & & f_{R,n} = \alpha_n a^{-2}(x_5) e^{-Mx_5} C_{-M}(x_5, m_n) \end{array} \tag{B.28}$$

  The IR boundary conditions

  $$f_R(L) = 0 \quad \text{xor} \quad f_L(L) = 0 \tag{B.29}$$

  determine allowed values of m_n. The constants α_n are fixed by the normalization conditions

  $$\int_0^L a^3 |f_{L,n}|^2 = 1 = \int_0^L a^3 |f_{R,n}|^2 \tag{B.30}$$

  Thanks to the equations of motion connecting f_L and f_R both left and right profile normalization yield the same value of α.

- Brane kinetic terms.
  We consider a brane kinetic term for the chirality that would be non-vanishing in the absence thereof (can i say it more simply?)

  $$\mathcal{L}_{\mathrm{UV}} = i l_0 a^3 \delta(0) \overline{\psi}_L \gamma^\mu \partial_\mu \psi_L \quad \text{xor} \quad \mathcal{L}\mathrm{UV} = i l_0 a^3 \delta(0) \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R \tag{B.31}$$

  $$\mathcal{L}_{\mathrm{UV}} = i l_L a^3 \delta(L) \overline{\psi}_L \gamma^\mu \partial_\mu \psi_L \quad \text{xor} \quad \mathcal{L}\mathrm{UV} = i l_L a^3 \delta(L) \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R \tag{B.32}$$

The UV boundary conditions now read

$$f_R(0) = m_n l_0 a_0^{-1} f_L(0) \quad \text{xor} \quad f_L(0) = -m_n l_0 a_0^{-1} f_R(0) \tag{B.33}$$

The profiles

$$\begin{aligned} f_{L,n} &= \alpha_n a^{-2}(x_5) e^{M x_5} \left(C_M(x_5, m_n) - m_n l_0 S_M(x_5, m_n) \right) \\ f_{R,n} &= \alpha_n a^{-2}(x_5) e^{-M x_5} \left(S_{-M}(x_5, m_n) + m_n l_0 C_{-M}(x_5, m_n) \right) \end{aligned} \tag{B.34}$$

xor

$$\begin{aligned} f_{L,n} &= -\alpha_n a^{-2}(x_5) e^{M x_5} \left(S_M(x_5, m_n) + m_n l_0 C_M(x_5, m_n) \right) \\ f_{R,n} &= \alpha_n a^{-2}(x_5) e^{-M x_5} \left(C_{-M}(x_5, m_n) - m_n l_0 S_{-M}(x_5, m_n) \right) \end{aligned} \tag{B.35}$$

The IR boundary conditions

$$f_{R,n}(L) = -m_n l_L a_L^{-1} f_{L,n}(L) \quad \text{xor} \quad f_{L,n}(L) = m_n l_L a_L^{-1} f_{R,n}(L) \tag{B.36}$$

determine allowed values of m_n. The constants α_n are fixed by the normalization conditions

$$\int_0^L a^3 \left(1 + l_0 \delta(0) + l_L \delta(0)\right) |f_{L,n}|^2 = 1 = \int_0^L a^3 |f_{R,n}|^2 \tag{B.37}$$

xor the 3 permutations thereof.

- Mass mixing with a UV boundary fermion

Consider now a UV boundary fermion $\tilde{\psi}_R$ mixing via a Dirac mass term with a bulk fermion

$$\mathcal{L}_{\rm UV} = i a^3 \delta(0) \overline{\tilde{\psi}_R} \gamma^\mu \partial_\mu \tilde{\psi}_R + -(a^4 \delta(0) \sqrt{\tilde{M}} \overline{\psi_L} \psi_R + \text{h.c.}) \tag{B.38}$$

The new fermion also has KK expansion $\tilde{\psi}_R(x) = \tilde{f}_{R,n} \psi_R(x)$ but its profiles are not profiles but a collection of numbers. Diagonalization in the KK basis fixes $\tilde{f}_{R,n} = a_0 \sqrt{\tilde{M}}/m_n f_{L,n}(0)$. The UV boundary condition for the bulk fermion happens to be

$$f_{R,n}(0) = -a_0 \frac{\tilde{M}}{m_n} f_{L,n}(0) \tag{B.39}$$

- Mass mixing via a UV boundary fermion

Consider now a UV boundary fermion $\tilde{\psi}_R$ mixing via a Dirac mass term with two bulk fermions

$$\mathcal{L}_{\rm UV} = i a^3 \delta(0) \overline{\tilde{\psi}_R} \gamma^\mu \partial_\mu \tilde{\psi}_R + -a^4 \delta(0) (\sqrt{\tilde{M}_1} \overline{\psi_L^1} \psi_R + \sqrt{\tilde{M}_2} \overline{\psi_L^2} \psi_R + \text{h.c.}) \tag{B.40}$$

Diagonalization in the KK basis fixes the brane fermion "profile",

$$\tilde{f}_{R,n} = a_0 \sqrt{\tilde{M}_1}/m_n f_{L,n}^1(0) + a_0 \sqrt{\tilde{M}_2}/m_n f_{L,n}^2(0) \tag{B.41}$$

The UV boundary condition for the bulk fermion happens to be

$$\begin{aligned} f^1_{R,n}(0) &= -a_0\frac{\tilde{M}_1}{m_n}f^1_{L,n}(0) - a_0\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}f^2_{L,n}(0) \\ f^2_{R,n}(0) &= -a_0\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}f^1_{L,n}(0) - a_0\frac{\tilde{M}_2}{m_n}f^2_{L,n}(0) \end{aligned} \tag{B.42}$$

The bulk fermion profiles can be written as

$$\begin{aligned} f^1_{L,n} &= a^{-2}(x_5)e^{M_1x_5}\left(\alpha^1_n[C_{M_1}(x_5,m_n) + \frac{\tilde{M}_1}{m_n}S_{M_1}(x_5,m_n)] + \alpha^2_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}S_{M_1}(x_5,m_n)\right) \\ f^1_{R,n} &= a^{-2}(x_5)e^{-M_1x_5}\left(\alpha^1_n[S_{-M_1}(x_5,m_n) - \frac{\tilde{M}_1}{m_n}C_{-M_1}(x_5,m_n)] - \alpha^2_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}C_{-M_1}(x_5,m_n)\right) \\ f^2_{L,n} &= a^{-2}(x_5)e^{M_2x_5}\left(\alpha^1_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}S_{M_2}(x_5,m_n) + \alpha^2_n[C_{M_2}(x_5,m_n) + \frac{\tilde{M}_2}{m_n}S_{M_2}(x_5,m_n)]\right) \\ f^2_{R,n} &= a^{-2}(x_5)e^{-M_2x_5}\left(-\alpha^1_n\frac{\sqrt{\tilde{M}_1\tilde{M}_2}}{m_n}C_{-M_2}(x_5,m_n) + \alpha^2_n[S_{-M_2}(x_5,m_n) - \frac{\tilde{M}_2}{m_n}C_{-M_2}(x_5,m_n)]\right) \end{aligned} \tag{B.43}$$

- Boundary Dirac mass terms.
  Now, consider IR brane localized Dirac mass terms

$$\mathcal{L} = -a^4\delta(L)\tilde{m}\overline{\psi}_L\chi_R + \text{h.c.} \quad \text{xor} \quad \mathcal{L} = -a^4\delta(L)\tilde{m}\overline{\psi}_R\chi_L + \text{h.c.} \tag{B.44}$$

where χ is some other bulk fermion. The equation of motion are modfied as

$$\begin{aligned} \left(\partial_5 + 2\frac{a'}{a} + M\right)f^\psi_R &= za^{-1}f^\psi_L - \tilde{m}\delta(L)f^\chi_R \\ \left(-\partial_5 - 2\frac{a'}{a} + M\right)f^\psi_L &= za^{-1}f^\psi_R - \tilde{m}\delta(L)f^\chi_R \end{aligned} \tag{B.45}$$

This results in IR boundary conditions

$$f^\psi_R(L) = \tilde{m}f^\chi_R(L) \quad \text{xor} \quad f^\psi_L(L) = -\tilde{m}f^\chi_L(L) \tag{B.46}$$

Appendix C Fun with Bessel

a.b

The Bessel equation:

$$x^2f''(x) + xf'(x) + (x^2 - n^2)f(x) = 0 \tag{C.47}$$

has two independent solutions called the Bessel $J_n(x)$ and the Neumann $Y_n(x)$. Furthermore, the equation

$$x^2f''(x) - \alpha xf'(x) + ((bx)^2 - \beta)f(x) = 0 \tag{C.48}$$

is solved by

$$f(x) = x^{(1+\alpha)/2} Z_n(bx) \tag{C.49}$$

where $Z_n = J_n, Y_n$ and $n^2 = (1+\alpha)^2/4 + \beta$. Yet another incarnation of the Bessel equation is useful. The equation

$$\left[\partial_5^2 - \alpha k \partial_5 + (z^2 a^{-2} - \beta k^2)\right] f = 0 \tag{C.50}$$

with $a = e^{-kx_5}$ is solved by

$$f = a^{-\alpha/2} Z_n\left(\frac{z}{ka}\right) \qquad n^2 = \alpha^2/4 + \beta \tag{C.51}$$

The modified Bessels I_n and K_n are two independent solution of the Euclidean Bessel equation

$$x^2 f''(x) + x f'(x) - (x^2 + n^2) f(x) = 0 \tag{C.52}$$

They are related to ordinary Bessels as

$$I_n(x) = i^{-n} J_n(ix) \qquad K_n(x) = -\frac{\pi}{2} i^n [Y_n(ix) - i J_n(ix)] \tag{C.53}$$

Asymptotic behaviour for $x \ll 1$

$$\begin{aligned} J_n(x) &\approx \frac{(x/2)^n}{\Gamma(n+1)} & I_n(x) &\approx \frac{(x/2)^n}{\Gamma(n+1)} \\ Y_n(x) &\approx -\frac{\Gamma(n)2^n}{\pi x^n} & K_n(x) &\approx \frac{\Gamma(n)2^{n-1}}{x^n}(?) \\ Y_0(x) &\approx \frac{2}{\pi}(\log(x/2) + \gamma) & K_0(x) &\approx -(\log(x/2) + \gamma) \end{aligned} \tag{C.54}$$

Asymptotic behaviour for $x \gg 1$

$$\begin{aligned} J_n(x) &\approx \sqrt{\frac{2}{\pi x}} \cos(x - n\pi/2 - \pi/4) & I_n(x) &\approx \frac{1}{\sqrt{2\pi x}} e^x \\ Y_n(x) &\approx \sqrt{\frac{2}{\pi x}} \sin(x - n\pi/2 - \pi/4) & K_n(x) &\approx \sqrt{\frac{\pi}{2x}} e^{-x} \end{aligned} \tag{C.55}$$

The Wronskian formula

$$\begin{aligned} Y_n(x) J_{n+1}(x) - J_n(x) Y_{n+1}(x) &= 2/(\pi x) \\ K_n(x) I_{n+1}(x) - I_n(x) K_{n+1}(x) &= 1/x \end{aligned} \tag{C.56}$$

Recursion relations:

$$\begin{aligned} J_n'(x) + \frac{n}{x} J_n(x) &= J_{n-1}(x) \\ -J_n'(x) + \frac{n}{x} J_n(x) &= J_{n+1}(x) \end{aligned} \tag{C.57}$$

The same formulae hold for Y_n and I_n. However some dumbhead defined the McDonald function K_n such that the first of the recursion relation has a flipped sign:

$$\begin{aligned} K_n'(x) + \frac{n}{x} K_n(x) &= -K_{n-1}(x) \\ -K_n'(x) + \frac{n}{x} K_n(x) &= K_{n+1}(x) \end{aligned} \tag{C.58}$$

Kill McDonald, indeed.

References

agdema [1] K. Agashe, A. Delgado, M. J. May and R. Sundrum, "RS1, custodial isospin and precision tests," JHEP **0308** (2003) 050 [arXiv:hep-ph/0308036].

agcopo [2] K. Agashe, R. Contino and A. Pomarol, "The minimal composite Higgs model," Nucl. Phys. B **719** (2005) 165 [arXiv:hep-ph/0412089].

agcoda [3] K. Agashe, R. Contino, L. Da Rold and A. Pomarol, "A custodial symmetry for Z b anti-b," Phys. Lett. B **641** (2006) 62 [arXiv:hep-ph/0605341].

caposa [4] M. Carena, E. Ponton, J. Santiago and C. E. M. Wagner, "Light Kaluza-Klein states in Randall-Sundrum models with custodial SU(2)," Nucl. Phys. B **759** (2006) 202 [arXiv:hep-ph/0607106].

hisa [5] J. Hirn and V. Sanz, "The fifth dimension as an analogue computer for strong interactions at the LHC," arXiv:hep-ph/0612239.