Determining Inflaton Interactions via Gravitational Waves

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June 12, 2023

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The 16th International Conference on Interconnections between Particle Physics and Cosmology, June 12-16th 2023, Daejeon, Korea

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Basabendu Barman, Anish Ghoshal, BG, and Anna Socha, "Measuring Inflaton Couplings via Primordial Gravitational Waves", arXiv:2305.00027

Post-inflationary evolution of the Universe

$$S = \int d^4x\,\sqrt{-g}\,\left[-\frac{M_{Pl}^2}{2}\,\mathcal{R} + \mathcal{L}_\phi + \mathcal{L}_{\rm SM} + \mathcal{L}_{\rm int}\right]$$

$$ds^2 = dt^2 - a^2(t)\,d\vec{x}^2$$

The inflaton Lagrangian

$$\mathcal{L}_\phi \supset \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

with the α-attractor T-model of inflation potential

$$V(\phi) = \Lambda^4\,\tanh^{2n}\left(\frac{|\phi|}{M}\right) \simeq \Lambda^4 \begin{cases} 1 & |\phi| \gg M, \\ \left(\frac{|\phi|}{M}\right)^{2n} & |\phi| \ll M\,, \end{cases}$$

with $M = \sqrt{6\,\alpha}M_{Pl}$ for $\alpha = 1/6$.

$$\dot{\rho}_\phi + 3(1+\bar{w})H\rho_\phi = -(1+\bar{w})\,\Gamma_\phi\,\rho_\phi,$$
$$\dot{\rho}_{\rm SM} + 4H\rho_{\rm SM} = +(1+\bar{w})\,\Gamma_\phi\,\rho_\phi\,,$$
$$H^2 = \frac{\rho_{\rm tot}}{3M_{Pl}^2}\;\rho_{\rm tot} = \rho_\phi + \rho_{\rm SM}$$

with

$$\bar{w} \equiv \frac{\langle p_\phi \rangle}{\langle \rho_\phi \rangle} = \frac{n-1}{n+1}$$

We adopt the following parametrization of the inflaton width Γ_ϕ:

$$\Gamma_\phi = \Gamma_\phi^e \left(\frac{a_e}{a}\right)^\beta \,,$$

where a_e is the initial value of the scale factor, denoting the end of inflation, Γ_ϕ^e denotes the inflaton width at $a = a_e$, while β is assumed to be a constant parameter.

$$H^{\rm RH}(a) \simeq H_e \left(\frac{a_e}{a}\right)^{\frac{3n}{n+1}}$$

$$\rho_\phi^{\rm RH}(a) \simeq \rho_e \left(\frac{a_e}{a}\right)^{\frac{6n}{n+1}}$$

$$\rho_{\rm SM}^{\rm RH}(a) \simeq \left(\frac{6\,n}{n+1}\right) \cdot \Gamma_\phi^e\, H_e\, M_{Pl}^2 \begin{cases} \frac{n+1}{n+4-\beta\,(n+1)} \left[\left(\frac{a_e}{a}\right)^{\beta+\frac{3n}{n+1}} - \left(\frac{a_e}{a}\right)^4\right], & \beta \neq \frac{n+4}{n+1} \\ \left(\frac{a_e}{a}\right)^4 \ln\left(\frac{a}{a_e}\right), & \beta = \frac{n+4}{n+1}, \end{cases}$$

which can further be written as

$$\rho_{\rm SM}^{\rm RH}(a) \approx \rho_{\rm rh} \left(\frac{a_{rh}}{a}\right)^4 \begin{cases} \frac{1-(a_e/a)^{\frac{\beta\,(n+1)-(n+4)}{n+1}}}{1-(a_e/a_{rh})^{\frac{\beta\,(n+1)-(n+4)}{n+1}}}, & \beta \neq \frac{n+4}{n+1} \\ \frac{\ln(a/a_e)}{\ln(a_{rh}/a_e)}, & \beta = \frac{n+4}{n+1} \end{cases}$$

$$\rho_\phi^{\rm RH}(a_{rh}) = \rho_{\rm SM}^{\rm RH}(a_{rh}) \equiv \rho_{\rm rh}.$$

$$T^{\mathrm{RH}}(a) \propto \left(\Gamma_\phi^e H_e M_{Pl}^2\right)^{1/4} \times \begin{cases} \frac{n+1}{n+4-\beta(n+1)} \left[\left(\frac{a_e}{a}\right)^{\beta+\frac{3n}{n+1}} - \left(\frac{a_e}{a}\right)^4 \right]^{1/4}, & \beta \neq \frac{n+4}{n+1}, \\ \frac{a_e}{a}[\ln(a/a_e)]^{1/4}, & \beta = \frac{n+4}{n+1} \end{cases}$$

$$a_{rh} = a_e \begin{cases} \left(\frac{n+4-\beta(n+1)}{2n}\frac{H_e}{\Gamma_\phi^e}\right)^{\frac{n+1}{n(3-\beta)-\beta}}, & \beta \ll \frac{n+4}{n+1}, \\ \left(\frac{H_e}{\Gamma_\phi^e}\frac{n-2}{n}\right)^{\frac{n+1}{2(n-2)}} \mathcal{W}^{\frac{n+1}{2(2-n)}}\left(\frac{H_e}{\Gamma_\phi^e}\frac{n-2}{n}\right), & \beta = \frac{n+4}{n+1}, \\ \left(\frac{\beta-4+n(\beta-1)}{2n}\frac{H_e}{\Gamma_\phi^e}\right)^{\frac{n+1}{2(n-2)}}, & \beta \gg \frac{n+4}{n+1}, \end{cases}$$

where $\mathcal{W}[z]$ denotes the Lambert $\mathcal{W}$-function.

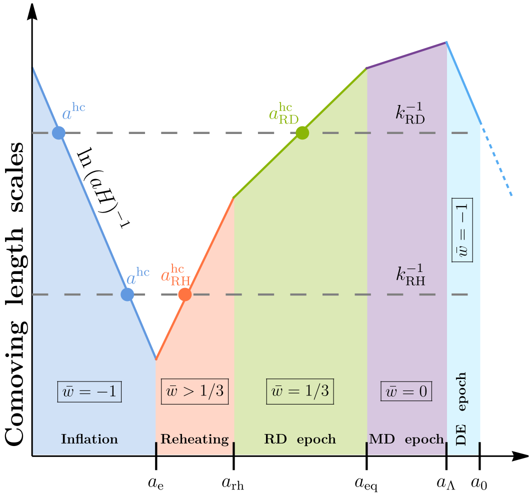

Figure 1: Evolution of cosmological comoving length scales with the scale factor at different epochs of the evolution of the universe. Here "RD" stands for radiation domination, "MD" implies matter domination, and "DE" indicates dark energy domination. In each case corresponding equation of state is also mentioned.

Inflaton-matter interactions

Direct interactions:

$$\mathcal{L}_{\text{int}} \supset \mathcal{L}_{SS\phi} + \mathcal{L}_{\psi\psi\phi} + \mathcal{L}_{VV\phi} + \mathcal{L}_{aa\phi} \,,$$

parametrized as

$$\begin{aligned}
\mathcal{L}_{SS\phi} &\supset g_{S\phi}\, SS\,\phi\,,\\
\mathcal{L}_{\psi\psi\phi} &\supset g_{\psi\phi}\,\overline{\psi}\,\psi\,\phi\,,\\
\mathcal{L}_{VV\phi} &\supset g_{V\phi}\, V_\mu\, V^\mu\,\phi\,,\\
\mathcal{L}_{aa\phi} &\supset h_{a\phi}\,\partial_\mu a\,\partial^\mu a\,\phi\,,
\end{aligned}$$

Parameters:

$$\{\Gamma^{e}_{\phi\to f}\,, \Lambda,\, n\}\,, \qquad \text{or} \qquad \{g_{i\phi}\,, n\,, \Lambda\}\,, \qquad \text{with} \qquad \alpha = 1/6$$

where $f = SS,\ \psi\psi,\ \mathcal{VV}\,, aa$.

$$\ddot{\phi} + 3\,H\,\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0\,,$$

During the period of reheating the oscillating inflaton field reads

$$\phi(t) = \varphi(t)\;\cdot\;\mathcal{P}(t)$$

$\mathcal{P}(t)$ is a quasi-periodic, fast-oscillating, $\varphi(t)$ is a slowly-varying envelope defined by $\rho_\phi \equiv V(\varphi)$

$$\varphi(t) = M_P \left(\frac{\rho_\phi(t)}{\Lambda^4}\right)^{\frac{1}{2n}} \qquad \mathcal{P}(t) = \sum_{l=-\infty}^{\infty} \mathcal{P}_l e^{-il\omega t}$$

The generic solution for $\mathcal{P}(t)$ can be written in terms of the inverse of the regularized incomplete beta function $\mathcal{I}_z^{-1}(i,j)$

$$\mathcal{P}(a) = \left[\mathcal{I}_z^{-1}\!\left(\frac{1}{2n},\frac{1}{2}\right)\right]^{\frac{1}{2n}} \quad \text{with} \quad \mathcal{T} \equiv \frac{2\pi}{\omega} = \frac{\sqrt{4\pi}}{m_\phi}\sqrt{\frac{2n-1}{n}}\,\frac{\Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{n+1}{2n}\right)}$$

$$\Gamma_\phi = \Gamma_\phi^e \left(\frac{a_e}{a}\right)^\beta ,$$

For massless final state:

$$\Gamma^e_{\phi\to SS} = \frac{1+n}{2n}\frac{\omega_e}{2\pi}\left(\frac{g_{S\phi}M_{Pl}}{\Lambda^2}\right)^2\left(\frac{\rho_e}{\Lambda^4}\right)^{\frac{1-n}{2n}}\sum_{l=1}^{\infty} l|\mathcal{P}_l|^2 ,$$

$$\Gamma^e_{\phi\to\psi\bar{\psi}} = \omega_e\cdot\frac{1+n}{2n}\frac{g^2_{\psi\phi}}{4\pi}\left(\frac{\omega_e M_{Pl}}{\Lambda^2}\right)^2\left(\frac{\rho_e}{\Lambda^4}\right)^{\frac{n-1}{2n}}\sum_{l=1}^{\infty} l^3|\mathcal{P}_l|^2$$

$$\Gamma^e_{\phi\to\mathcal{V}\mathcal{V}} = 2\cdot\Gamma^e_{\phi\to SS}(g_{S\phi}\leftrightarrow g_{V\phi})$$

$$\Gamma^e_{\phi\to aa} = \frac{1+n}{2n}\frac{\omega_e}{8\pi}\left(\frac{\omega_e^2 M_{Pl}h_{a\phi}}{\Lambda^2}\right)^2\left(\frac{\rho_e}{\Lambda^4}\right)^{\frac{3n-3}{2n}}\sum_{l=1}^{\infty} l^5|\mathcal{P}_l|^2,$$

and

$$\beta_S = \frac{3(1-n)}{1+n},\qquad \beta_\psi = \frac{3(n-1)}{1+n},\qquad \beta_V = \beta_S,\qquad \beta_a = \frac{9(n-1)}{1+n}$$

$$\omega = \omega_e\cdot\left(\frac{\rho_\phi}{\Lambda^4}\right)^{\frac{n-1}{2n}} \simeq \omega_e\left(\frac{\rho_e}{\Lambda^4}\right)^{\frac{n-1}{2n}}\left(\frac{a_e}{a}\right)^{\frac{3(n-1)}{n+1}},\quad \omega_e \equiv \sqrt{2\,n^2\,\pi}\,\frac{\Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)}\,\frac{\Lambda^2}{M_{Pl}}.$$

Primordial gravitational waves

GWs are described as the tensor metric perturbations in a spatially-flat FLRW Universe

$$ds^2 = dt^2 - a^2(t)\,\left(\delta_{ij} + h_{ij}(t,\mathrm{x})\right)\,dx^i\,dx^j\,.$$

with the transverse-traceless (TT) conditions, $h^i_i = \partial^i\,h_{ij} = 0$.

The linearized Einstein equations over the FRLW background

$$\ddot{h}_{ij} + 3\,H\,\dot{h}_{ij} - \frac{\nabla^2}{a^2}\,h_{ij} = 16\,\pi\,G\,\Pi^{TT}_{ij}\,,$$

with Π^{TT}_{ij} being the transverse and traceless part of the anisotropic stress tensor.

$$h_{ij}(t\,,\mathrm{x}) = \sum_{\lambda=+\,,\times}\int\frac{d^3\mathrm{k}}{(2\,\pi)^3}\,h^\lambda(t\,,\mathrm{k})\,\epsilon^\lambda_{ij}(\mathrm{k})\,\exp\left(i\,\mathrm{k}\cdot\mathrm{x}\right)\,,$$

where $\epsilon^\lambda_{ij}(\mathrm{k})$ are spin-2 polarization tensors.

$$\left(h^\lambda_\mathrm{k}\right)'' + 2\,\frac{a'}{a}\,\left(h^\lambda_\mathrm{k}\right)' + k^2\,h^\lambda_\mathrm{k} = 16\,\pi\,G\,a^2\,\Pi^\lambda_\mathrm{k}\,,$$

where *prime* denotes a derivative with respect to the conformal time

- *Super-Hubble scale:*
  For modes far outside the Hubble horizon, i.e., $k \ll aH$, one can write GWs mode EoM as

$$\left(h_{\mathrm{k}}^{\lambda}\right)'' + 2\,\frac{a'}{a}\,\left(h_{\mathrm{k}}^{\lambda}\right)' \approx 0 \;\Rightarrow\; a^{-2}\;\left(a^2\,h_{\mathrm{k}}^{\lambda\prime}\right)' \approx 0\,,$$

  where we have ignored the source term.

$$h_{\mathrm{k}}^{\lambda}(\tau) = \mathcal{C}_1 + \mathcal{C}_2 \int \frac{d\tau'}{a(\tau')^2} \simeq \mathcal{C}_1\;,$$

  where $\mathcal{C}_{1,2}$ are constants of integration.
- *Sub-Hubble scale:*
  After the end of inflation, modes eventually re-enter the horizon $(k > a\,H)$ and start to oscillate. The EoM can be solved in the WKB from

$$h_{\mathrm{k}}^{\lambda} \simeq \frac{C}{a(\tau)}\;\exp\left(\pm ik\,\tau\right),$$

  where C is some arbitrary constant.

The energy density carried by the GWs is given by

$$\rho_{\mathrm{GW}} = \frac{1}{64\pi G a^2} \left\langle (\partial_\tau h_{ij})^2 + (\nabla h_{ij})^2 \right\rangle$$

$$\rho_{\mathrm{GW}} = \frac{1}{32\,\pi\,G} \int d\;\ln k \left(\frac{k}{a}\right)^2 \left[\frac{k^3}{\pi^2} \sum_\lambda \left|h_{\mathrm{k}}^\lambda\right|^2\right].$$

$$\Omega_{\mathrm{GW}}(\tau, k) \equiv \frac{1}{\rho_c} \frac{d\rho_{\mathrm{GW}}}{d\log k}$$

$$V(\phi) = \Lambda^4 \tanh^{2n}\left(\frac{|\phi|}{M}\right) \simeq \Lambda^4 \begin{cases} 1 & |\phi| \gg M, \\ \left(\frac{|\phi|}{M}\right)^{2n} & |\phi| \ll M, \end{cases}$$

$$\Omega_{\rm GW}^{(0)}(k) = \frac{1}{24}\left(\frac{k}{a_0 H_0}\right)^2 \mathcal{P}_{T,\rm prim}\left(\frac{a^{\rm hc}}{a_0}\right)^2$$

$$\Downarrow$$

$$\Omega_{\rm GW}^{(0)}(k) \propto k^{\frac{2n-4}{2n-1}} \propto f^{\frac{2n-4}{2n-1}}$$

with $f = \frac{k}{2\pi}\frac{1}{a_0}$.

$$n = 2 \quad (\bar{w} = 1/3) \quad \Rightarrow \quad \Omega_{\rm GW}^{(0)\,RD}(k) \propto f^0$$

$$V(\phi) = \Lambda^4 \tanh^{2n}\left(\frac{|\phi|}{M}\right) \simeq \Lambda^4 \begin{cases} 1 & |\phi| \gg M, \\ \left(\frac{|\phi|}{M}\right)^{2n} & |\phi| \ll M, \end{cases}$$

$$\Gamma_\phi = \Gamma_\phi^e \left(\frac{a_e}{a}\right)^\beta ,$$

$n > 2$

$$\Omega_{\rm GW}^{(0),\rm RH}(f) \propto f^{\frac{2n-4}{2n-1}} \Lambda^{\frac{4(2-n)}{4n-2}} \times \begin{cases} \left(\frac{\Lambda^2}{M_{Pl}\Gamma_\phi^e}\right)^{\frac{3n(n-2)}{(1-2n)(\beta(1+n)-3n)}} & \beta \ll \frac{n+4}{n+1} \\ \left(\frac{(n\sqrt{2})^n \Lambda^2}{\sqrt{2}\, M_{Pl}\, \Gamma_\phi^e}\right)^{\frac{n-2}{4n-2}} & \beta = \frac{n+4}{n+1} \\ \left(\frac{\Lambda^2}{M_{Pl}\Gamma_\phi^e}\right)^{\frac{3n}{4n-2}} & \beta \gg \frac{n+4}{n+1} \end{cases}$$

$$\Omega_{\mathrm{GW}}^{(0)}(f) \sim \Omega_{\mathrm{GW}}^{(0),\mathrm{RD}} \begin{cases} 1\,, & f < f_c \\ f^{\frac{2n-4}{2n-1}}\,, & f_c < f < f_{\mathrm{max}} \\ 0\,, & f > f_{\mathrm{max}} \end{cases}$$

where f_c and f_{max} are calculable functions of Γ_ϕ^e.

Constraints, results and discussion

Constraints:

- $\Delta N_{\rm eff}$

$$\rho_{\rm rad}(T \ll m_e) = \rho_\gamma + \rho_\nu + \rho_{\rm GW} = \left[1 + \frac{7}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 N_{\rm eff}\right]\rho_\gamma$$

$$\Delta N_{\rm eff} \equiv N_{\rm eff} - N_{\rm eff}^{\rm SM} = \frac{8}{7}\left(\frac{11}{4}\right)^{\frac{4}{3}}\left(\frac{\rho_{\rm GW}(T)}{\rho_\gamma(T)}\right)$$

$$\left(\frac{h^2\,\rho_{\rm GW}}{\rho_c}\right)\Bigg|_0 = \int_{f_{\rm BBN}}^{f_{\rm max}} \frac{df}{f}\, h^2\,\Omega_{\rm GW}^{(0)}(f) \lesssim 5.6\times 10^{-6}\,\Delta N_{\rm eff}$$

$$\Omega_{\rm GW}\,h^2(f) \lesssim 2\times 5.6\times 10^{-6}\left(\frac{n-2}{2\,n-1}\right)\Delta N_{\rm eff} \lesssim 5.6\times 10^{-6}\,\Delta N_{\rm eff}$$

- BBN: $T_{\rm rh}$

$$T_{\rm rh} \propto \left(M_{Pl}\Gamma_\phi^e \Lambda^2\right)^{\frac{1}{4}} \times \begin{cases} \left(\frac{\Lambda^2}{\Gamma_\phi^e M_{Pl}}\right)^{\frac{\beta(n+1)+3n}{4(\beta(n+1)-3n)}}, & \beta \ll \frac{n+4}{n+1}, \\ \left(\frac{\Lambda^2}{\Gamma_\phi^e M_{Pl}}\right)^{\frac{n+1}{2(n-2)}}, & \beta = \frac{n+4}{n+1}, \\ \left(\frac{\Lambda^2}{\Gamma_\phi^e M_{Pl}}\right)^{\frac{n+1}{4-2n}}, & \beta \gg \frac{n+4}{n+1}, \end{cases}$$

$$T_{\rm rh} \gtrsim 4 \text{ MeV}$$

- Inflation: Λ

$$\Lambda \lesssim 10^{16} \text{ GeV}$$

Constraints, results and discussion

Results:

$$\Gamma_\phi = \Gamma_\phi^e \left(\frac{a_e}{a}\right)^\beta$$

- Scenario I: $\phi \to SS$

$$\Gamma^e_{\phi\to SS} \simeq \frac{1}{\sqrt{6\,\pi}} \left(\frac{3}{2}\right)^{\frac{1}{2\,n}} \frac{(n+1)\,n^{1-n}}{2^{n/2}} \left(\frac{\Gamma\left[\frac{n+1}{2n}\right]}{\Gamma\left[\frac{1}{2n}\right]}\right) \frac{g^2_{S\phi}\,M_{Pl}}{\Lambda^2} \sum \ell\,|\mathcal{P}_\ell|^2 \,,$$

with $\beta_S = \frac{3(1-n)}{n+1}$.

- Scenario II: $\phi \to \psi\psi$

$$\Gamma^e_{\phi\to\psi\bar\psi} \simeq g^2_{\psi\phi}\,\frac{n\,(1+n)}{4}\sqrt{\frac{3\,\pi}{2}} \left(\frac{2}{3}\right)^{\frac{1}{2\,n}} \left(\sqrt{2}\,n\right)^n \left(\frac{\Gamma\left[\frac{n+1}{2n}\right]}{\Gamma\left[\frac{1}{2n}\right]}\right)^3 \frac{\Lambda^2}{M_{Pl}} \sum \ell^3\,|\mathcal{P}_\ell|^2 \,,$$

with $\beta_\psi = 3\,(n-1)/(n+1)$.

- Scenario III: $\phi \to aa$

$$\Gamma^e_{\phi\to aa} \simeq \frac{h^2_{a\phi}}{3} \left(\frac{2}{3}\right)^{\frac{3}{2\,n}} \left(\frac{3\,\pi}{8}\right)^{\frac{3}{2}} 2^{\frac{3n}{2}} \left(\frac{\Gamma\left[\frac{n+1}{2n}\right]}{\Gamma\left[\frac{1}{2n}\right]}\right)^5 \frac{\Lambda^6}{M^3_{Pl}} \sum \ell^5\,|\mathcal{P}_\ell|^2 \,,$$

Discussion:

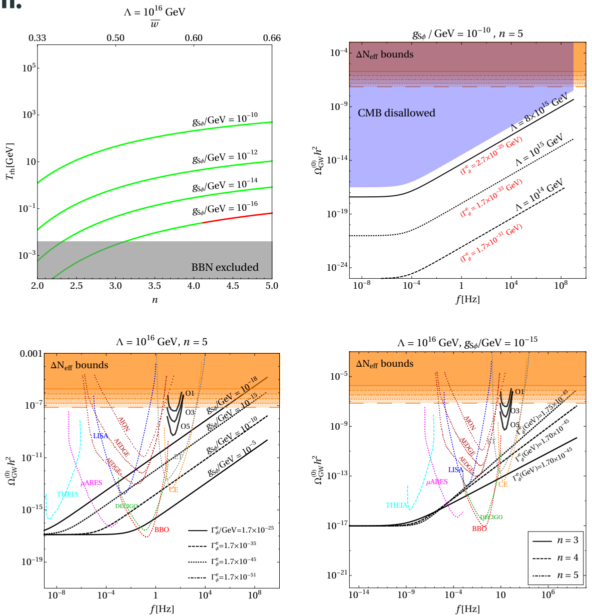

Figure 2: $\phi \to SS$. Constraints: (a) The red part of the curve is discarded from present bound on $\Delta N_{\rm eff}$ due to PLANCK, (b) The grey-shaded region is in conflict with the BBN limit on $T_{\rm rh}$, (c) The orange-shaded regions are discarded from $\Delta N_{\rm eff}$ due to overproduction of GWs, (d) In the bottom and upper-right panels the parameters Λ, n and $g_{S\phi}$ are chosen so that the BBN limit $T_{\rm rh} > 4$ MeV is satisfied.

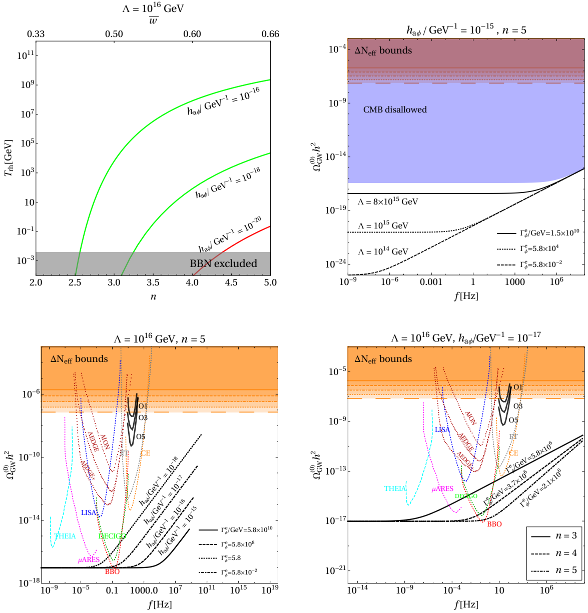

Figure 3: $\phi \to \psi\psi$. $n = 7/2$ corresponds to $\beta_\psi = (n+4)/(n+1)$.

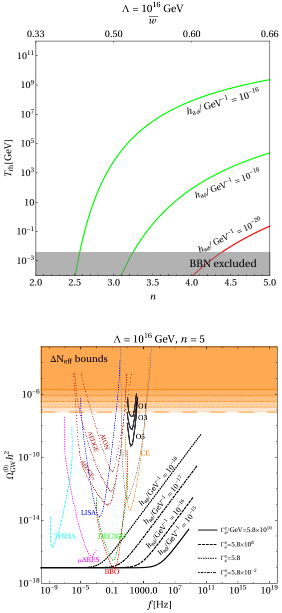

Figure 4: $\phi \to aa$.

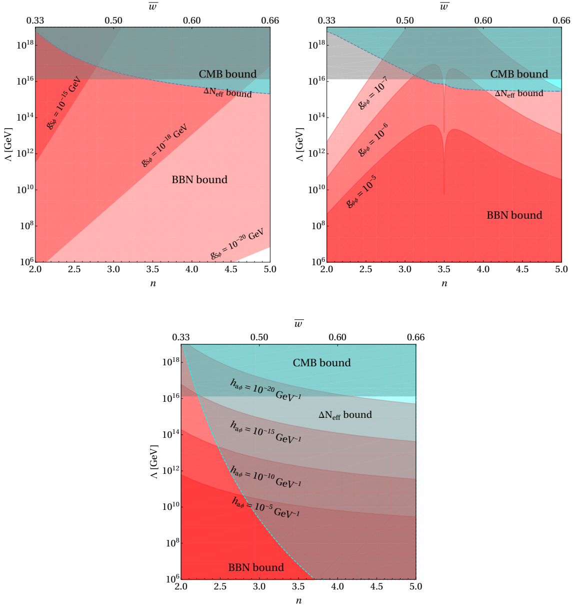

Figure 5: $\phi \to SS$ (top left), $\phi \to \psi\psi$ (top right) and $\phi \to aa$ (bottom) scenario. In all cases the red shaded region is disallowed from BBN bound on reheating temperature $T_{\rm rh} < 4$ MeV, the cyan dashed region is forbidden from PLANCK observed $\Delta N_{\rm eff}$ bound on GW overproduction at $f = f_{\rm max}$, and the "CMB bound" discards scale of inflation $\Lambda > 1.34 \times 10^{16}$ GeV from constraint on tensor to scalar ratio.

Summary

- Possibility of probing inflaton couplings to the visible sector by primordial gravitational waves (GWs) of inflationary origin has been investigated.
- The inflaton-matter "decay width" turns out to be time dependent

$$\Gamma_\phi = \Gamma_\phi^e \left(\frac{a_e}{a}\right)^\beta \qquad \text{with} \qquad \Gamma_\phi^e = \Gamma_\phi^e(g_{i\phi})$$

- The primordial power spectrum and the spectral energy density of primordial GW have been calculated:

$$\Omega_{\rm GW}^{(0),\rm RH}(f) \propto f^{\frac{2n-4}{2n-1}}\, \Lambda^{\frac{4(2-n)}{4n-2}} \times \left(\frac{\Lambda^2}{M_{Pl}\Gamma_\phi^e}\right)^{\frac{3n(n-2)}{(1-2n)(\beta(1+n)-3n)}} \qquad \beta \ll \frac{n+4}{n+1}$$

- Couplings as small as $g_{s\phi} \sim 10^{-15}$ GeV, $g_{\psi\phi} \sim 10^{-5}$ or $h_{a\phi} \sim 10^{-17}\ \text{GeV}^{-1}$ turn out to be sensitive to future gravitational wave detectors.

Back-up slides

$$\Omega^{(0),\mathrm{RH}}_{\mathrm{GW}}(f) \simeq \Omega^{(0),\mathrm{RD}}_{\mathrm{GW}}\, \widetilde{\mathcal{F}}(g_\star) \left[\frac{2\pi}{\sqrt{3}}\,\frac{T_0}{M_{Pl}}\,\left(\frac{30}{\pi^2\, g^{\mathrm{rh}}_{\star\rho}}\right)^{-\frac{1}{4}}\right]^{\frac{4-2n}{2n-1}} f^{\frac{2n-4}{2n-1}} \left[\frac{3n}{n+1}\,(2n^2)^n\,\Lambda^4\right]^{\frac{2-n}{4n-2}}$$

$$\times \begin{cases} \left(\frac{2n}{n+1}\right)^{\frac{n-2}{4n-2}} \left[\frac{n+4-\beta(n+1)}{2n}\right]^{\frac{3n(2-n)}{(2n-1)(\beta(n+1)3n}} \left[\frac{(n\sqrt{2})^n\,\Lambda^2}{\sqrt{2}\,M_{Pl}\,\Gamma^e_\phi}\right]^{\frac{3n(n-2)}{(1-2n)(\beta(1+n)-3n)}}, & \beta \ll \frac{n+4}{n+1}, \\ \vartheta^{\frac{n+1}{2n-1}}\,\mathcal{W}^{\frac{n+1}{1-2n}}(\vartheta) \left[\ln\left(\vartheta^{\frac{n+1}{2n-4}}\,\mathcal{W}^{\frac{n+1}{4-2n}}(\vartheta)\right)\right]^{\frac{2-n}{4n-2}} \left[\frac{(n\sqrt{2})^n\,\Lambda^2}{\sqrt{2}\,M_{Pl}\,\Gamma^e_\phi}\right]^{\frac{n-2}{4n-2}}, & \beta = \frac{n+4}{n+1} \\ \left(\frac{n+1}{2n}\right)^{\frac{2-n}{4n-2}} \left[\frac{\beta(n+1)-(n+4)}{2n}\right]^{\frac{3n}{4n-2}} \left[\frac{(n\sqrt{2})^n\,\Lambda^2}{\sqrt{2}\,M_{Pl}\,\Gamma^e_\phi}\right]^{\frac{3n}{4n-2}}, & \beta \gg \frac{n+4}{n+1}, \end{cases}$$

with

$$\vartheta \equiv \left[\frac{(n\sqrt{2})^n\,\Lambda^2}{\sqrt{2}\,M_{Pl}\,\Gamma^e_\phi}\right]\,\frac{n-2}{n}$$

$$\Omega_{\mathrm{GW}}(\tau\,,k) \equiv \frac{1}{\rho_c}\,\frac{d\rho_{\mathrm{GW}}}{d\log k} = \frac{1}{12}\,\left(\frac{k}{a(\tau)\,H(\tau)}\right)^2 \mathcal{P}_{T\mathrm{prim}}\,\mathcal{T}(\tau,k)$$

where the primordial power spectrum is given by the Hubble parameter at the time when the corresponding mode crosses the horizon during inflation ($k = aH$):

$$\mathcal{P}_{T,\mathrm{prim}} \equiv \frac{k^3}{\pi^2}\,\sum_\lambda \left|h^\lambda_{\mathbf{k}\,,\mathrm{prim}}\right|^2 = \left.\frac{2\,H^2}{\pi^2\,M_{Pl}^2}\right|_{k=a\,H}$$

For the transfer function one finds

$$\mathcal{T}(\tau\,,k) \equiv \left|\frac{h^\lambda_{\mathbf{k}}(\tau)}{h^\lambda_{\mathbf{k},\mathrm{prim}}}\right|^2 = \frac{1}{2}\,\left(\frac{a^{\mathrm{hc}}}{a}\right)^2$$