Andrzej Trautman
Institute of Theoretical Physics
University of Warsaw

The rudiments of twistor theory

On the Road to Reality with Roger Penrose
Warszawa, 17 May 2010

Introduction

Roger Penrose played a most important role in the development of GRT during the 2nd half of XX century; twistors are his discovery that influenced both mathematics and physics

Introduction

Roger Penrose played a most important role in the development of GRT during the 2nd half of XX century; twistors are his discovery that influenced both mathematics and physics

This talk: elementary introduction prepared for those participants who had little contact with the subject. My arrogance and conceit...

Rudiments: many important results, generalizations and directions of research will not even be mentioned.

Basic idea: to associate, in a natural way, a complex analytic manifold $\mathscr{H}$ with a real manifold $\mathscr{M}$ so that the geometry and physics on $\mathscr{M}$ are expressed by holomorphic data on $\mathscr{H}$.

Basic idea: to associate, in a natural way, a complex analytic manifold $\mathscr{H}$ with a real manifold $\mathscr{M}$ so that the geometry and physics on $\mathscr{M}$ are expressed by holomorphic data on $\mathscr{H}$.

Penrose 1967 $\mathscr{M}$=Minkowski space $\mathscr{H} = \mathbb{C}\mathsf{P}_3$

Basic idea: to associate, in a natural way, a complex analytic manifold $\mathscr{H}$ with a real manifold $\mathscr{M}$ so that the geometry and physics on $\mathscr{M}$ are expressed by holomorphic data on $\mathscr{H}$.

Penrose 1967 $\mathscr{M}$=Minkowski space $\mathscr{H} = \mathbb{C}\mathsf{P}_3$

Atiyah, Hitchin, Singer 1978

$\mathscr{M}$=self-dual 4-dim Riemannian manifold

$\mathscr{H}$=bundle of directions of 2-spinors

in particular, if $\mathscr{M} = \mathbb{S}_4$ then $\mathscr{H} = \mathbb{C}\mathsf{P}_3$

Basic idea: to associate, in a natural way, a complex analytic manifold $\mathscr{H}$ with a real manifold $\mathscr{M}$ so that the geometry and physics on $\mathscr{M}$ are expressed by holomorphic data on $\mathscr{H}$.

Penrose 1967 $\mathscr{M}$=Minkowski space $\mathscr{H} = \mathbb{C}\mathsf{P}_3$

Atiyah, Hitchin, Singer 1978

$\mathscr{M}$=self-dual 4-dim Riemannian manifold

$\mathscr{H}$=bundle of directions of 2-spinors

in particular, if $\mathscr{M} = \mathbb{S}_4$ then $\mathscr{H} = \mathbb{C}\mathsf{P}_3$

Hitchin 1982 minitwistors $\mathscr{M} = \mathbb{R}^3$ $\mathscr{H} = T\mathbb{C}\mathsf{P}_1$

(set of all oriented lines in $\mathscr{M}$)

Penrose's motivation:

(i) Description of *Reality* proceeds from phenomenology (continues functions suffice) through differentiable functions of classical physics to *real analytic* functions of fundamental physics where elliptic eqs appear

Penrose's motivation:

(i) Description of *Reality* proceeds from phenomenology (continues functions suffice) through differentiable functions of classical physics to *real analytic* functions of fundamental physics where elliptic eqs appear

complex numbers and *holomorphic* functions in quantum theories and also in special solutions of wave and Einstein eqs; Kerr theorem; algebraic geometry as an even more rigid structure?

Penrose's motivation:

(i) Description of *Reality* proceeds from phenomenology (continues functions suffice) through differentiable functions of classical physics to *real analytic* functions of fundamental physics where elliptic eqs appear

complex numbers and *holomorphic* functions in quantum theories and also in special solutions of wave and Einstein eqs; Kerr theorem; algebraic geometry as an even more rigid structure?

(ii) Space(-time) points are not physical, (thin) *rays of light* relatively easy to realize

(iii) Importance of *massless* particles and fields — *conformal* geometry; action of conformal group requires *compactification* of flat space (recall $x \mapsto 1/x$ is conformal)

(iii) Importance of *massless* particles and fields — *conformal* geometry; action of conformal group requires *compactification* of flat space (recall $x \mapsto 1/x$ is conformal)

(iv) Need to find a new geometry to *connect gravitation with quantum theory*

(iii) Importance of *massless* particles and fields — *conformal* geometry; action of conformal group requires *compactification* of flat space (recall $x \mapsto 1/x$ is conformal)

(iv) Need to find a new geometry to *connect gravitation with quantum theory*

(v) The *special role of dim 4*: curvatures are 2-forms; only in dim 4 curvatures can be self-dual and there is the related decomposition $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$; the special place of dim 4 confirmed by Donaldson's discovery of exotic smooth structures on $\mathbb{R}^4$

The celestial sphere

can be identified with the complex projective line

Consider light **ray** (null line) $\ell(t, x, y, z) \in \mathbb{R}\mathsf{P}_3$,

$$t^2 - x^2 - y^2 - z^2 = 0.$$

Here $\ell(v)$ denotes the line spanned by the vector $v \in V^\times = V \setminus \{0\}$. (Most authors write $[v]$ instead of $\ell(v)$)

The celestial sphere

can be identified with the complex projective line

Consider light **ray** (null line) $\ell(t, x, y, z) \in \mathbb{R}\mathsf{P}_3$,

$$t^2 - x^2 - y^2 - z^2 = 0.$$

Here $\ell(v)$ denotes the line spanned by the vector $v \in V^\times = V \setminus \{0\}$. (Most authors write $[v]$ instead of $\ell(v)$)

Every such ray contains one point with $t = 1$ so that the set of all rays through one point – the celestial sphere – is identified with

$$\mathbb{S}_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

For every $(x, y, z) \in \mathbb{S}_2$ the equation

$$\begin{pmatrix} 1+z & x-\mathrm{i}y \\ x+\mathrm{i}y & 1-z \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0, \qquad (\xi, \eta) \in S = \mathbb{C}^2$$

has solutions that form a complex line in $\mathbb{C}^2$.

For every $(x, y, z) \in \mathbb{S}_2$ the equation

$$\begin{pmatrix} 1+z & x-\mathrm{i}y \\ x+\mathrm{i}y & 1-z \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0, \qquad (\xi, \eta) \in S = \mathbb{C}^2$$

has solutions that form a complex line in $\mathbb{C}^2$.

This gives a bijection (diffeomorphism)

$$\mathbb{S}_2 \to \mathbb{C}\mathsf{P}_1 : \quad (x, y, z) \mapsto \ell(\xi, \eta)$$

and induces a complex structure on $\mathbb{S}_2$ that agrees with its metric so that the 2-sphere is a Hermitian (even Kähler) manifold.

The action of the group $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Spin}(1,3)$ on the space S of spinors induces conformal (Möbius) transformations of the 2-sphere.

The action of the group $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Spin}(1,3)$ on the space S of spinors induces conformal (Möbius) transformations of the 2-sphere. Draw lessons:

(i) to describe conformal transformation in 2 dimensions one has to use the spin (Lorentz) group in 4 space-time dimensions; this generalizes: use $\mathrm{Spin}(p+1,q+1)$ to obtain conformal transformations of a compactified flat space with metric of signature (p,q);

The action of the group $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Spin}(1,3)$ on the space S of spinors induces conformal (Möbius) transformations of the 2-sphere. Draw lessons:

(i) to describe conformal transformation in 2 dimensions one has to use the spin (Lorentz) group in 4 space-time dimensions; this generalizes: use $\mathrm{Spin}(p+1, q+1)$ to obtain conformal transformations of a compactified flat space with metric of signature (p, q);

(ii) Minkowski space is exceptional in the sense that only in dimension 4 the celestial sphere is a Hermitian manifold

The set of all rays in Minkowski space is a 5-dim manifold $\mathbb{R}^3 \times \mathbb{S}_2$; as such it cannot be complex; twistors provide an ingenious extension of that manifold (after compactification) to $\mathbb{C}\mathsf{P}_3$ and explain the role of the additional dimension. Connection with origin of the name *twistor*.

The set of all rays in Minkowski space is a 5-dim manifold $\mathbb{R}^3 \times \mathbb{S}_2$; as such it cannot be complex; twistors provide an ingenious extension of that manifold (after compactification) to $\mathbb{C}\mathsf{P}_3$ and explain the role of the additional dimension. Connection with origin of the name *twistor*.

Solutions of wave equations depend on functions of **3** variables (Cauchy data). One can expect that analytic solutions of wave equations can be obtained from data on the **3**-dim twistor space $\mathbb{C}\mathsf{P}_3$.

Twistors: definitions

To represent conformal transformations of the compactified Minkowski space $\mathscr{M}$ one has to consider the group $\mathrm{Spin}(2,4)$; the corresponding spaces of Weyl (chiral, reduced, half-) spinors are the complex 4-dim. vector spaces T and T^* of *twistors*.

Twistors: definitions

To represent conformal transformations of the compactified Minkowski space $\mathscr{M}$ one has to consider the group $\mathrm{Spin}(2,4)$; the corresponding spaces of Weyl (chiral, reduced, half-) spinors are the complex 4-dim. vector spaces T and T^* of *twistors*.

There is a volume element, $\mathrm{vol} \in \bigwedge^4 \mathsf{T}$ (or $\mathrm{vol}^* \in \bigwedge^4 \mathsf{T}^*$) so that $\mathrm{Aut}\,(\mathsf{T}, \mathrm{vol}) = \mathrm{SL}(\mathsf{T}) \cong \mathrm{Spin}(6, \mathbb{C})$.

The volume element defines a Hodge isomorphism $\star : \bigwedge^2 \mathsf{T} \to \bigwedge^2 \mathsf{T}^*$.

There is no scalar product in T, but the 6-dim vector space $\mathsf{W} = \bigwedge^2 \mathsf{T}$ has a quadratic form – the Pfaffian – defined by $\mathrm{Pf}(w)\,\mathrm{vol} = \frac{1}{2} w \wedge w$.

There is no scalar product in T, but the 6-dim vector space $\mathsf{W} = \bigwedge^2 \mathsf{T}$ has a quadratic form – the Pfaffian – defined by $\mathrm{Pf}(w)\,\mathrm{vol} = \frac{1}{2} w \wedge w$.

If $\mathrm{vol} = e_1 \wedge \cdots \wedge e_4$, then the basis $(e_\alpha)_{\alpha=1,\dots,4}$ is said to be unimodular; with respect to such a basis $\mathrm{Pf}(w) = w^{23}w^{14} + w^{31}w^{24} + w^{12}w^{34}$. (When shown this, physicists think of $\mathbf{E} \cdot \mathbf{B}$ and recognize the formula $\mathrm{Pf}(w)^2 = \det w$).

By virtue of

$$w \circ \star w = \mathrm{Pf}(w)\, \mathrm{id}_{\mathsf{T}}$$

a representation of the Clifford algebra of $(\mathsf{W}, \mathrm{Pf})$ in the space $\mathsf{T} \oplus \mathsf{T}^*$ of Dirac spinors is obtained from

$$\mathsf{W} \to \mathrm{End}(\mathsf{T} \oplus \mathsf{T}^*) : \quad w \mapsto \begin{pmatrix} 0 & w \\ \star w & 0 \end{pmatrix}$$

There is the null cone

$$\mathsf{NW} = \{w \in \mathsf{W}^\times \mid \mathrm{Pf}(w) = 0\}$$

There is the null cone

$$\mathsf{NW} = \{w \in \mathsf{W}^\times \mid \mathrm{Pf}(w) = 0\}$$

The (Klein) quadric

$$\mathsf{QW} = \{\ell(w) \in \mathsf{PW} \mid w \in \mathsf{NW}\}$$

is a 4-dimensional complex compact manifold with a conformal geometry induced from $(\mathsf{NW}, \mathrm{Pf}\,|\mathsf{NW})$ by local sections of the line bundle $\mathsf{NW} \to \mathsf{QW}$.

There is the null cone

$$\mathsf{NW} = \{w \in \mathsf{W}^\times \mid \mathrm{Pf}(w) = 0\}$$

The (Klein) quadric

$$\mathsf{QW} = \{\ell(w) \in \mathsf{PW} \mid w \in \mathsf{NW}\}$$

is a 4-dimensional complex compact manifold with a conformal geometry induced from $(\mathsf{NW}, \mathrm{Pf}\,|\mathsf{NW})$ by local sections of the line bundle $\mathsf{NW} \to \mathsf{QW}$.

The group $\mathrm{SL}(\mathsf{T})$ acts by conformal transformations on the quadric. Since $w \wedge w = 0$ is equivalent to $w = \tau_1 \wedge \tau_2$ for some $\tau_1, \tau_2 \in \mathsf{T}$, the quadric can be identified with the Grassmannian $\mathrm{Gr}(2, \mathsf{T})$ of 2-planes in T.

Breaking the symmetry

Consider two spaces of *spinors* (S, ϵ) and (S', ϵ'), where S is 2-dim complex and

$$\epsilon : \mathsf{S} \to \mathsf{S}^*, \quad \epsilon^* = -\epsilon, \quad \epsilon(e_A) = \epsilon_{AB} e^B$$

where $(e_A)_{A=1,2}$ is a basis in S; similarly for S'.

Breaking the symmetry

Consider two spaces of *spinors* (S, ϵ) and (S', ϵ'), where S is 2-dim complex and

$$\epsilon : \mathsf{S} \to \mathsf{S}^*, \quad \epsilon^* = -\epsilon, \quad \epsilon(e_A) = \epsilon_{AB} e^B$$

where $(e_A)_{A=1,2}$ is a basis in S; similarly for S'.

The decomposition

$$\mathsf{T} = \mathsf{S} \oplus \mathsf{S}'$$

breaks the symmetry in the description of the geometry, like the stereographic projection.

It allows one to distinguish a 4-dim affine space, an open and dense subspace of the quadric.

There is a decomposition

$$\Lambda^2(\mathsf{S} \oplus \mathsf{S}') = \Lambda^2\,\mathsf{S} \oplus \Lambda^2\,\mathsf{S}' \oplus (\mathsf{S} \otimes \mathsf{S}')$$

There is a decomposition

$$\bigwedge^2(\mathsf{S} \oplus \mathsf{S}') = \bigwedge^2 \mathsf{S} \oplus \bigwedge^2 \mathsf{S}' \oplus (\mathsf{S} \otimes \mathsf{S}')$$

The 1-dim subspaces $\bigwedge^2 \mathsf{S}$ and $\bigwedge^2 \mathsf{S}'$ of W are both null with respect to Pf, but their sum is not, and the 4-dim vector space $\mathsf{S} \otimes \mathsf{S}'$ is the orthogonal complement of that sum.

There is a decomposition

$$\bigwedge^2(\mathsf{S} \oplus \mathsf{S}') = \bigwedge^2 \mathsf{S} \oplus \bigwedge^2 \mathsf{S}' \oplus (\mathsf{S} \otimes \mathsf{S}')$$

The 1-dim subspaces $\bigwedge^2 \mathsf{S}$ and $\bigwedge^2 \mathsf{S}'$ of W are both null with respect to Pf, but their sum is not, and the 4-dim vector space $\mathsf{S} \otimes \mathsf{S}'$ is the orthogonal complement of that sum.

Every element of the form $s \otimes s' \in \mathsf{S} \otimes \mathsf{S}'$ is null.

There is a decomposition

$$\wedge^2(\mathsf{S} \oplus \mathsf{S}') = \wedge^2 \mathsf{S} \oplus \wedge^2 \mathsf{S}' \oplus (\mathsf{S} \otimes \mathsf{S}')$$

The 1-dim subspaces $\wedge^2 \mathsf{S}$ and $\wedge^2 \mathsf{S}'$ of W are both null with respect to Pf, but their sum is not, and the 4-dim vector space $\mathsf{S} \otimes \mathsf{S}'$ is the orthogonal complement of that sum.

Every element of the form $s \otimes s' \in \mathsf{S} \otimes \mathsf{S}'$ is null.

Let $w_0 \in \wedge^2 \mathsf{S}$ and $w_\infty \in \wedge^2 \mathsf{S}'$ be such that $w_0 \wedge w_\infty = \mathrm{vol}$, then the injection

$$\mathsf{S} \otimes \mathsf{S}' \to \mathsf{QW}, \quad w \mapsto \ell(w_0 + w - \mathrm{Pf}(w) w_\infty)$$

is conformal and generalizes the stereographic map $\mathbb{C} \to \mathbb{C}\mathsf{P}_1$.

The complement of the image of $\mathsf{S} \otimes \mathsf{S}'$ in the quadric is the compactified null cone with vertex at w_∞.

The complement of the image of $\mathsf{S} \otimes \mathsf{S}'$ in the quadric is the compactified null cone with vertex at w_∞.

The group $\mathrm{SL}(\mathsf{T})$ mixes S and S'; for example, the interchange $w_0 \leftrightarrows w_\infty$ induces the inversion

$$\ell(w_0 + w - \mathrm{Pf}(w) w_\infty) \mapsto \ell(w_\infty + w - \mathrm{Pf}(w) w_0) =$$

$$= \ell(w_0 - w/\mathrm{Pf}(w) - \mathrm{Pf}(w/\mathrm{Pf}(w)) w_\infty)$$

Real structure

To continue on the Road to *Reality* one introduces the *real* Minkowski space (and its conformal compactification $\mathscr{M}$) by identifying S' with $\bar{\mathsf{S}}$ or $\bar{\mathsf{S}}^*$ (use $\bar{\epsilon}$ to go from one to the other).

Real structure

To continue on the Road to *Reality* one introduces the *real* Minkowski space (and its conformal compactification $\mathscr{M}$) by identifying S' with $\bar{\mathsf{S}}$ or $\bar{\mathsf{S}}^*$ (use $\bar{\epsilon}$ to go from one to the other).

Put $e_{\bar{A}} = \overline{e_A}$ so that $(e_{\bar{A}})_{A=1,2}$ is a basis in $\bar{\mathsf{S}}$. The restriction of the Pfaffian to the real vector space

$$\mathrm{Re}(\mathsf{S} \otimes \bar{\mathsf{S}}) = \{x^{A\bar{B}} e_A \otimes e_{\bar{B}} \mid \overline{x^{A\bar{B}}} = x^{B\bar{A}}\}$$

has signature $(1, 3)$. Every real null vector is of the form $s \otimes \bar{s}$.

If (V, g) is another Minkowski vector space with a basis $(e_\mu)_{\mu=0,\ldots,3}$, then there is an isometry

$$\sigma : \mathsf{V} \to \mathrm{Re}(\mathsf{S} \otimes \bar{\mathsf{S}}), \;\; \sigma(e_\mu) = \sigma_\mu{}^{A\bar{B}} e_A \otimes e_{\bar{B}}$$

If (V, g) is another Minkowski vector space with a basis $(e_\mu)_{\mu=0,\dots,3}$, then there is an isometry

$$\sigma : \mathsf{V} \to \mathrm{Re}(\mathsf{S} \otimes \bar{\mathsf{S}}), \;\; \sigma(e_\mu) = \sigma_\mu{}^{A\bar{B}} e_A \otimes e_{\bar{B}}$$

The isometry property is expressed by

$$g_{\mu\nu} = \sigma_\mu{}^{A\bar{B}} \sigma_\nu{}^{C\bar{D}} \epsilon_{AC} \epsilon_{\bar{B}\bar{D}}$$

(Penrose usually omits the sigmas; abstract index notation)

Back to twistors: if

$$\mathsf{T} = \mathsf{S} \oplus \bar{\mathsf{S}}^*, \quad \text{then} \quad \bar{\mathsf{T}}^* = \bar{\mathsf{S}}^* \oplus \mathsf{S}$$

If $s \in \mathsf{S}$ and $s' \in \mathsf{S}^*$, then there is the twistor $\tau = (s, \bar{s}')$ (in Penrose's notation: $\mathsf{Z}^\alpha = (\omega^A, \pi_{B'})$)

Back to twistors: if

$$\mathsf{T} = \mathsf{S} \oplus \bar{\mathsf{S}}^*, \quad \text{then} \quad \bar{\mathsf{T}}^* = \bar{\mathsf{S}}^* \oplus \mathsf{S}$$

If $s \in \mathsf{S}$ and $s' \in \mathsf{S}^*$, then there is the twistor $\tau = (s, \bar{s}')$ (in Penrose's notation: $\mathsf{Z}^\alpha = (\omega^A, \pi_{B'})$)

and the natural swap map

$$c : \mathsf{T} \to \bar{\mathsf{T}}^*, \quad c(s, \bar{s}') = (\bar{s}', s)$$

which is is Hermitian, $\bar{c}^* = c$.

This map extends to $\mathsf{W} = \bigwedge^2 \mathsf{T}$: if $w \in \bigwedge^2 \mathsf{T}$ is considered as a map from T^* to T, then the composition $c^* \bar{w} c$ is an antisymmetric map from T to T^*, i.e. an element of $\mathsf{W}^* = \bigwedge^2 \mathsf{T}^*$ and

$$\mathrm{Re}\,\mathsf{W} = \{w \in \mathsf{W} \mid \star w = c^* \bar{w} c\}$$

is a real 6-dim vector space with a quadratic form $\mathrm{Pf}\,|\,\mathrm{Re}\,\mathsf{W}$ of signature $(2, 4)$.

The form ($\langle , \rangle$ means evaluation)

$$C : \mathsf{T} \times \mathsf{T} \to \mathbb{C}, \quad C(\tau_1, \tau_2) = \langle \bar{\tau}_1, c(\tau_2) \rangle$$

is (pseudo) Hermitian: if $\tau = (s, \bar{s}')$, then

$$C(\tau, \tau) = \langle s, s' \rangle + \langle \bar{s}, \bar{s}' \rangle \quad \text{is real}$$

The form C has signature $(2, 2)$, therefore
$\mathrm{Aut}\,(\mathsf{T}, \mathrm{vol}, C) \cong \mathrm{SU}(2, 2) = \mathrm{Spin}(2, 4)$.

The form ($\langle , \rangle$ means evaluation)

$$C : \mathsf{T} \times \mathsf{T} \to \mathbb{C}, \quad C(\tau_1, \tau_2) = \langle \bar{\tau}_1, c(\tau_2) \rangle$$

is (pseudo) Hermitian: if $\tau = (s, \bar{s}')$, then

$$C(\tau, \tau) = \langle s, s' \rangle + \langle \bar{s}, \bar{s}' \rangle \quad \text{is real}$$

The form C has signature $(2, 2)$, therefore $\mathrm{Aut}\,(\mathsf{T}, \mathrm{vol}, C) \cong \mathrm{SU}(2, 2) = \mathrm{Spin}(2, 4)$.

A twistor $\tau = (s, \bar{s}')$ is **null** if $C(\tau, \tau) = 0 \Leftrightarrow \langle s, s' \rangle$ is **pure imaginary**

Note: if instead of $\mathsf{S}' = \bar{\mathsf{S}}^*$ one assumes $\mathsf{S} = \bar{\mathsf{S}}$ and $\mathsf{S}' = \bar{\mathsf{S}}'$ (real spaces), then one gets a reduction of $\mathrm{SL}(4, \mathbb{C})$ to $\mathrm{SL}(4, \mathbb{R}) = \mathrm{Spin}(3, 3)$.

Note: if instead of $\mathsf{S}' = \bar{\mathsf{S}}^*$ one assumes $\mathsf{S} = \bar{\mathsf{S}}$ and $\mathsf{S}' = \bar{\mathsf{S}}'$ (real spaces), then one gets a reduction of $\mathrm{SL}(4, \mathbb{C})$ to $\mathrm{SL}(4, \mathbb{R}) = \mathrm{Spin}(3, 3)$.

The twistor equation

Identifying $\mathsf{S} \otimes \bar{\mathsf{S}}$ with $\mathrm{Hom}(\bar{\mathsf{S}}^*, \mathsf{S})$ one has, for every $x \in \mathsf{V}$, the map $\sigma(x) : \bar{\mathsf{S}}^* \to \mathsf{S}$ which is Hermitian, $\overline{\sigma(x)^*} = \sigma(x)$ (Pauli matrices are Hermitian)

Note: if instead of $\mathsf{S}' = \bar{\mathsf{S}}^*$ one assumes $\mathsf{S} = \bar{\mathsf{S}}$ and $\mathsf{S}' = \bar{\mathsf{S}}'$ (real spaces), then one gets a reduction of $\mathrm{SL}(4, \mathbb{C})$ to $\mathrm{SL}(4, \mathbb{R}) = \mathrm{Spin}(3, 3)$.

The twistor equation

Identifying $\mathsf{S} \otimes \bar{\mathsf{S}}$ with $\mathrm{Hom}(\bar{\mathsf{S}}^*, \mathsf{S})$ one has, for every $x \in \mathsf{V}$, the map $\sigma(x) : \bar{\mathsf{S}}^* \to \mathsf{S}$ which is Hermitian, $\overline{\sigma(x)^*} = \sigma(x)$ (Pauli matrices are Hermitian)

Consider the field of spinors $\phi : \mathsf{V} \to \mathsf{S}$ associated with $\tau = (s, \bar{s}')$ and given by

$$\phi(x) = s - \mathrm{i}\sigma(x)\bar{s}'$$

This is a general solution of the *twistor equation* (indices win!):

$$\nabla^{(A}{}_{\bar{C}}\phi^{B)} = 0, \quad \phi = e_A\phi^A, \quad \nabla_{A\bar{B}} = \sigma^\mu{}_{A\bar{B}}\nabla_\mu$$

Here $\nabla_\mu = \partial/\partial x^\mu$, but the equation generalizes to Riemannian manifolds. The equation is conformally invariant and its integrability imposes severe restrictions on the tensor W of conformal curvature.

The twistor equation is part of the decomposition into irreducible parts

∇ on spinor = Weyl–Dirac operator+Penrose twistor operator

analogous to

∇ on vector = div + curl + eq. for conformal Killing vectors

Close relation between solutions of those equations. Killing spinors.

If the equation $\phi(x) = 0$ has a real solution $x \in \mathsf{V}$, then the twistor $\tau = (s, \bar{s}')$ is null.

If the equation $\phi(x) = 0$ has a real solution $x \in \mathsf{V}$, then the twistor $\tau = (s, \bar{s}')$ is null. If τ is null and generic, in the sense that $\langle s, s' \rangle \neq 0$, then there is the ray

$$\mathbb{R} \to \mathsf{V}, \quad t \mapsto l + tk$$

such that

$$(\diamondsuit) \qquad \sigma(l + tk) = \mathrm{i}\frac{s \otimes \bar{s}}{\langle s, s' \rangle} + t\epsilon^{-1}(s') \otimes \bar{\epsilon}^{-1}(\bar{s}')$$

and $\phi(l + tk) = 0$.

If the equation $\phi(x) = 0$ has a real solution $x \in \mathsf{V}$, then the twistor $\tau = (s, \bar{s}')$ is null. If τ is null and generic, in the sense that $\langle s, s' \rangle \neq 0$, then there is the ray

$$\mathbb{R} \to \mathsf{V}, \;\; t \mapsto l + tk$$

such that

$$(\Diamond) \qquad\qquad \sigma(l + tk) = \mathrm{i}\frac{s \otimes \bar{s}}{\langle s, s' \rangle} + t\epsilon^{-1}(s') \otimes \bar{\epsilon}^{-1}(\bar{s}')$$

and $\phi(l + tk) = 0$. If $\tau = (0, \bar{s}')$, then there is the ray $t\epsilon^{-1}(s') \otimes \bar{\epsilon}^{-1}(\bar{s}')$. The null twistor $(s, 0)$ defines a ray on the null cone at infinity.

Replacing $(s, \bar{s}')$ by $\lambda(s, \bar{s}')$, $\lambda \in \mathbb{C}^\times$, does not change the ray $\Diamond$ which is defined by an element of

$$\mathsf{PT}_0 = \{\ell(\tau) \in \mathsf{PT} \mid C(\tau, \tau) = 0\},$$

a 5-dim real submanifold of the projective twistor space PT with an induced Cauchy–Riemann structure.

Replacing $(s, \bar{s}')$ by $\lambda(s, \bar{s}')$, $\lambda \in \mathbb{C}^\times$, does not change the ray $\Diamond$ which is defined by an element of

$$\mathsf{PT}_0 = \{\ell(\tau) \in \mathsf{PT} \mid C(\tau, \tau) = 0\},$$

a 5-dim real submanifold of the projective twistor space PT with an induced Cauchy–Riemann structure. If $\tau = (s, \bar{s}')$ is not null, then $\langle s, s' \rangle$ is complex and $\Diamond$ describes a ray in $\mathbb{C} \otimes \mathsf{V}$ parallel to the real vector k and passing through the tip of a complex null vector l. The collection of all null twistors orthogonal to such a τ defines, in the real space V, an (Ivor) *Robinson "congruence"* (1-dim foliation) of rays which is **twisting** in the sense that the distribution of 3-planes orthogonal to the vectors of the congruence is not integrable (Shadows rotate...)

Replacing $(s, \bar{s}')$ by $\lambda(s, \bar{s}')$, $\lambda \in \mathbb{C}^\times$, does not change the ray $\diamondsuit$ which is defined by an element of

$$\mathsf{PT}_0 = \{\ell(\tau) \in \mathsf{PT} \mid C(\tau, \tau) = 0\},$$

a 5-dim real submanifold of the projective twistor space PT with an induced Cauchy–Riemann structure. If $\tau = (s, \bar{s}')$ is not null, then $\langle s, s' \rangle$ is complex and $\diamondsuit$ describes a ray in $\mathbb{C} \otimes \mathsf{V}$ parallel to the real vector k and passing through the tip of a complex null vector l. The collection of all null twistors orthogonal to such a τ defines, in the real space V, an (Ivor) *Robinson "congruence"* (1-dim foliation) of rays which is **twisting** in the sense that the distribution of 3-planes orthogonal to the vectors of the congruence is not integrable (Shadows rotate...)

The Penrose correspondences

The real quadric

$$\mathscr{M} = \{\ell(w) \in \mathsf{QW} \mid w \in \mathrm{Re}\,\mathsf{W}\}$$

provides a conformal compactification of Minkowski space. It is diffeomorphic to $\mathbb{S}_1 \times \mathbb{S}_3$. Its null geodesics are **rays**.

The Penrose correspondences

The real quadric

$$\mathscr{M} = \{\ell(w) \in \mathsf{QW} \mid w \in \mathrm{Re}\,\mathsf{W}\}$$

provides a conformal compactification of Minkowski space. It is diffeomorphic to $\mathbb{S}_1 \times \mathbb{S}_3$. Its null geodesics are **rays**. An element $\ell(\tau_1 \wedge \tau_2)$ of QW is in $\mathscr{M}$ iff

$$\star(\tau_1 \wedge \tau_2) = c^*(\bar{\tau}_1 \wedge \bar{\tau}_2)c \in \Lambda^2 T^*$$

Evaluating both sides of the last equation on τ_1 and τ_2 one obtains, the necessary and sufficient conditions for $\ell(\tau_1 \wedge \tau_2)$ to be in $\mathscr{M}$:

$$C(\tau_1, \tau_1) = 0,\ C(\tau_2, \tau_2) = 0 \ \text{ and } \ C(\tau_1, \tau_2) = 0$$

For every $\lambda_1, \lambda_2 \in \mathbb{C}^\times$ the twistor $\lambda_1\tau_1 + \lambda_2\tau_2$ is null and orthogonal to τ_1 and τ_2. The set of all directions of these twistors is the null cone (celestial sphere) of the point $\ell(\tau_1 \wedge \tau_2) \in \mathscr{M}$.

For every $\lambda_1, \lambda_2 \in \mathbb{C}^\times$ the twistor $\lambda_1\tau_1 + \lambda_2\tau_2$ is null and orthogonal to τ_1 and τ_2. The set of all directions of these twistors is the null cone (celestial sphere) of the point $\ell(\tau_1 \wedge \tau_2) \in \mathscr{M}$.

There are natural, bijective **Penrose correspondences**

$\mathscr{M}$	PT_0
ray	point
intersecting rays	orthogonal points
point	complex projective line
	(celestial sphere)

Complexification "unifies" null rays in Lorentz spaces and complex structures in Euclidean spaces

Let V be a real vector space of dim $2n$ with a scalar product h that is either positive-definite (Euclidean) or of signature $(1, 2n-1)$ (Lorentzian).

Complexification "unifies" null rays in Lorentz spaces and complex structures in Euclidean spaces

Let V be a real vector space of dim $2n$ with a scalar product h that is either positive-definite (Euclidean) or of signature $(1, 2n-1)$ (Lorentzian).

The complexification $\mathbb{C} \otimes \mathsf{V}$ contains subspaces which are *totally null* and of maximal dimension, i.e. n.

Complexification "unifies" null rays in Lorentz spaces and complex structures in Euclidean spaces

Let V be a real vector space of dim $2n$ with a scalar product h that is either positive-definite (Euclidean) or of signature $(1, 2n-1)$ (Lorentzian).

The complexification $\mathbb{C} \otimes \mathsf{V}$ contains subspaces which are *totally null* and of maximal dimension, i.e. n.

If N is such an *mtn* space, then the complex conjugate space $\bar{N}$ is also mtn; their intersection $N \cap \bar{N}$ is a totally null space that is (the complexification of a) real.

null space that is (the complexification of a) real. Therefore,

$$\dim(N \cap \bar{N}) = \begin{cases} 0 & \text{if } (\mathsf{V}, h) \text{ is Euclidean} \\ 1 & \text{if } (\mathsf{V}, h) \text{ is Lorentzian} \end{cases}$$

null space that is (the complexification of a) real. Therefore,

$$\dim(N \cap \bar{N}) = \begin{cases} 0 & \text{if } (\mathsf{V}, h) \text{ is Euclidean} \\ 1 & \text{if } (\mathsf{V}, h) \text{ is Lorentzian} \end{cases}$$

Dually, one describes an *mtn* N by the (direction of an) n-form F such that $v \in N \Leftrightarrow v \lrcorner F = 0$. Since $*F$ corresponds in this way to $N^\perp$, $N^\perp = N$ for *mtn*s, one has $*F \| F$ and, since $** = \pm\,\mathrm{id}$, there are two kinds of *mtn*s (the α and β planes of classical projective geometry).

null space that is (the complexification of a) real. Therefore,

$$\dim(N \cap \bar{N}) = \begin{cases} 0 & \text{if } (\mathsf{V}, h) \text{ is Euclidean} \\ 1 & \text{if } (\mathsf{V}, h) \text{ is Lorentzian} \end{cases}$$

Dually, one describes an *mtn* N by the (direction of an) n-form F such that $v \in N \Leftrightarrow v \lrcorner F = 0$. Since $*F$ corresponds in this way to $N^\perp$, $N^\perp = N$ for *mtn*s, one has $*F \| F$ and, since $** = \pm\,\mathrm{id}$, there are two kinds of *mtn*s (the α and β planes of classical projective geometry).

In the Euclidean case, N defines a *complex structure* J in V,

$$\mathbb{C} \otimes V = N \oplus \bar{N} \qquad J|N = \sqrt{-1}\,\mathrm{id}_N$$

which is orthogonal, $h(Jx, Jx) = h(x, x)$ for every $x \in \mathsf{V}$. Conversely, every such J defines an mtn. Mathematicians write $\mathbb{C} \otimes \mathsf{V} = \mathsf{V}^{1,0} \oplus \mathsf{V}^{0,1}$.

which is orthogonal, $h(Jx, Jx) = h(x, x)$ for every $x \in \mathsf{V}$. Conversely, every such J defines an mtn. Mathematicians write $\mathbb{C} \otimes \mathsf{V} = \mathsf{V}^{1,0} \oplus \mathsf{V}^{0,1}$.

In the Lorentzian case, N defines a ray $K = \mathrm{Re}(N \cap \bar{N})$; moreover, it defines also an orthogonal complex structure in the $(2n-2)$-dim *screen space* $K^{\perp}/K$. In four dimensions, the screen space is 2-dim and orientation is enough to define a complex structure in this Euclidean space; for this reason, physicists restrict their attention to K.

Cartan used *mtn*s to define simple (pure) spinors. If $\gamma : \mathsf{V} \to \mathrm{End}\, S$ defines a representation of the Clifford algebra of (V, h) in the space $\mathsf{S} = \mathsf{S}_+ \oplus \mathsf{S}_-$ of Dirac spinors and $0 \neq s \in \mathsf{S}$, then the vector space

$$(\heartsuit) \qquad \{v \in \mathbb{C} \otimes \mathsf{V} \mid \gamma(v)s = 0\}$$

is totally null; if $(\heartsuit)$ is *mtn*, then the line $\ell(s)$ is said to consist of *simple* spinors.

Cartan used *mtn*s to define simple (pure) spinors. If $\gamma : \mathsf{V} \to \mathrm{End}\, S$ defines a representation of the Clifford algebra of (V, h) in the space $\mathsf{S} = \mathsf{S}_+ \oplus \mathsf{S}_-$ of Dirac spinors and $0 \neq s \in \mathsf{S}$, then the vector space

$$(\heartsuit) \qquad \{v \in \mathbb{C} \otimes \mathsf{V} \mid \gamma(v)s = 0\}$$

is totally null; if $(\heartsuit)$ is *mtn*, then the line $\ell(s)$ is said to consist of *simple* spinors. In dimensions 4 and 6 every Weyl spinor is simple. Since in dimension 4 the spaces of Weyl spinors $\mathsf{S}_\pm$ are complex 2-dim, the corresponding manifold of *mtn*s of one chirality is diffeomorphic to $\mathbb{C}\mathsf{P}_1$.

Integrability

These observations become interesting when applied to the tangent spaces of a $2n$-dim Riemannian or Lorentzian manifold $(\mathcal{M}, g)$.

Integrability

These observations become interesting when applied to the tangent spaces of a $2n$-dim Riemannian or Lorentzian manifold $(\mathcal{M}, g)$.

Let now $\mathcal{N}$ be a field (distribution) on $\mathcal{M}$ of *mtn* subspaces of $\mathbb{C} \otimes T\mathcal{M}$ and let $\mathcal{F}$ be a field of n-forms providing the dual description of the distribution. The distribution is said to be integrable if

(int) $$[\mathrm{Sec}\,\mathcal{N}, \mathrm{Sec}\,\mathcal{N}] \subset \mathrm{Sec}\,\mathcal{N}$$

This is equivalent to the existence of a field μ of 1-forms such that

$$\mathrm{d}\mathcal{F} = \mu \wedge \mathcal{F}$$

Can one get rid of μ, by rescaling of $\mathcal{F}$, to obtain a Maxwell field in dim 4 ? (Robinson, Tafel)

This is equivalent to the existence of a field μ of 1-forms such that

$$\mathrm{d}\mathcal{F} = \mu \wedge \mathcal{F}$$

Can one get rid of μ, by rescaling of $\mathcal{F}$, to obtain a Maxwell field in dim 4 ? (Robinson, Tafel)

In the Riemannian case $\mathcal{N}$ defines an almost complex structure $\mathcal{J}$ and (int) is equivalent to the integrability of $\mathcal{J}$. If (int) holds, then $(\mathcal{M}, g, \mathcal{J})$ is a Hermitian manifold.

In the Lorentzian case, if (int) holds,

(i) the distribution $\mathcal{K} = \mathrm{Re}(\mathcal{N} \cap \bar{\mathcal{N}})$ defines a foliation of $\mathcal{M}$ by a family of rays (null geodesics) and the distribution $\mathcal{K}^{\perp}/\mathcal{K}$ with its conformal structure induced by g is invariant with respect to the flows generated by sections of $\mathcal{K} \to \mathcal{M}$;

In the Lorentzian case, if (int) holds,

(i) the distribution $\mathcal{K} = \mathrm{Re}(\mathcal{N} \cap \bar{\mathcal{N}})$ defines a foliation of $\mathcal{M}$ by a family of rays (null geodesics) and the distribution $\mathcal{K}^{\perp}/\mathcal{K}$ with its conformal structure induced by g is invariant with respect to the flows generated by sections of $\mathcal{K} \to \mathcal{M}$; if the foliation is regular, i.e. quotient $\mathcal{M}/\mathcal{K}$ is a manifold $\mathcal{L}$ etc., then

(ii) the distribution $\mathcal{N}$ induces a Cauchy–Riemann structure on $\mathcal{L}$.

Shear-free congruences of rays and the Kerr thm

In particular, in the 4-dim case, the invariance of the conformal structure of the bundle $\mathcal{K}^{\perp}/\mathcal{K}$ of screen spaces is the shear-free property of congruence of rays;

Shear-free congruences of rays and the Kerr thm

In particular, in the 4-dim case, the invariance of the conformal structure of the bundle $\mathcal{K}^{\perp}/\mathcal{K}$ of screen spaces is the shear-free property of congruence of rays; the corresponding CR 3-dim manifold $\mathcal{L}$ is characterized by a bundle $\mathcal{H} \subset T\mathcal{L}$ with complex structures in its fibres. In the case of Minkowski space, every foliation of $\mathscr{M}$ by rays is given, locally, by a 3-dim submanifold $\mathcal{L}$ of PT_0.

Shear-free congruences of rays and the Kerr thm

In particular, in the 4-dim case, the invariance of the conformal structure of the bundle $\mathcal{K}^{\perp}/\mathcal{K}$ of screen spaces is the shear-free property of congruence of rays; the corresponding CR 3-dim manifold $\mathcal{L}$ is characterized by a bundle $\mathcal{H} \subset T\mathcal{L}$ with complex structures in its fibres. In the case of Minkowski space, every foliation of $\mathscr{M}$ by rays is given, locally, by a 3-dim submanifold $\mathcal{L}$ of PT_0.

The manifold $\mathcal{L}$ has an analytic CR structure – and so corresponds to a shear-free congruence of rays – iff it is the intersection of PT_0 with a complex hypersurface in PT (Penrose form of the Kerr thm).

For example, a Robinson congruence is given as the intersection of PT_0 with the hypersurface

$$\{\ell(\tau) \in \mathsf{PT} \mid C(\tau', \tau) = 0\} \subset \mathsf{PT}$$

where $\tau' \in \mathsf{T}^\times$ is **not null**.

For example, a Robinson congruence is given as the intersection of PT_0 with the hypersurface

$$\{\ell(\tau) \in \mathsf{PT} \mid C(\tau', \tau) = 0\} \subset \mathsf{PT}$$

where $\tau' \in \mathsf{T}^\times$ is **not null**.

In 4-dim Einstein spaces

The theorems of

Goldberg and Sachs in GRT

Plebański and Przanowski in Euclid. sign.

connect degeneracy of W to integrability of $\mathcal{N}$

An aside: the Robinson congruence

To describe the Robinson congruence in Minkowski space, consider first coordinates (U, r, X, Y) in (V, g) such that the metric is

$$g = 2\,\mathrm{d}U\,\mathrm{d}r - (\mathrm{d}X^2 + \mathrm{d}Y^2)$$

Introduce new real coordinates (u, r, x, y)

$$X + \mathrm{i}Y = (r + \mathrm{i}a)(x + \mathrm{i}y), \quad U = u + \tfrac{1}{2}r(x^2 + y^2), \quad a \in \mathbb{R}$$

so that

$$g = 2\kappa\,\mathrm{d}r - (r^2 + a^2)(\mathrm{d}x^2 + \mathrm{d}y^2), \quad \kappa = \mathrm{d}u + a(x\,\mathrm{d}y - y\,\mathrm{d}x)$$

The 2-form $F = Z(u, x, y)\kappa \wedge \mathrm{d}(x + \mathrm{i}y)$ is self-dual, $(*F = \mathrm{i}F)$, and $\kappa \wedge \mathrm{d}\kappa = 2a\,\mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y$.

The 2-form $F = Z(u, x, y)\kappa \wedge \mathrm{d}(x + \mathrm{i}y)$ is self-dual, ($*F = \mathrm{i}F$), and $\kappa \wedge \mathrm{d}\kappa = 2a\,\mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y$.

For $a \neq 0$, the null vector field $\partial/\partial r$ generates a twisting Robinson congruence. Maxwell's equations $\mathrm{d}F = 0$ reduce to $L_a(Z) = 0$, where

$$L_a = a(x + \mathrm{i}y)\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

The 2-form $F = Z(u,x,y)\kappa \wedge \mathrm{d}(x+\mathrm{i}y)$ is self-dual, $(*F = \mathrm{i}F)$, and $\kappa \wedge \mathrm{d}\kappa = 2a\,\mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y$.

For $a \neq 0$, the null vector field $\partial/\partial r$ generates a twisting Robinson congruence. Maxwell's equations $\mathrm{d}F = 0$ reduce to $L_a(Z) = 0$, where

$$L_a = a(x+\mathrm{i}y)\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

For $a = 0$ (no twist) $L_0(Z) = 0$ is the C-R equation so that a general solution of $\mathrm{d}F = 0$ is given by $F = Z\,\mathrm{d}u \wedge \mathrm{d}(x+\mathrm{i}y)$, where Z is a holomorphic function of $x + \mathrm{i}y$, depending smoothly on u.

For $a \neq 0$ the situation is drastically different. The equation $L_1(Z) = 0$ has two independent solutions $Z_1 = x + \mathrm{i}y$ and $Z_2 = u + \frac{1}{2}\mathrm{i}(x^2 + y^2)$. The embedding

$$\mathbb{R}^3 \to \mathbb{C}^2 : \quad (u, x, y) \mapsto (Z_1(x, y), Z_2(u, x, y))$$

gives a realization of the CR structure $(\mathbb{R}^3, \kappa, \mathrm{d}(x + \mathrm{i}y))$ on a hypersurface in $\mathbb{C}^2$.

For $a \neq 0$ the situation is drastically different. The equation $L_1(Z) = 0$ has two independent solutions $Z_1 = x + \mathrm{i}y$ and $Z_2 = u + \frac{1}{2}\mathrm{i}(x^2 + y^2)$. The embedding

$$\mathbb{R}^3 \to \mathbb{C}^2 : \quad (u, x, y) \mapsto (Z_1(x, y), Z_2(u, x, y))$$

gives a realization of the CR structure $(\mathbb{R}^3, \kappa, \mathrm{d}(x + \mathrm{i}y))$ on a hypersurface in $\mathbb{C}^2$.

Hans Lewy (1956) constructed a function $A(u, x, y)$ of class C^∞ such that the equation $L_1(Z) = A$ has not even local solutions. But there are such solutions if A is of class C^ω.

The Penrose transform

is a sophisticated and precise isomorphism between the space of analytic solutions of field equations for particles of arbitrary spin and the sheaf cohomology classes of appropriate holomorphic bundles on $\mathbb{PT}$.

The Penrose transform

is a sophisticated and precise isomorphism between the space of analytic solutions of field equations for particles of arbitrary spin and the sheaf cohomology classes of appropriate holomorphic bundles on $\mathbb{PT}$.

Initiated by Penrose (1968) and completed by him and Ward, Wells, Eastwood, Baston,... in the 1980s.

Here rudiments only.

Observation (Whittaker, Bateman 1904):

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} f(k_\rho x^\rho, l_\sigma x^\sigma) = f_{11} k_\mu k_\nu + 2 f_{12} k_\mu l_\nu + f_{22} l_\mu l_\nu$$

so that if the vectors k and l are null and $\perp$ to each other, then $\Box f = 0$.

Observation (Whittaker, Bateman 1904):

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} f(k_\rho x^\rho, l_\sigma x^\sigma) = f_{11} k_\mu k_\nu + 2 f_{12} k_\mu l_\nu + f_{22} l_\mu l_\nu$$

so that if the vectors k and l are null and $\perp$ to each other, then $\Box f = 0$.

Consider the null twistor $s, \bar{s}'$, where

$$s = \sigma(x)\bar{s}', \quad \sigma(x) = \begin{pmatrix} u & \bar{z} \\ z & v \end{pmatrix}, \quad \bar{s}' = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

so that

$$\det \sigma(\mathrm{d}x) = \mathrm{d}u \, \mathrm{d}v - \mathrm{d}\bar{z} \, \mathrm{d}z$$

$$s = (u + \lambda\bar{z}, z + \lambda v) = (k_\rho x^\rho, l_\sigma x^\sigma)$$

and the vectors k and l are null and orthogonal. Therefore

$$\oint f(u + \lambda\bar{z}, z + \lambda v, \lambda)\,\mathrm{d}\lambda$$

is a solution of the wave equation. f is a holomorphic function of 3 variables; there is a natural way of interpreting it as such a function on PT.

$$s = (u + \lambda \bar{z}, z + \lambda v) = (k_\rho x^\rho, l_\sigma x^\sigma)$$

and the vectors k and l are null and orthogonal. Therefore

$$\oint f(u + \lambda \bar{z}, z + \lambda v, \lambda)\, \mathrm{d}\lambda$$

is a solution of the wave equation. f is a holomorphic function of 3 variables; there is a natural way of interpreting it as such a function on PT.

Let $\mu^1 = -\lambda$ and $\mu^2 = 1$, the field

$$\phi^{A_1 \dots A_s} = \oint_C f(u + \lambda\bar{z}, z + \lambda v, \lambda)\mu^{A_1} \dots \mu^{A_s} \, \mathrm{d}\lambda$$

satisfies the (Fierz–Pauli) equation

$$\nabla_{A_1 \bar{B}} \phi^{A_1 \dots A_s} = 0$$

The resulting field is a sum (integral) of fields of type N in the sense of the Cartan–Petrov–Penrose classification. Penrose shows that algebraically special fields can also be so obtained by a suitable choice of f: for example, a field of type N results from choosing f that contains only a simple pole inside the contour C.

Adding to f a function holomorphic inside C does not change ϕ: need for cohomology considerations.

Twistors in proper Riemannian geometry

Pure mathematicians have extended the ideas of Penrose to the geometry of Riemannian manifolds with a positive-definite metric tensor.

Assume (M, g) is 4-dim oriented proper Riemannian manifold.

Twistors in proper Riemannian geometry

Pure mathematicians have extended the ideas of Penrose to the geometry of Riemannian manifolds with a positive-definite metric tensor.

Assume (M, g) is 4-dim oriented proper Riemannian manifold. The Riemann tensor is decomposed into irreducible parts

$$Riem = R + Ric_0 + W_+ + W_-$$

where Ric_0 is the traceless part of the Ricci tensor, W_+ and W_- are the self-dual and anti-self-dual parts of the Weyl tensor of conformal curvature.

Let J be a complex structure in $T_x\mathsf{M}$; it defines an orientation in $T_x\mathsf{M}$; call J positive if this orientation agrees with that of the manifold; negative otherwise.

Let J be a complex structure in T_xM; it defines an orientation in T_xM; call J positive if this orientation agrees with that of the manifold; negative otherwise.

There are two bundles P_+ and P_- over M such that the fibre of P_+ (P_-) at $x \in$ M is the set of all positive (negative) complex structures in T_xM. All these fibres are diffeomorphic to $\mathbb{C}\mathsf{P}_1$.

Let J be a complex structure in T_xM; it defines an orientation in T_xM; call J positive if this orientation agrees with that of the manifold; negative otherwise.

There are two bundles P_+ and P_- over M such that the fibre of P_+ (P_-) at $x \in$ M is the set of all positive (negative) complex structures in T_xM. All these fibres are diffeomorphic to $\mathbb{C}\mathsf{P}_1$.

One introduces on both P_+ and P_- an almost complex structure as follows. Take, e.g., P_- and use the Levi–Civita connection on (M, g) to decompose $T_J\mathsf{P}_- = H_J \oplus V_J$, where V_J is the space tangent to the fibre of $\pi : \mathsf{P}_- \to \mathsf{M}$ at $J \in \mathsf{P}_-$ and there is the isomorphism $T_J\pi : H_J \to T_{\pi(J)}\mathsf{M}$.

Define $\mathcal{J}$ on P_- so that $\mathcal{J}|V_J$ is given by the complex structure of the fibre and $\mathcal{J}|H_J = T_{\pi(J)}^{-1} \circ J \circ T_{\pi(J)}$.

Define $\mathcal{J}$ on P_- so that $\mathcal{J}|V_J$ is given by the complex structure of the fibre and $\mathcal{J}|H_J = T_{\pi(J)}^{-1} \circ J \circ T_{\pi(J)}$.

Theorem (M. F. Atiyah, N. J. Hitchin and I. M. Singer 1978) If (M, g) is self-dual, i.e. $W_- = 0$, then $\mathcal{J}$ is integrable, i.e. P_- is a complex manifold.

Proof is based on the twistor equation.

Example The sphere $\mathbb{S}_4$ is conformally flat and there are two (isomorphic) complex manifolds $\mathsf{P}_\pm = \mathbb{C}\mathsf{P}_3$, part of the fibration $\mathbb{C}\mathsf{P}_1 \to \mathbb{C}\mathsf{P}_3 \to \mathbb{S}_4$.

Define $\mathcal{J}$ on P_- so that $\mathcal{J}|V_J$ is given by the complex structure of the fibre and $\mathcal{J}|H_J = T^{-1}_{\pi(J)} \circ J \circ T_{\pi(J)}$.

Theorem (M. F. Atiyah, N. J. Hitchin and I. M. Singer 1978) If (M, g) is self-dual, i.e. $W_- = 0$, then $\mathcal{J}$ is integrable, i.e. P_- is a complex manifold.

Proof is based on the twistor equation.

Example The sphere $\mathbb{S}_4$ is conformally flat and there are two (isomorphic) complex manifolds $\mathsf{P}_\pm = \mathbb{C}\mathsf{P}_3$, part of the fibration $\mathbb{C}\mathsf{P}_1 \to \mathbb{C}\mathsf{P}_3 \to \mathbb{S}_4$.

Roger Penrose's twistors
provide an important vehicle
for science to move on
The Road to Reality