Gravitational Dark Matter

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◇ A. Ahmed, B.G., Anna Socha, "Gravitational Vector Dark Matter", in preparation

Motivations

The existence of DM has been inferred only from gravitational effects.

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + \mathcal{L}_{\rm SM} + \mathcal{L}_{\rm DM} + \mathcal{L}_{\rm int} \right],$$

with $\kappa \equiv M_{Pl}^{-1} = \sqrt{8\pi G_{\rm N}}$

$$\mathcal{L}_{\rm DM} = -\frac{1}{4} X_{\mu\nu} X^{\mu\nu} + \frac{1}{2} m_X^2 X_\mu X^\mu \,,$$

where $X_{\mu\nu} \equiv \partial_\mu X_\nu - \partial_\nu X_\mu$.

Quantization of a vector field in a curved background

The action for an Abelian vector fields reads:

$$\mathcal{L}_{DM} = \int d^4x\sqrt{-g}\left(-\frac{1}{4}g_{\mu\alpha}g_{\nu\beta}X^{\mu\nu}X^{\alpha\beta} + \frac{1}{2}m_X^2 g_{\mu\nu}X^{\mu}X^{\nu}\right).$$

The background metric is the FLRW:

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2.$$

Extremizing the above action with respect to X_μ one obtains

$$\partial_\mu(g^{\mu\rho}g^{\nu\sigma}X_{\rho\sigma}) + \frac{1}{\sqrt{-g}}(\partial_\mu\sqrt{-g})g^{\mu\rho}g^{\nu\sigma}X_{\rho\sigma} + m_X^2 g^{\mu\nu}X_\mu = 0,$$

so that

$$\vec{\nabla}\cdot\dot{\vec{X}} - \nabla^2 X_0 + m_X^2 a^2 X_0 = 0,$$

$$\ddot{\vec{X}} + H\dot{\vec{X}} - \frac{1}{a^2}\left[\nabla^2\vec{X} - \vec{\nabla}(\vec{\nabla}\cdot\vec{X})\right] + m_X^2\vec{X} = \vec{\nabla}(\dot{X}_0 + HX_0).$$

That could be simplified:

$$\vec{\nabla}\cdot\dot{\vec{X}} - \nabla^2 X_0 + m_X^2 a^2 X_0 = 0,$$

$$\ddot{\vec{X}} + H\dot{\vec{X}} - \frac{1}{a^2}\nabla^2\vec{X} + m_X^2\vec{X} = -2H\nabla X_0.$$

It is convenient to adopt the Fourier transform

$$X_\mu(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\tilde{X}_\mu(t,\vec{k})e^{i\vec{k}\cdot\vec{x}}\,,$$

where the reality of the $X_\mu(t,\vec{x})$ field implies $\tilde{X}_\mu(t,\vec{k}) = \tilde{X}_\mu^*(t,-\vec{k})$. Then we get

$$\tilde{X}_0 = \frac{-i\vec{k}\cdot\partial_t\tilde{\vec{X}}}{k^2 + a^2 m_X^2},$$

$$\partial_t^2\tilde{\vec{X}} + H\partial_t\tilde{\vec{X}} + \left(\frac{k^2}{a^2} + m_X^2\right)\tilde{\vec{X}} = -2\vec{k}\frac{\vec{k}\cdot\partial_t\tilde{\vec{X}}}{k^2 + m_X^2 a^2}H.$$

Note that the X_0 is an auxiliary field and has no dynamics.

The $\tilde{\vec{X}}$ can be decomposed in a basis of helicity states:

$$\tilde{\vec{X}}(t,\vec{k}) = \sum_{\lambda=\pm,L} \vec{\epsilon}_\lambda(\vec{k})\tilde{X}_\lambda(t,\vec{k}),$$

where $\tilde{X}_\pm$ and $\tilde{X}_L$ denote two transversely-polarized modes and a single longitudinally-polarized mode, respectively. Then

$$\ddot{\tilde{X}}_\pm + H\dot{\tilde{X}}_\pm + \left(\frac{k^2}{a^2} + m_X^2\right)\tilde{X}_\pm = 0,$$
$$\ddot{\tilde{X}}_L + H\left(1 + \frac{2k^2}{k^2 + a^2 m_X^2}\right)\dot{\tilde{X}}_L + \left(\frac{k^2}{a^2} + m_X^2\right)\tilde{X}_L = 0\,.$$

Switching to the conformal time $(dt = a(\tau)d\tau)$ we get

$$\tilde{X}''_\pm + \overbrace{(k^2 + a^2 m_X^2)}^{\omega_\pm^2}\tilde{X}_\pm = 0,$$
$$\tilde{X}''_L + \frac{2k^2}{k^2 + a^2 m_X^2}\frac{a'}{a}\tilde{X}'_L + (k^2 + a^2 m_X^2)\tilde{X}_L = 0\,.$$

For the longitudinal mode, $\tilde{X}_L$, it is convenient to perform a field redefinition

$$\tilde{X}_L = \frac{\sqrt{k^2 + a^2 m_X^2}}{a m_X} \mathcal{X}_L \,,$$

so that

$$\mathcal{X}_L'' + \omega_L^2(\tau)\mathcal{X}_L = 0,$$

with

$$\omega_L^2(\tau) \equiv k^2 + m_X^2 a^2 - \frac{k^2}{k^2 + m_X^2 a^2}\frac{a''}{a} + 3\frac{k^2 m_X^2 a'^2}{(k^2 + m_X^2 a^2)^2}.$$

Quantization:

$$\hat{\vec{\tilde{X}}}_L(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left\{ \vec{\epsilon}_L(\vec{k}) \hat{a}_{\vec{k}} \tilde{\mathcal{X}}_L(\tau, \vec{k}) e^{i\vec{k}\cdot\vec{x}} + \vec{\epsilon}_L(\vec{k}) \hat{a}^\dagger_{\vec{k}} \tilde{\mathcal{X}}^*_L(\tau, \vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right\},$$

$$\hat{\vec{\tilde{X}}}_\pm(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left\{ \vec{\epsilon}_\pm(\vec{k}) \hat{b}_{\vec{k},\pm} \tilde{\mathcal{X}}_\pm(\tau, \vec{k}) e^{i\vec{k}\cdot\vec{x}} + \vec{\epsilon}_\pm{}^*(\vec{k}) \hat{b}^\dagger_{\vec{k},\pm} \tilde{\mathcal{X}}^*_\pm(\tau, \vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right\}.$$

and

$$[\hat{\vec{\tilde{X}}}_L(\tau, \vec{x}), \hat{\vec{\tilde{\Pi}}}_L(\tau, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}),$$

$$[\hat{\vec{\tilde{X}}}_\pm(\tau, \vec{x}), \hat{\vec{\tilde{\Pi}}}_\pm(\tau, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}),$$

with

$$[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}] = \delta^{(3)}(\vec{k} - \vec{k'})$$
$$[\hat{b}_{\vec{k},\lambda}, \hat{b}^{\dagger}_{\vec{k}',\lambda'}] = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k} - \vec{k'})$$

Note that the Wronskian:

$$W[v, v^*] \equiv v'v^* - v'^*v$$

is time-independent and normalized as follows

$$W[\tilde{\mathcal{X}}_L, \tilde{\mathcal{X}}_L^*] = W[\tilde{\mathcal{X}}_\pm, \tilde{\mathcal{X}}_\pm^*] = -i. \quad (1)$$

To solve equations of motion for the two transversely-polarized mode and the single longitudinally-polarized mode we impose the Bunch-Davies initial conditions:

$$\lim_{\tau \to -\infty} \tilde{\mathcal{X}}_L(\tau, \vec{k}) = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \quad \lim_{\tau \to -\infty} \tilde{\mathcal{X}}_\pm(\tau, \vec{k}) = \frac{1}{\sqrt{2k}} e^{-ik\tau}.$$

Gravitational production of dark matter

The action for the vector DM in a background metric $\bar{g}_{\mu\nu}$ is given by

$$\mathcal{L}_{\rm DM} = \int d^4x \sqrt{-\bar{g}} \left(-\frac{1}{4} \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} X^{\mu\nu} X^{\alpha\beta} - \frac{1}{2} m_X^2 \bar{g}_{\mu\nu} X^{\mu} X^{\nu} \right) ,$$

where the background metric is of the FLRW form with the line element

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2 .$$

The energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-\bar{g}}} \frac{\delta(\sqrt{-\bar{g}} \mathcal{L}_{\rm DM})}{\delta \bar{g}_{\mu\nu}},$$

one can find the energy density of the vector DM as

$$\rho_X = T_{00} = \frac{1}{2a^2} (|\dot{\vec{X}} - \nabla X_0|^2 + \frac{1}{a^2} |\vec{\nabla} \times \vec{X}|^2 + a^2 m_X^2 X_0^2 + m_X^2 \vec{X}^2) .$$

$$\langle\rho_L\rangle = \frac{1}{2\pi^2 a^4}\int dk k^2\left[|\tilde{\mathcal{X}}_L'|^2 + \left(\tilde{\mathcal{X}}_L'\tilde{\mathcal{X}}_L^* + \tilde{\mathcal{X}}_L'^*\tilde{\mathcal{X}}_L\right) + \frac{A'(\tau)}{A(\tau)}\left(\frac{A'(\tau)^2}{A^2(\tau)} + k^2 + a^2 m_X^2\right)|\tilde{\mathcal{X}}_L|^2\right],$$

$$\langle\rho_\pm\rangle = \frac{1}{2\pi^2 a^4}\int dk k^2\left[|\mathcal{X}_\pm'|^2 + \left(k^2 + a^2 m_X^2\right)|\mathcal{X}_\pm|^2\right],$$

where $\langle\rho_L\rangle \equiv \langle 0| : \hat{\rho}_L : |0\rangle$ and $\langle\rho_\pm\rangle \equiv \langle 0| : \hat{\rho}_\pm : |0\rangle$ and

$$A(\tau) \equiv \frac{\sqrt{k^2 + a^2(\tau) m_X^2}}{a(\tau) m_X}.$$

Equations of motion for longitudinal and transversely polarized Fourier modes:

$$\tilde{\mathcal{X}}_L'' + \omega_L^2(\tau)\tilde{\mathcal{X}}_L = 0, \qquad \mathcal{X}_\pm'' + \omega_\pm^2(\tau)\mathcal{X}_\pm = 0,$$

where the time-dependent frequencies are given by

$$\omega_L^2(\tau) = k^2 + a^2 m_X^2 - \frac{k^2}{k^2 + a^2 m_X^2}\left(\frac{a''}{a} - \frac{3a^2 m_X^2}{k^2 + a^2 m_X^2}\frac{a'^2}{a^2}\right),$$

$$\omega_\pm^2(\tau) = k^2 + a^2 m_X^2.$$

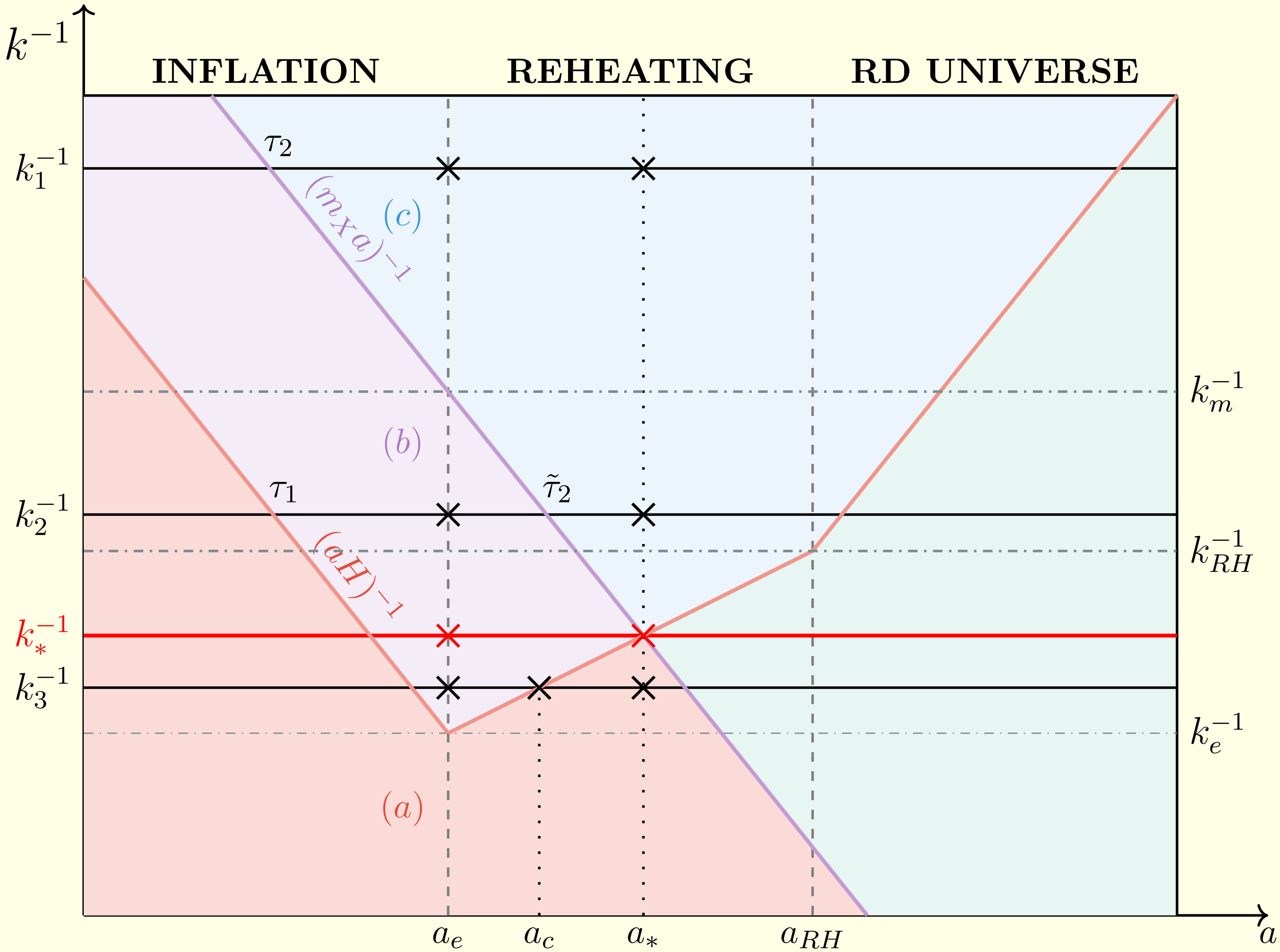


Figure 1: Scaling of the energy density as a function of the scale factor for heavy DM i.e $H_{\mathsf{RH}} < m_X$. The main contribution to the total energy density comes from the mode $k_* \equiv a(\tau : H = m_X)m_X$. Here $k_e \equiv a_e H_e$, $k_m \equiv a_e m_X$ and $k_{\mathsf{RH}} \equiv a_{\mathsf{RH}} H_{\mathsf{RH}}$.

- For vector DM mass $H_{\mathsf{RH}} < m_X < H_I$,

$$\frac{d\langle n_L(\tau : H = m_X)\rangle}{d\ln k} = \frac{1}{8\pi^2}\begin{cases} H_I^{\frac{2\left(3w^2+3w+2\right)}{(w+1)(3w+1)}} m_X^{\frac{2}{1+w}} \left(\frac{a_e}{k}\right)^{\frac{3(1-w)}{(1+3w)}}, & \text{for } k_* < k < k_e, \\ H_I^{\frac{2(1+3w)}{3(1+w)}} m_X^{\frac{1-3w}{3(1+w)}} \left(\frac{k}{a_e}\right)^2, & \text{for } k < k_*. \end{cases}$$

- For vector DM mass $H_{\mathsf{RME}} < m_X < H_{\mathsf{RH}}$,

$$\frac{d\langle n_L(\tau : H = m_X)\rangle}{d\ln k} = \frac{1}{8\pi^2}\begin{cases} m_X^{3/2} H_I^{\frac{3(w+3)}{2(3w+1)}} \gamma^{\frac{1-3w}{1+w}} \left(\frac{a_e}{k}\right)^{\frac{3(1-w)}{1+3w}}, & \text{for } k_{\mathsf{RH}} < k < k_e, \\ m_X^{3/2} H_I^{5/2} \gamma^{\frac{-1+3w}{3(1+w)}} \left(\frac{a_e}{k}\right), & \text{for } k_* < k < k_{\mathsf{RH}}, \\ H_I \gamma^{\frac{2(1-3w)}{3(1+w)}} \left(\frac{k}{a_e}\right)^2, & \text{for } k < k_*. \end{cases}$$

$$\gamma \equiv \sqrt{\frac{H_{\mathsf{RH}}}{H_I}} \quad \text{and} \quad p = w\rho$$

Note that the number density per ln frequency has a peak structure if and only if $w \in (-\frac{1}{3}, 1)$. In this case, $d\langle n_L(H = m_X)\rangle / d \ln k$ is dominated by modes with $k = k_* \equiv a(\tau : H = m_X) m_X$ and $k_e \equiv a_e H_I$.

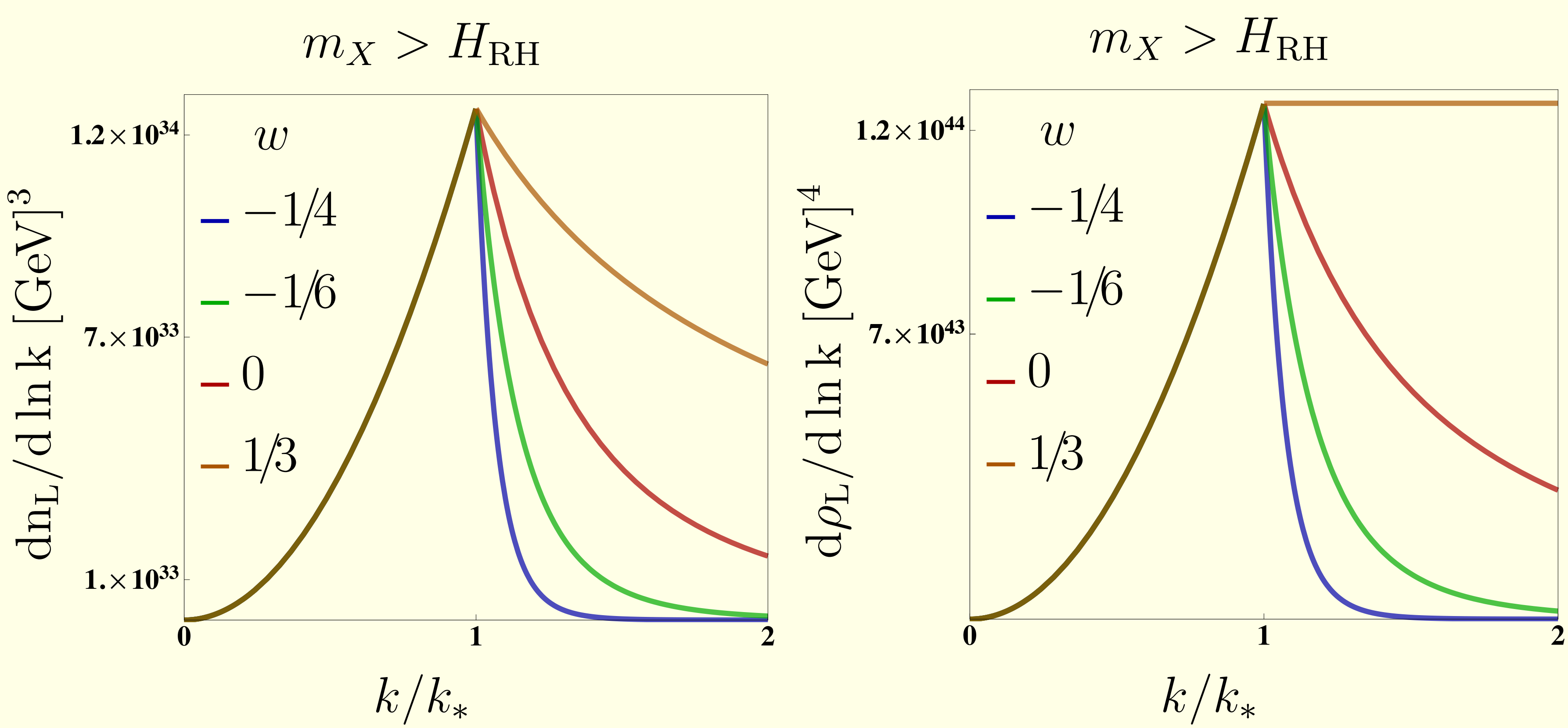


Figure 2: The energy density momentum distribution for different values of the equation-of state parameter w, for $m_X > H_{RH}$, $k_* \equiv a(\tau : H = m_X) m_X$.

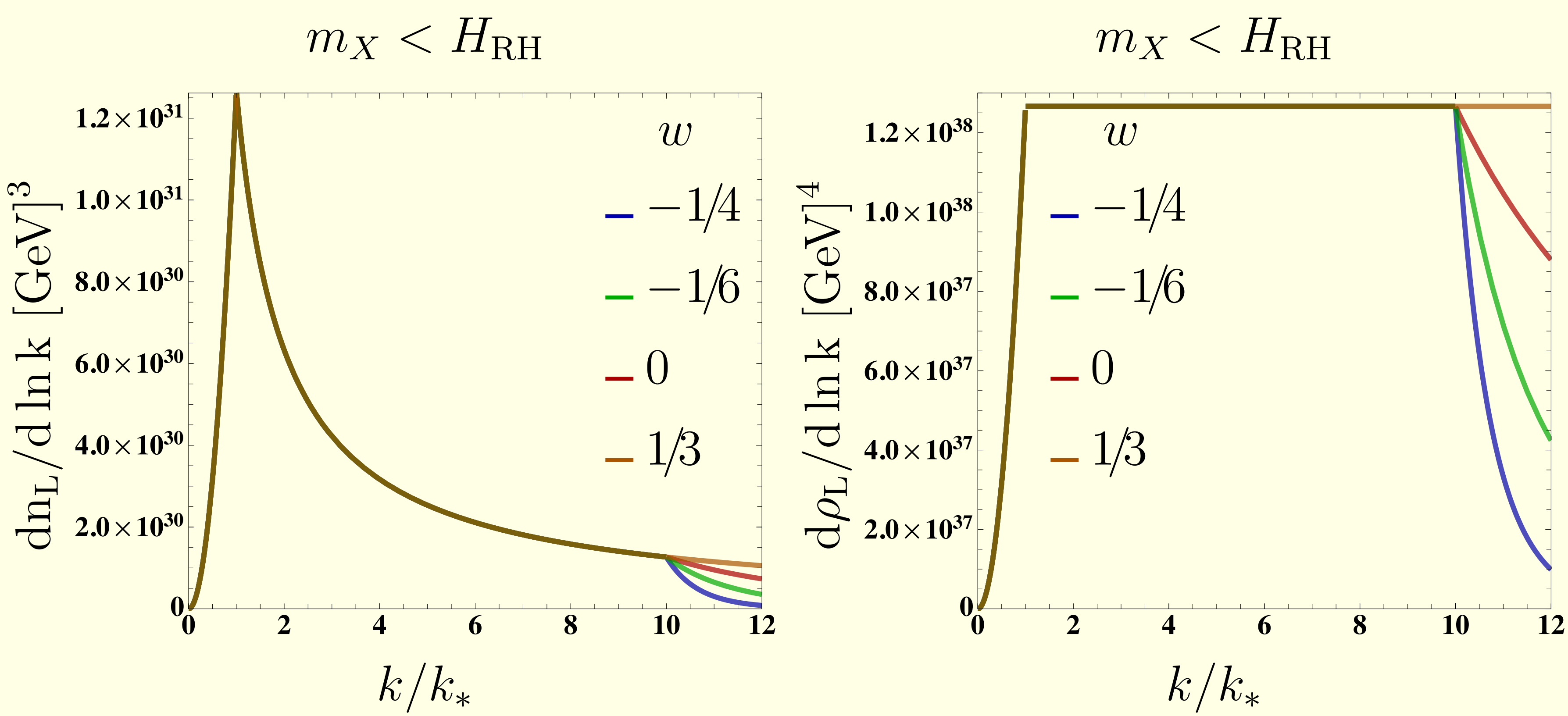


Figure 3: The energy density momentum distribution for different values of the equation-of state parameter w, for $m_X < H_{RH}$, $k_* \equiv a(\tau : H = m_X) m_X$.

Finally the DM present-day relic abundance can be calculated as,

$$
\begin{aligned}
\Omega_X h^2 &= T_* \frac{s_0 h^2}{4\rho_c \kappa^{-2}} \frac{n_*}{m_X} \\
&= \frac{1}{8\pi^2}\left(\frac{45}{128\pi^2 g_*(T_{\mathsf{RH}})}\right)^{1/4} \frac{s_0 h^2}{\kappa^{-3/2}\rho_c} \\
&\times \begin{cases} \left(\frac{1}{2} + \frac{1+3w}{3(1-w)}\right)\gamma^{1/2} H_I^{9/4} m_X^{1/4}, & H_{\mathsf{RH}} < m_X < H_I \\ \left(\frac{3}{2} H_I^2 m_X^{1/2} + \frac{1+3w}{3(1-w)} H_I^{3/2} m_X \gamma^{-1}\right), & H_{\mathsf{RME}} < m_X < H_{\mathsf{RH}} \end{cases}
\end{aligned} \tag{4}
$$

where $s_0 = 2970\ \mathrm{cm}^{-3}$, $\rho_c = 1.054 \times 10^{-5} h^2$ GeVcm^{-3}, $\kappa^{-1} = M_{Pl} = 2.435 \times 10^{18}$ GeV and $\Omega_X^{\mathrm{obs}} h^2 = 0.1198 \pm 0.002$. Furthermore, we assume that $g_*(T_{\mathsf{RH}}) \simeq 106$, i.e. no extra d.o.f. beyond the SM.

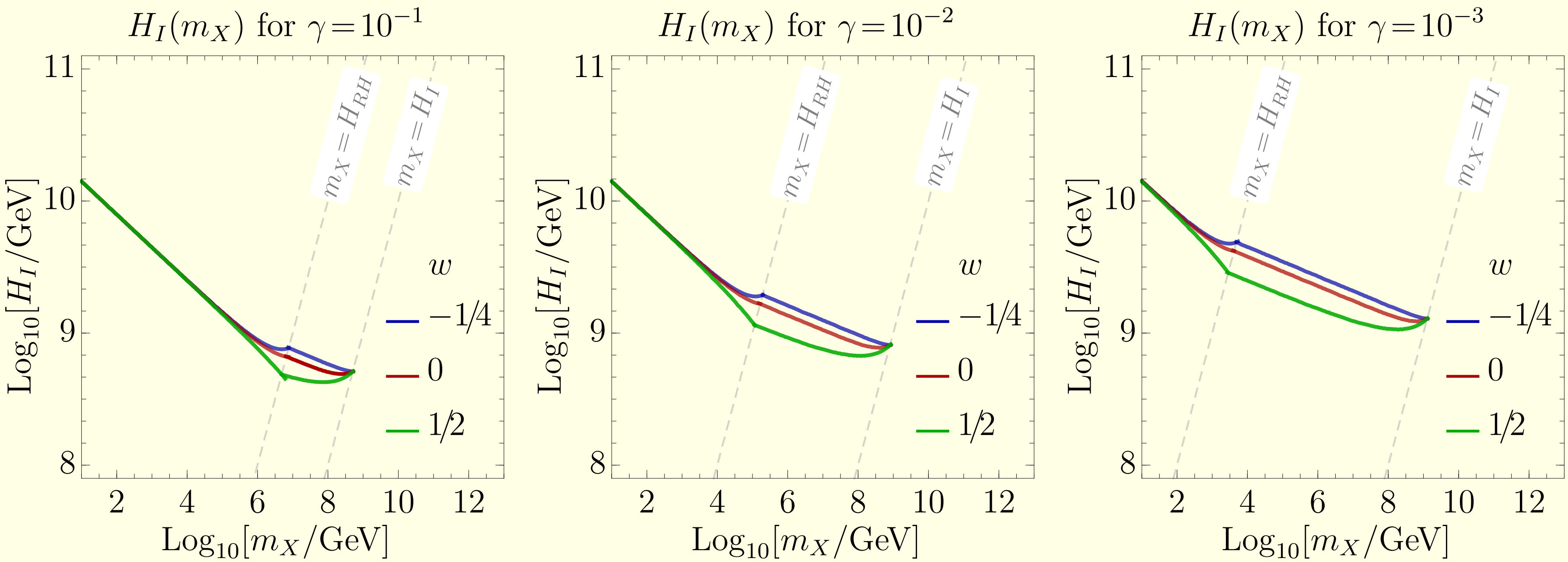


Figure 4: Relations between the Hubble rate at the end of inflation H_I and vector DM mass m_X that reproduces the observed relic abundance $\Omega_{\mathrm{DM}} h^2$ for the gravitational production only.

Reheating and perturbative production of dark matter

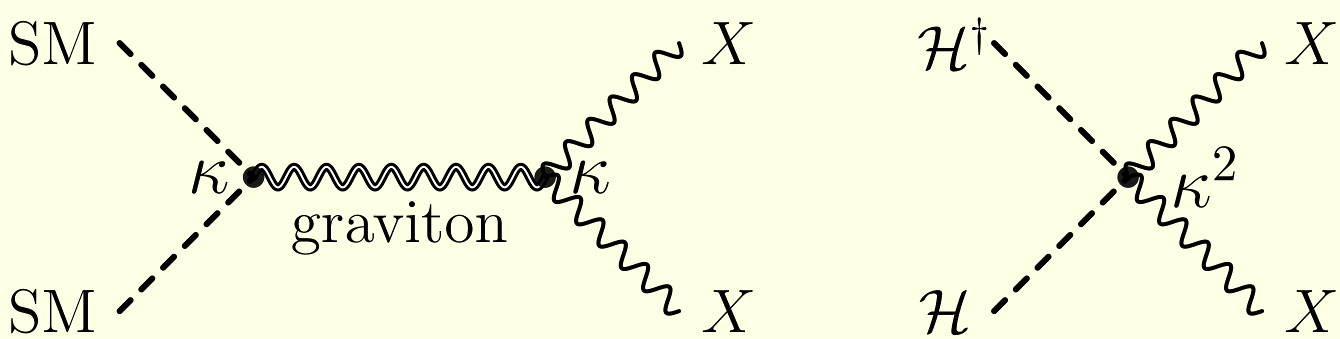
$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + \mathcal{L}_{\rm SM} + \mathcal{L}_{\rm DM} + \mathcal{L}_{\rm int}\right] = \int d^4x \mathcal{L}_{\rm eff},$$

$$\mathcal{L}_{\rm eff} = \mathcal{L}^{(4)}_{\rm SM} + \mathcal{L}^{(4)}_{\rm DM} + \mathcal{L}^{(5)}_{\rm int} + \mathcal{L}^{(6)}_{\rm int} + \mathcal{O}(\kappa^3) \quad \text{for} \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\mathcal{L}^{(4)}_{\rm DM} = -\frac{1}{4}X_{\mu\nu}X^{\mu\nu} + \frac{1}{2}m_X^2 X_\mu X^\mu \,,$$

$$\mathcal{L}^{(5)}_{\text{int}} = \frac{\kappa}{2}h^{\mu\nu}\left[T^{\rm SM}_{\mu\nu} + T^{\rm DM}_{\mu\nu}\right],$$

$$\mathcal{L}^{(6)}_{\text{int}} = \frac{\kappa^2}{2}\mathcal{C}_X m_X^2 |\mathcal{H}|^2 X_\mu X^\mu \qquad \leftarrow \qquad \mathcal{C}_X \kappa^2 (D^\mu\Phi)^*(D_\mu\Phi)\,\mathcal{H}^\dagger\mathcal{H}$$



Amplitudes for the annihilation of SM particles (with spin $i=0,1/2,1$ before SSB, i.e. $m_i=0$) to VDM squared and summed over all spins read

$$\sum_{\text{spins}}|\mathcal{M}_{0\to 1}|^2=\frac{\kappa^4}{16s^2}\Big[3(m_X^2-t)^2(m_X^2-s-t)^2+4\mathcal{C}_X^2 s^2\Big(12m_X^4-4m_X^2 s+s^2\Big)$$

$$+4\mathcal{C}_X s\Big(6m_X^6-5m_X^4 s-12m_X^4 t+m_X^2 s^2+4m_X^2 st+6m_X^2 t^2+s^2 t+st^2\Big)\Big],$$

$$\sum_{\text{spins}}|\mathcal{M}_{1/2\to 1}|^2=-\frac{\kappa^4}{32s^2}\Big[12m_X^8-12m_X^6(s+4t)+m_X^4\left(5s^2+48st+72t^2\right)$$

$$-2m_X^2\left(2s^3+11s^2t+30st^2+24t^3\right)+t(s+t)\left(5s^2+12st+12t^2\right)\Big],$$

$$\sum_{\text{spins}}|\mathcal{M}_{1\to 1}|^2=\frac{\kappa^4}{8s^2}\Big[\Big(m_X^4-2m_X^2t+s^2+st+t^2\Big)\Big(3m_X^4-6m_X^2t+s^2+3st+3t^2\Big)\Big],$$

The freeze-in:

$$\sigma(s)_{1\to 1} = \frac{\kappa^4}{7680\pi\sqrt{s}\sqrt{s-4m_X^2}}\Big[6[40\mathcal{C}_X(6\mathcal{C}_X-1)+3]m_X^4 +$$

$$2[40\mathcal{C}_X(1-6\mathcal{C}_X)+3]m_X^2 s + [20\mathcal{C}_X(6\mathcal{C}_X-1)+3]s^2\Big],$$

$$\sigma(s)_{1/2\to 1} = \frac{\kappa^4\left(48m_X^4+56m_X^2 s+13s^2\right)}{15360\pi\sqrt{s\left(s-4m_X^2\right)}},$$

$$\sigma(s)_{1\to 1} = \frac{\kappa^4\left(48m_X^4+56m_X^2 s+13s^2\right)}{3840\pi\sqrt{s}\sqrt{s-4m_X^2}}.$$

The corresponding thermally averaged cross sections:

$$\langle \sigma v \rangle_{1\to 1} = \frac{\kappa^4}{23040\pi K_2^2(m_X/T)} \left\{ K_2^2\left(\frac{m_X}{T}\right)\left[6\left(m_X^2+4T^2\right)+160T^2\mathcal{C}_X\left(6\mathcal{C}_X-1\right)\right] + \right.$$
$$+ m_X K_1\left(\frac{m_X}{T}\right)\left[m_X\left(40\left(6\mathcal{C}_X-1\right)\mathcal{C}_X+9\right)K_1\left(\frac{m_X}{T}\right) + \right.$$
$$\left.\left. + 4T\left(20\left(6\mathcal{C}_X-1\right)\mathcal{C}_X+3\right)K_2\left(\frac{m_X}{T}\right)\right]\right\}$$

$$\langle \sigma v \rangle_{1/2\to 1} = \frac{\kappa^4}{11520\pi K_2^2(m_X/T)} \left\{ K_2^2\left(\frac{m_X}{T}\right)\left[9m_X^2+26T^2\right] + \right.$$
$$\left. + m_X K_1\left(\frac{m_X}{T}\right)\left[11m_X+13TK_2\left(\frac{m_X}{T}\right)\right]\right\}$$

$$\langle \sigma v \rangle_{1\to 1} = \frac{\kappa^4}{2880\pi K_2^2(m_X/T)} \left\{ K_2^2\left(\frac{m_X}{T}\right)\left[9m_X^2+26T^2\right] + \right.$$
$$\left. + m_X K_1\left(\frac{m_X}{T}\right)\left[11m_X+13TK_2\left(\frac{m_X}{T}\right)\right]\right\} = 4\left\langle \sigma v \right\rangle_{1/2\to 0}$$

The total cross-section for the vector DM production can be written as a sum of those three contributions from all the SM particles as,

$$\langle \sigma v \rangle = N_0 \, \langle \sigma v \rangle_{0 \to 1} + N_{1/2} \, \langle \sigma v \rangle_{1/2 \to 1} + N_1 \, \langle \sigma v \rangle_{1 \to 1} \, ,$$

where $N_0 = 4, N_{1/2} = 45$, and $N_1 = 12$ denote numbers of degrees of freedom (before SSB).

Inflaton decay to DM

$$\mathcal{L}^{\phi}_{\rm int} = -\mathcal{C}_{\phi X} \kappa \frac{m_X^2}{2} \phi X_\mu X^\mu \qquad \leftarrow \qquad \mathcal{C}_{\phi X} \kappa (D^\mu \Phi)^* (D_\mu \Phi) \phi$$

$$\Downarrow$$

$$\Gamma_{\phi \to XX} = \frac{\kappa^2 \mathcal{C}_{\phi X}^2}{128 \pi m_\phi} \sqrt{1 - 4 \frac{m_X^2}{m_\phi^2}} \left(m_\phi^4 - 4 m_X^2 m_\phi^2 + 12 m_X^4 \right) ,$$

The time evolution of the energy density ρ_i and number density n_i of each considered species is encoded in the following system of three coupled Boltzmann equations

$$
\begin{aligned}
\dot{\rho}_\phi + 3(1+w)H\rho_\phi &= -\Big(\langle\Gamma^\phi_R\rangle + \langle\Gamma^\phi_X\rangle\Big)\rho_\phi\,, \\
\dot{\rho}_R + 4H\rho_R &= \langle\Gamma^\phi_R\rangle\rho_\phi + 2\langle E_X\rangle\langle\sigma|v|\rangle\Big(n_X^2 - \bar{n}_X^2\Big)\,, \\
\dot{n}_X + 3Hn_X &= \langle\Gamma^\phi_X\rangle\frac{\rho_\phi}{m_\phi} - \langle\sigma|v|\rangle\Big(n_X^2 - \bar{n}_X^2\Big), \\
H^2 &= \frac{\kappa^2}{3}\left(\rho_\phi + \rho_X + \rho_R\right),
\end{aligned}
$$

where $\bar{n}_X$ denotes the equilibrium number density of the vector DM X_μ, while w parametrizes the equation of state of the inflaton field.

We introduce rescaled dimensionless variables to solve the above Boltzmann equations,

$$\Phi = \rho_\phi \frac{a^{3(1+w)}}{T_{\mathsf{RH}}^4}, \qquad \mathcal{R} = \rho_R \frac{a^4}{T_{\mathsf{RH}}^4}, \qquad \mathcal{N}_X = n_X \frac{a^3}{T_{\mathsf{RH}}^3}.$$

It is convenient to use the scale factor, rather than time:

$$\frac{da}{aH} = dt.$$

The Hubble parameter expressed in terms of the new variables is given by

$$H^2 = \frac{\kappa^2}{3}\frac{T_{\mathsf{RH}}^4}{a^3}\left[\Phi a^{-3w} + \mathcal{R} a^{-1} + \mathcal{N}_X \frac{\langle E_X \rangle}{T_{\mathsf{RH}}}\right].$$

We can also rewrite Boltzmann equations as follows

$$\begin{aligned}
\frac{d\Phi}{da} &= -\frac{\langle \Gamma_R^\phi \rangle + \langle \Gamma_X^\phi \rangle}{aH}\Phi, \\
\frac{d\mathcal{R}}{da} &= \frac{\langle \Gamma_R^\phi \rangle}{H}\Phi a^{-3w} + \frac{\langle \sigma v \rangle}{H a^3} 2\langle E_X \rangle T_{\mathsf{RH}}^2 \left(\mathcal{N}_X^2 - \bar{\mathcal{N}}_X^2\right) \qquad (5) \\
\frac{d\mathcal{N}_X}{da} &= \frac{\langle \Gamma_X^\phi \rangle}{H}\frac{T_{\mathsf{RH}}}{m_\phi}\Phi a^{-(1+3w)} - \frac{\langle \sigma v \rangle}{H a^4}\left(\mathcal{N}_X^2 - \bar{\mathcal{N}}_X^2\right).
\end{aligned}$$

We adopt the following assumptions:

(i) For the inflaton decay, the dominant decay channel is $\phi \to RR$, while $\phi \to XX$ is sub-dominant, i.e. $\Gamma_R^\phi \gg \Gamma_X^\phi$, so that the standard cosmology is not affected substantially,

(ii) During the reheating, i.e. period between H_I^{-1} and $\Gamma_R^{\phi\,-1}$ the dominant part of the total energy density was in the form of the inflaton field.

(iii) The reheating period is followed by the RD epoch during which the total energy density is dominated by the ρ_R.

The second assumption allows us to fix the initial conditions for the set of new variables,

$$\Phi_I = \frac{3H_I^2}{\kappa^2 T_{\text{RH}}^4}, \qquad \mathcal{R}_I = 0, \qquad \mathcal{N}_{X_I} = 0.$$

Temperature – scale factor relation

$$\mathcal{R} = \frac{\pi^2 g_*(T_{\mathsf{RH}})}{30\gamma^2} \begin{cases} \frac{2}{5-3w}\left(a^{\frac{5}{2}(1-3w/5)} - 1\right), & \text{for} \quad w \neq \frac{5}{3} \\ \ln a, & \text{for} \quad w = \frac{5}{3} \end{cases}$$

with $\gamma \equiv \sqrt{\frac{H_{\mathsf{RH}}}{H_I}}$.

The temperature of the system is related to radiation energy density $\mathcal{R}$ as follows

$$T = \left(\frac{30}{\pi^2 g_*(T)}\right)^{1/4} T_{\mathsf{RH}} \mathcal{R}^{1/4} a^{-1}.$$

Therefore, we can express the temperature in terms of the scale factor as,

$$T(a) = \left(\frac{90}{\pi^2 g_*(T)}\right)^{\frac{1}{4}} \sqrt{\frac{\gamma H_I}{\kappa}} \begin{cases} \left(\frac{2}{5-3w}\right)^{\frac{1}{4}} \left(a^{-\frac{3}{2}(1+w)} - a^{-4}\right)^{\frac{1}{4}}, & \text{for} \quad w \neq \frac{5}{3}, \\ \left(\frac{\ln a}{a^4}\right)^{\frac{1}{4}}, & \text{for} \quad w = \frac{5}{3}. \end{cases}$$

The Hubble rate – scale factor relation

$$H(a) = \begin{cases} H_I a^{-\frac{3}{2}(1+w)}, & \text{for} \quad a < a_{\rm RH} \\ H_{\rm RH}\left(\frac{a_{\rm RH}}{a}\right)^2, & \text{for} \quad a > a_{\rm RH} \end{cases}$$

DM relic abundance

(i) the DM production dominated by annihilation of SM particles in the thermal bath, i.e. via annihilation freeze-in mechanism,

(ii) the DM mostly produced through inflaton decays.

(i)

$$\frac{d\mathcal{N}_X^{\rm FI}}{da} \simeq \frac{\langle\sigma|v|\rangle}{Ha^4}\bar{\mathcal{N}}_X^{\rm FI},$$

$$\mathcal{N}_{X\infty}^{\rm FI} \simeq \frac{1}{T_{\rm RH}^3}\int_1^{a_{\rm RH}} da\,\frac{a^2}{H(a)}\langle\sigma v\rangle\bar{n}_X^2 + \frac{1}{T_{\rm RH}^3}\int_{a_{\rm RH}}^{a_f} da\,\frac{a^2}{H(a)}\langle\sigma v\rangle\bar{n}_X^2,$$

(ii)

$$\frac{d\mathcal{N}_X^{\phi}}{da} = 3\frac{\langle\Gamma_X^{\phi}\rangle H_I}{\kappa^2 m_\phi} T_{\mathsf{RH}}^{-3} a^{\frac{1}{2}(1-3w)},$$

$$\mathcal{N}_{X_{\mathsf{RH}}}^{\phi} = \frac{\mathcal{C}_{\phi X}^2}{64\pi(1-w)}\left(\frac{\pi^2 g_*(T_{\mathsf{RH}})}{90}\right)^{3/4}\sqrt{1-4\frac{m_X^2}{m_\phi^2}}\left(m_\phi^4 - 4m_X^2 m_\phi^2 + 12 m_X^4\right)$$
$$\times \frac{\kappa^{3/2}}{\gamma^3 H_I^{1/2} m_\phi^2}\left(a_{\mathsf{RH}}^{\frac{3}{2}(1-w)} - 1\right).$$

$$\Downarrow$$

$$\Omega_X h^2 = \frac{45}{2\pi^2 g_*(T_{\mathsf{RH}})}\frac{s_0 h^2}{\rho_c}\gamma^{\frac{4}{1+w}} m_X \mathcal{N}_X(T_0).$$

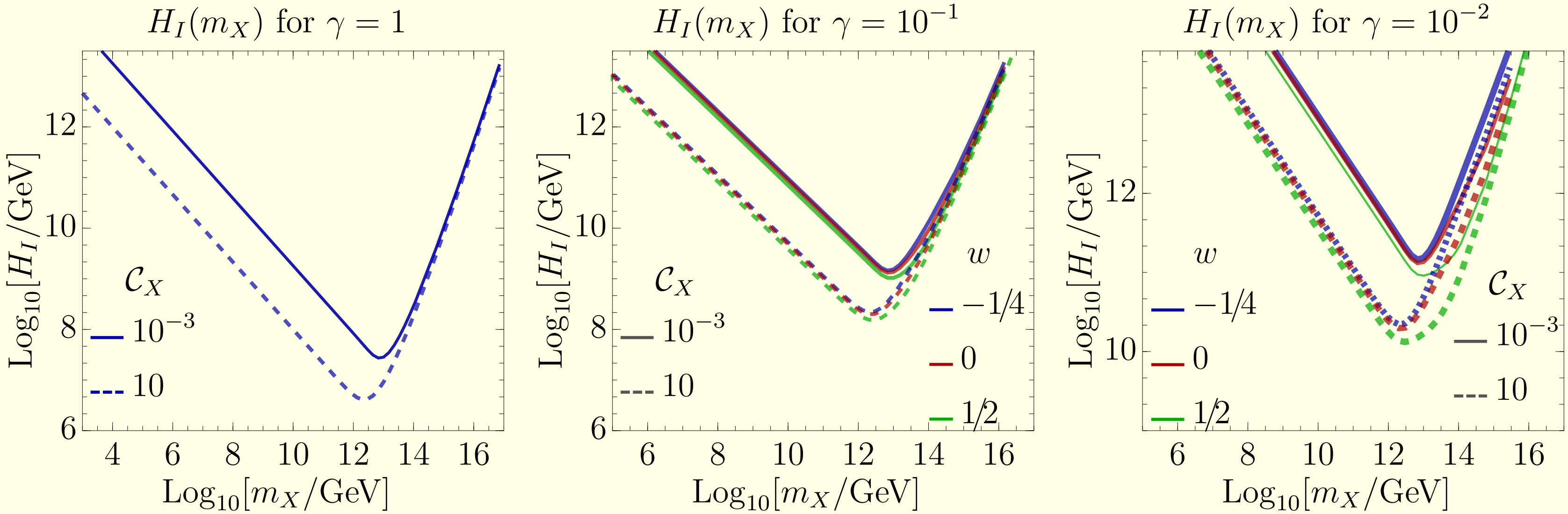


Figure 5: Relations between the Hubble rate at the end of inflation H_I and vector DM mass m_X that reproduces the observed relic abundance $\Omega_{\rm DM}h^2$ for the freeze-in through the dim-6 operator.

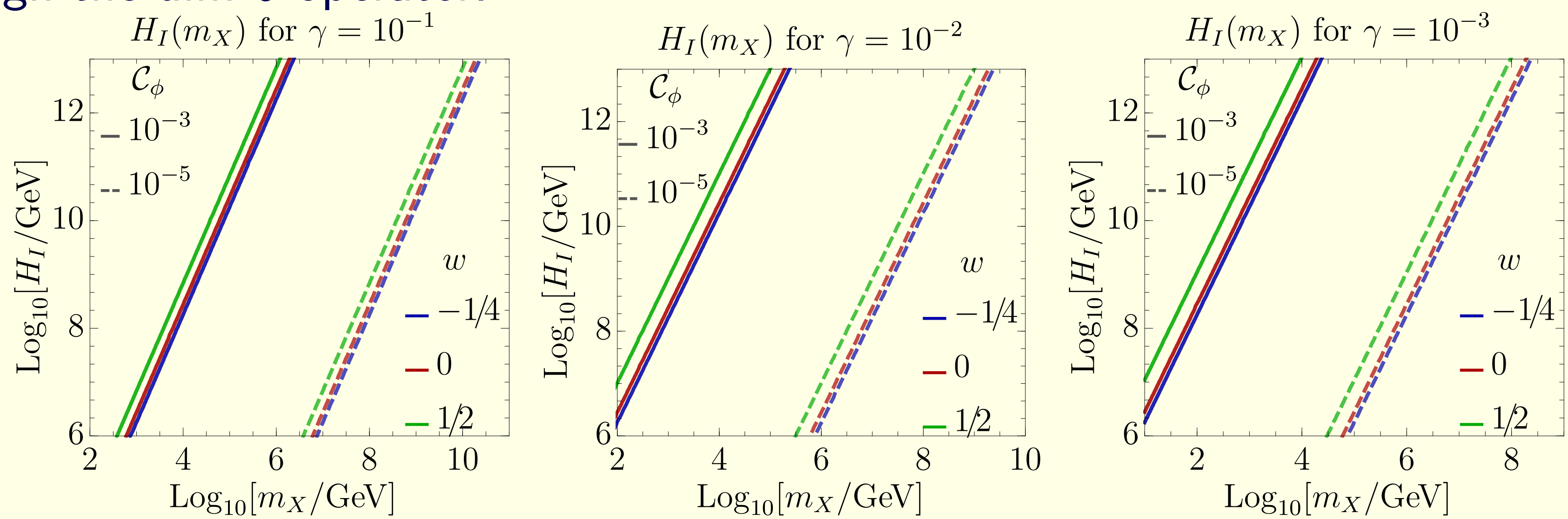


Figure 6: Relations between the Hubble rate at the end of inflation H_I and vector DM mass m_X that reproduces the observed relic abundance $\Omega_{\rm DM}h^2$ for the inflaton decay through the dim-5 operator.

Summary

- Evidence for DM is only of gravitational origin.
- DM that interacts with the SM exclusively through gravity is a viable option consistent with the observed DM abundance.
- DM production mechanisms:
  - non-perturbative - gravitational production,
  - perturbative - freeze-in via graviton exchange or dim-6 Planck mass suppressed effective operator, inflaton decays to DM pairs via dim-5 Planck mass suppressed effective operator.

  have been discussed.
- Effects of modified equation of state during reheating have been discussed.

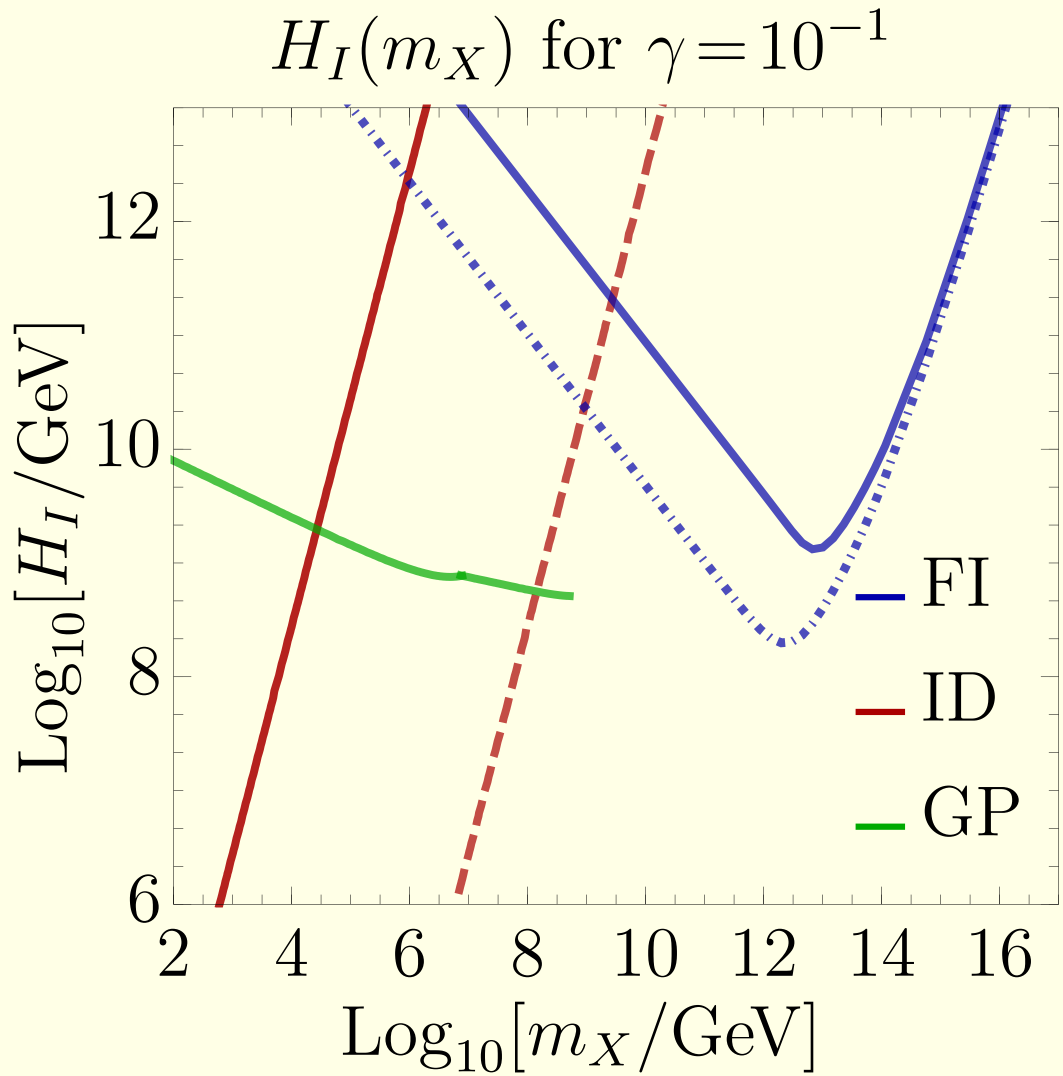


Figure 7: Comparison of the gravitational production, the inflaton decay and dim-6 operator annihilation. The observed relic abundance $\Omega_{\rm DM}h^2$ is reproduced. For solid and dashed lines we used $\mathcal{C}_X = 10^{-3}, 10$ and $\mathcal{C}_\phi = 10^{-3}, 10^{-5}$, respectively.