Tree-Level Unitarity in Presence of Warped Geometries

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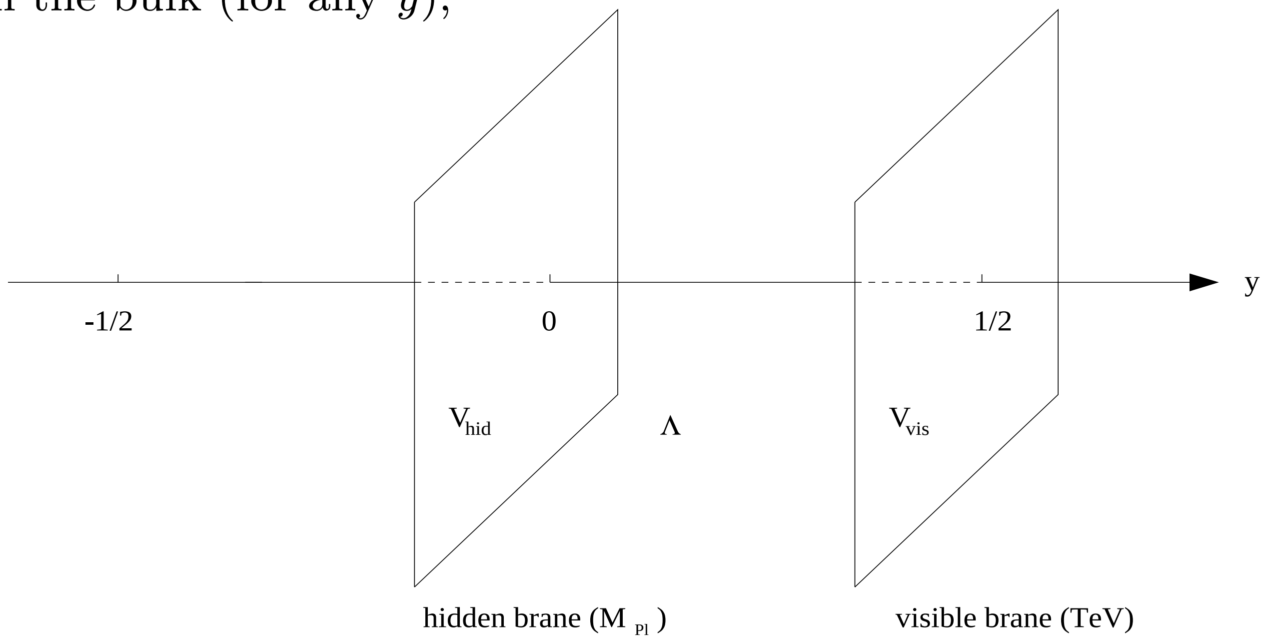
B.G. and Jack Gunion

The Randall-Sundrum Model

- 3 space, 1 time (x^μ), + 1 extra spatial dimension (y), orbifold: $y \equiv y+1$, $y \equiv -y$
- Standard Model particles on a "visible" brane (at $y = 1/2$),
- Planck mass scale physics on the "hidden" (at $y = 0$),
- Gravity in the bulk (for any y),



The full 5d action:

$$
\begin{aligned}
S &= -\int d^4x\, dy \sqrt{-\widehat{g}} \left(\frac{\widehat{R}}{\epsilon^2} + \Lambda \right) \\
&\quad + \int d^4x \sqrt{-g_{\rm hid}} (\mathcal{L}_{\rm hid} - V_{\rm hid}) + \int d^4x \sqrt{-g_{\rm vis}} (\mathcal{L}_{\rm vis} - V_{\rm vis})
\end{aligned}
$$

The strategy:

- Neglecting $\mathcal{L}_{hid}$ and $\mathcal{L}_{vis}$ we solve the Einstein's equations.

  The RS metric

$$\widehat{g}_{\widehat{\mu\nu}}(x,y) = \left(\begin{array}{c|c} e^{-2m_0 b_0 |y|}\eta_{\mu\nu} & 0 \\ \hline 0 & -b_0^2 \end{array} \right)$$

  is a solution of the Einstein's equations if:

$$V_{\rm hid} = -V_{\rm vis} = \frac{12 m_0}{\epsilon^2} \qquad \text{and} \qquad \Lambda = -\frac{12 m_0^2}{\epsilon^2}$$

- An expansion around the background metric:
  - $\eta_{\mu\nu} \to \eta_{\mu\nu} + \epsilon h_{\mu\nu}(x,y)$,
  - $b_0 \to b_0 + b(x)$,

$$h_{\mu\nu}(x,y) = \sum_n h^n_{\mu\nu}(x) \frac{\chi^n(y)}{\sqrt{b_0}} \qquad \Longrightarrow \qquad -\frac{1}{\widehat{\Lambda}_W} \sum_{n\neq 0} h^n_{\mu\nu} T^{\mu\nu} - \frac{\phi_0}{\Lambda_\phi} T^\mu_\mu$$

for

$$\widehat{\Lambda}_W \simeq \sqrt{2} M_{Pl} \Omega_0, \qquad \Lambda_\phi = \sqrt{3}\widehat{\Lambda}_W, \qquad \Omega_0 = e^{-m_0 b_0/2} \qquad \text{and} \qquad \phi_0(x) \equiv \sqrt{6} M_{Pl} e^{-m_0(b_0+b(x))/2}$$

Advantages:

- ”Solution” of the hierarchy problem:

$$\textbf{All (!) mass parameters of the 5d theory of } \boldsymbol{\mathcal{O}(M_{Pl})}\textbf{:}$$
$$\boldsymbol{\epsilon^{-2} \sim M_{Pl}^3,\ m_0 \lesssim M_{Pl},\ \widehat{v} \sim M_{Pl},\ 1/b_0 \sim m_0/70}$$

$$\Downarrow$$

$$\textbf{Effective 4d mass } \boldsymbol{v_0 = \Omega_0 \widehat{v} = e^{-m_0 b_0/2} \widehat{v} \sim} \textbf{ 1 TeV}$$

Drawbacks:

- (No stabilization $\Leftrightarrow$ massless radion) $\Longrightarrow$ Goldberger-Wise model
- Fine tuning of the cosmological constants

The Curvature-Higgs mixing

$$S_\xi = \xi \int d^4x \sqrt{g_{\rm vis}} R(g_{\rm vis}) H^\dagger H \,,$$

where $R(g_{\rm vis})$ is the Ricci scalar for the metric induced on the visible brane.

$$\mathcal{L} = -\tfrac{1}{2}\left\{1 + 6\gamma^2\xi\right\} \phi_0 \Box \phi_0 - \tfrac{1}{2}\phi_0 m_{\phi_0}^2 \phi_0 - \tfrac{1}{2} h_0 (\Box + m_{h_0}^2) h_0 - 6\gamma\xi\phi_0 \Box h_0 \,,$$

where $\phi_0(x) \equiv \sqrt{6} M_{Pl} e^{-m_0(b_0+b(x))/2}$ and

$$\gamma \equiv v_0/\Lambda_\phi \quad \text{for} \quad \Lambda_\phi \simeq \sqrt{6} M_{Pl} \Omega_0$$

The mixing angle θ

$$\tan 2\theta \equiv 12\gamma\xi Z \frac{m_{h_0}^2}{m_{\phi_0}^2 - m_{h_0}^2 (Z^2 - 36\xi^2\gamma^2)} \quad \text{for} \quad Z^2 \equiv 1 + 6\xi\gamma^2(1-6\xi) \equiv \beta - 36\xi^2\gamma^2 \,.$$

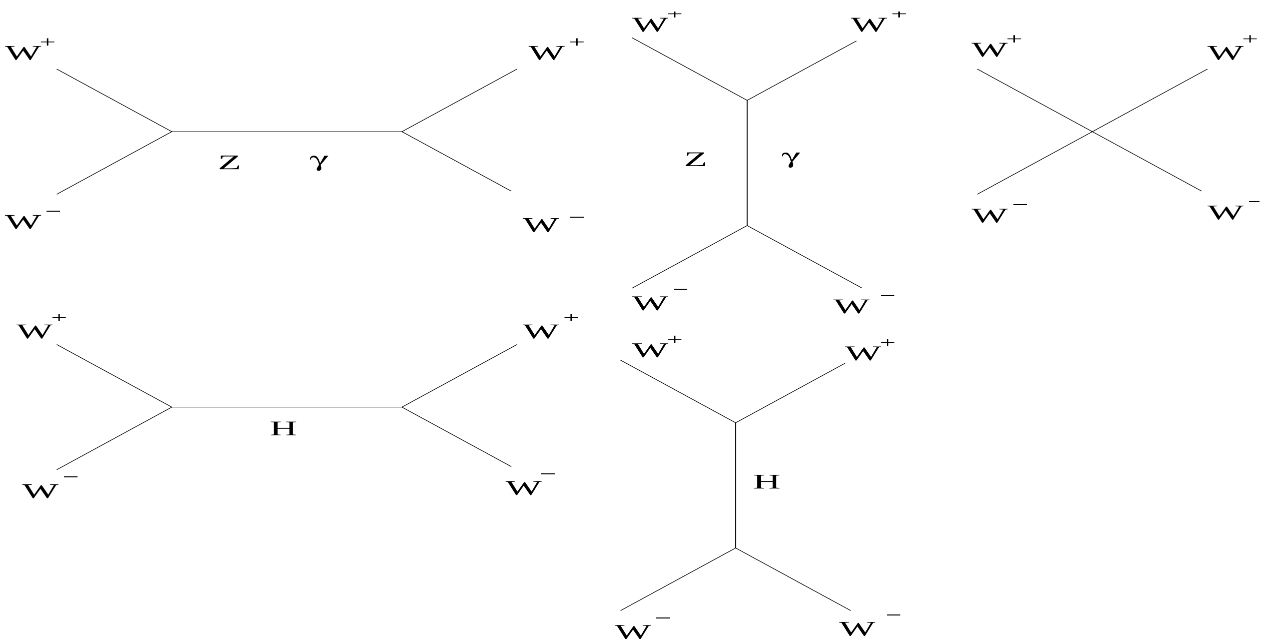
The states that diagonalize the kinetic energy and have canonical normalization are h and ϕ:

$$\begin{aligned} h_0 &= \left(\cos\theta - \frac{6\xi\gamma}{Z}\sin\theta\right) h + \left(\sin\theta + \frac{6\xi\gamma}{Z}\cos\theta\right)\phi \equiv dh + c\phi \\ \phi_0 &= -\cos\theta \frac{\phi}{Z} + \sin\theta \frac{h}{Z} \equiv a\phi + bh \,. \end{aligned}$$

The Lee-Quigg-Thacker bound for the Higgs boson mass

$$W_L^+ W_L^- \to W_L^+ W_L^-$$

$$T(s, \cos\theta) = 16\pi \sum_J (2J+1) a_J(s) P_J(\cos\theta) \quad \text{with} \quad a_J(s) = \frac{1}{32\pi} \int_{-1}^{1} T(s, \cos\theta) P_J(\cos\theta) d\cos\theta$$



For the SM for $W_L^+ W_L^- \to W_L^+ W_L^-$ we have

$$a_J = A_J \left(\frac{q}{m_W}\right)^4 + B_J \left(\frac{q}{m_W}\right)^2 + C_J$$

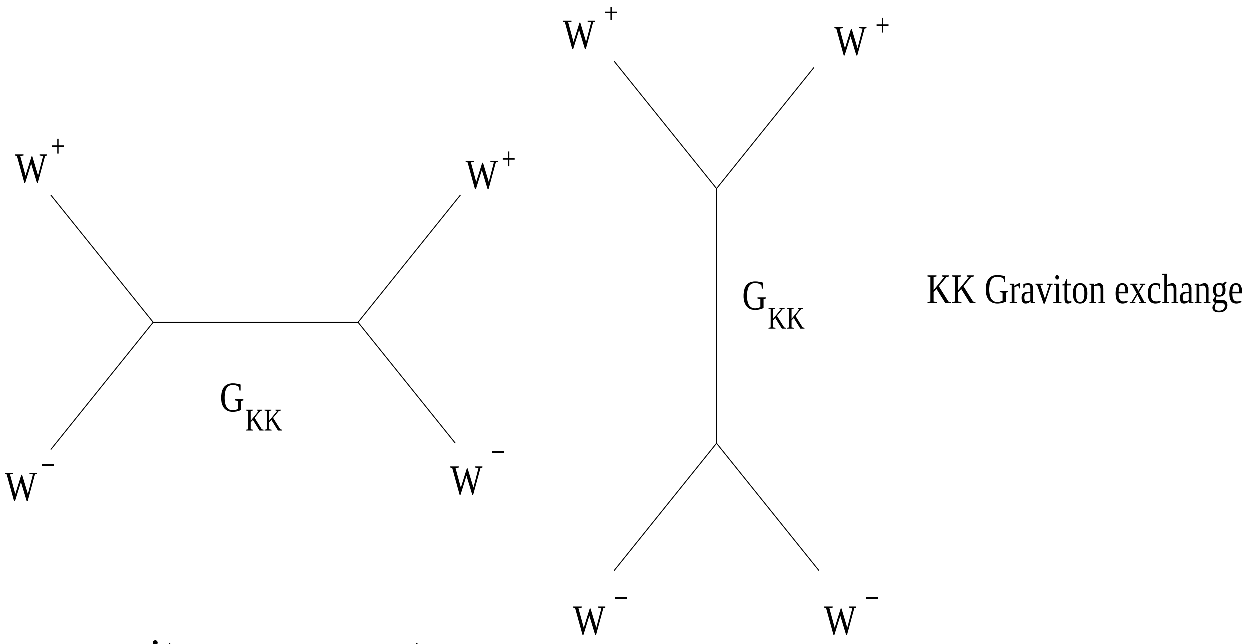
- divergent contributions for $J = 0, 1$ and 2
- A-terms vanish by the virtue of the gauge invariance for $J = 0, 1$ and 2
- for $J = 1$ and 0, the B-term is cancelled by the Higgs-boson exchange
- eventually a_J turns out to be m_H-dependent constant in the high-energy asymptotic region, that implies the Lee-Quigg-Thacker bound for the Higgs boson mass:

$$\mathbf{Im}(a_J) \geq |a_J|^2 \quad \Rightarrow \quad \mathbf{Re}(a_J) \leq \frac{1}{2}$$

$$\Downarrow$$

$$m_H \lesssim 870 \text{ GeV}$$

Tree-level unitarity in $W_L^+ W_L^- \to G_{KK}, H, \phi \to W_L^+ W_L^-$



- The massive graviton propagator

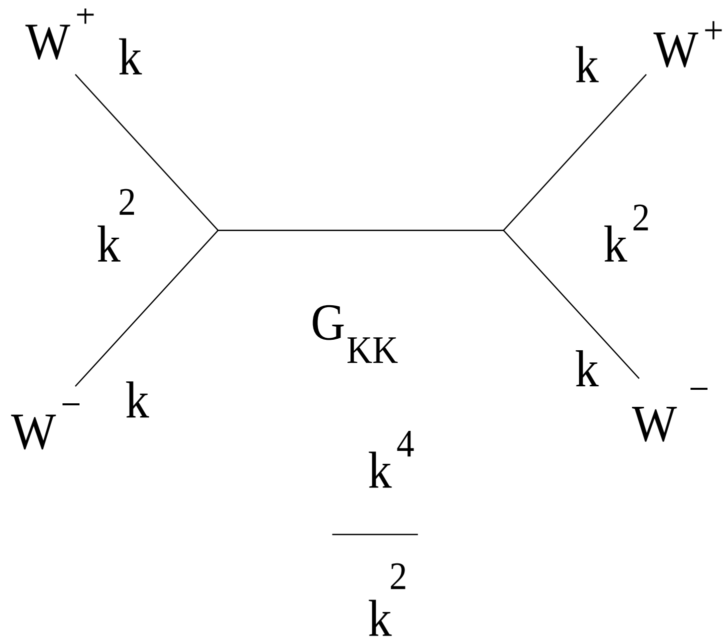
$$D^{\mu\nu,\alpha\beta}(k) = \frac{1}{2}\left(\bar{\eta}^{\mu\alpha}\bar{\eta}^{\nu\beta} + \bar{\eta}^{\mu\beta}\bar{\eta}^{\nu\alpha} - \frac{2}{3}\bar{\eta}^{\mu\nu}\bar{\eta}^{\alpha\beta}\right)\frac{i}{k^2 - m_G^2 + i\epsilon}\,,$$

  where $\bar{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{m_G^2}$ for $\eta^{\mu\nu}$ being the Minkowski metric.

- The graviton couples to the energy-momentum tensor $T_{\mu\nu}$, so the amplitude reads

$$T_{\mu\nu} D^{\mu\nu,\alpha\beta} T_{\alpha\beta} \quad \text{for} \quad T_{\mu\nu} \ni k_\mu^2, \ldots$$

- $\epsilon_\mu^{W_L}(k) = \frac{k_\mu}{m_W} + \mathcal{O}\left(\frac{m_W}{E}\right)$



KK Graviton exchange

$$\Downarrow$$

$$a_J \propto k^{10}$$

$$k_\mu T^{\mu\nu} = 0$$

$$\langle 0|T^{\mu\nu}|W_L^+ W_L^-\rangle =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6}[(1-2\beta_W)d^0_{0,0} + 2(\beta_W - 2)d^2_{0,0}]s & 0 & -\frac{1}{\sqrt{6}}(s + 4m_W^2)d^2_{1,0} \\ 0 & 0 & -\frac{1}{2}sd^0_{0,0} & 0 \\ 0 & -\frac{1}{\sqrt{6}}(s + 4m_W^2)d^2_{1,0} & 0 & -\frac{1}{6}[(1+\beta_W)d^0_{0,0} + 2(\beta_W - 2)d^2_{0,0}]s\,, \end{pmatrix}$$

in the reference frame in which off-shell graviton is at rest. The scattering angle is measured relatively to the direction of motion W^-, $d^J_{\mu\mu'}(\cos\theta) = d^J_{\mu\mu'}$ stands for the Wigner d function and $\beta_W \equiv 1 - 4m_W^2/s$.

$$\Downarrow$$

$$a_J \propto k^2$$

Note that the RS model is an effective theory (dim 5 operators: $\propto \frac{1}{\widehat{\Lambda}_W} h_{\mu\nu}T^{\mu\nu}$) having a cutoff $\mathcal{O}(1\mathrm{TeV})$, therefore the amplitude should satisfy the unitarity conditionsup to $\sqrt{s} \simeq 1$ TeV.

$$
\begin{aligned}
a_2 &= -\frac{1}{192\pi\hat{\Lambda}_W^2}\left\{\left[91+30\log\left(\frac{m_G^2}{s}\right)\right]s+\left[241+210\log\left(\frac{m_G^2}{s}\right)\right]m_G^2+32g^2v^2\right\}+\mathcal{O}(s^{-1})\\
a_1 &= -\frac{1}{384\pi\hat{\Lambda}_W^2}\left\{\left[73+36\log\left(\frac{m_G^2}{s}\right)\right]s+36\left[1+3\log\left(\frac{m_G^2}{s}\right)\right]m_G^2+37g^2v^2\right\}+\\
&\quad +\frac{1}{32\pi}\left[\frac{s}{v^2}+\frac{1}{2\cos^2\theta_W}(12\cos^2\theta_W-1)g^2\right]-\frac{1}{32\pi}R^2\left(\frac{s}{v^2}-g^2\right)+\mathcal{O}(s^{-1})\\
a_0 &= -\frac{1}{384\pi\Lambda_W^2}\left\{\left[11+12\log\left(\frac{m_G^2}{s}\right)\right]s-\left[10-12\log\left(\frac{m_G^2}{s}\right)\right]m_G^2+19g^2v^2\right\}+\\
&\quad +\frac{1}{32\pi}\frac{s}{v^2}-\frac{1}{32\pi}\left[R^2\left(\frac{s}{v^2}-g^2\right)+4\frac{\overline{m}_{\rm scal}^2}{v^2}\right]+\mathcal{O}(s^{-1})
\end{aligned}
$$

where $\overline{m}_{\rm scal}^2=g_{vvh}^2m_h^2+g_{vv\phi}^2m_\phi^2$ and $R^2\equiv g_{vvh}^2+g_{vv\phi}^2$ satisfies the following sum rule

$$
R^2=1+\left[\frac{\gamma(1-6\xi)}{Z}\right]^2\ \text{for}\ \gamma\equiv\frac{v}{\Lambda_\phi}\ \text{and}\ Z^2\equiv 1+6\xi\gamma^2(1-6\xi)
$$

$$
a_0=\frac{1}{32\pi}\left[(1-R^2)\frac{s}{v^2}+g^2R^2-4\frac{\overline{m}_{\rm scal}^2}{v^2}+\ \text{graviton contributions}\ \right],
$$

The very first term is responsible for the large violation of unitarity

$$
f\equiv(1-R^2)\frac{s}{v^2}=-\left(\frac{1-6\xi}{Z}\right)^2\frac{s}{\Lambda_\phi^2}
$$

$\boldsymbol{f\bar{f} \longrightarrow G_{KK}, H, \phi \longrightarrow Z_L Z_L}$

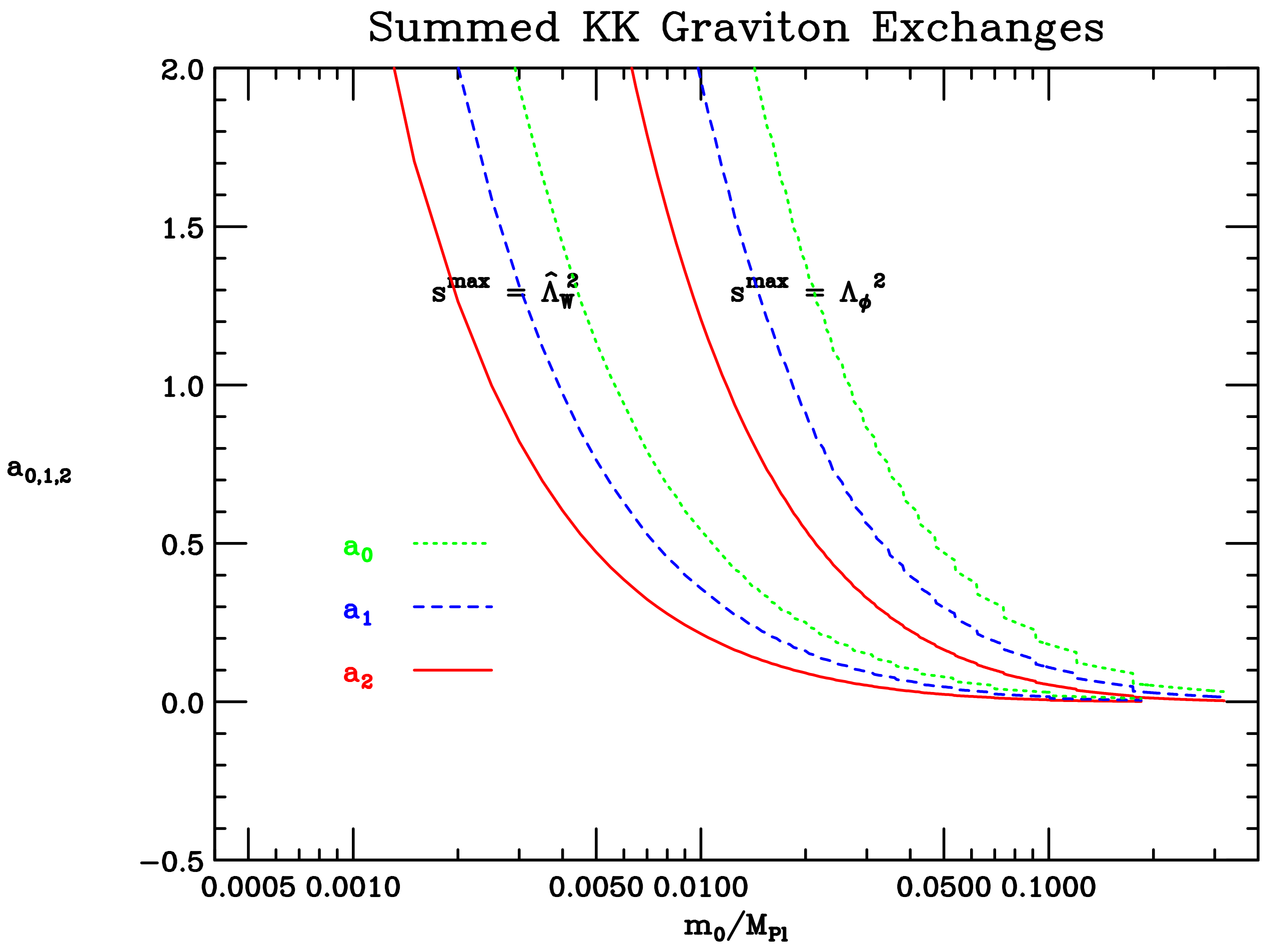
$(h_{\bar{f}},\ h_f)$	SM: t,u-channels	$(h-\phi)$ s-channel	G s-channel
$(h,\ h)$	$-2h\frac{m_f s^{1/2}}{v^2} + \mathcal{O}(s^0)$	$2hR^2\frac{m_f s^{1/2}}{v^2} + \mathcal{O}(s^0)$	$-h\frac{8}{3}\frac{m_f s^{1/2}}{\hat{\Lambda}_W^2}(d^2_{0,0} + \frac{1}{2}d^0_{0,0}) + \mathcal{O}(s^{-1/2})$
$(h,\ -h)$	0	0	$\sqrt{\frac{2}{3}}\frac{s}{\hat{\Lambda}_W^2}d^2_{1,0} + \mathcal{O}(s^0)$

Determination of the cutoff for the Randall-Sundrum model

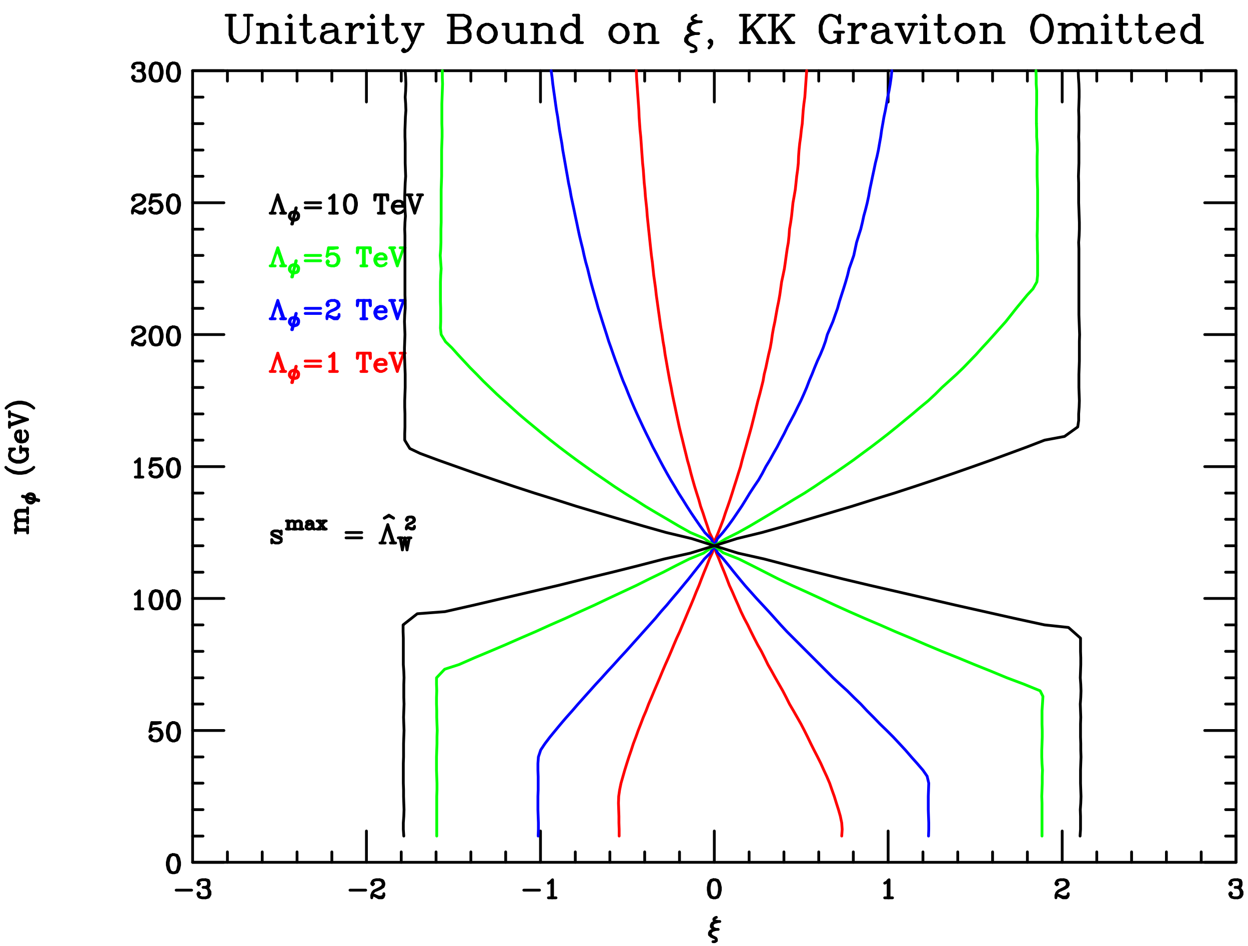
$$\begin{aligned} \widehat{\Lambda}_W &\simeq \sqrt{2} M_{Pl} \Omega_0 \, , \\ \Lambda_\phi &= \sqrt{6} M_{Pl} \Omega_0 = \sqrt{3} \widehat{\Lambda}_W \\ m_n &= m_0 x_n \Omega_0 \, , \end{aligned}$$

where $\Omega_0 M_{Pl} = e^{-m_0 b_0/2} M_{Pl}$ should be of order a TeV to solve the hierarchy problem. The x_n are the zeroes of the Bessel function J_1 ($x_1 \sim 3.8$, $x_2 \sim 7.0$). A useful relation following from the above equations is:

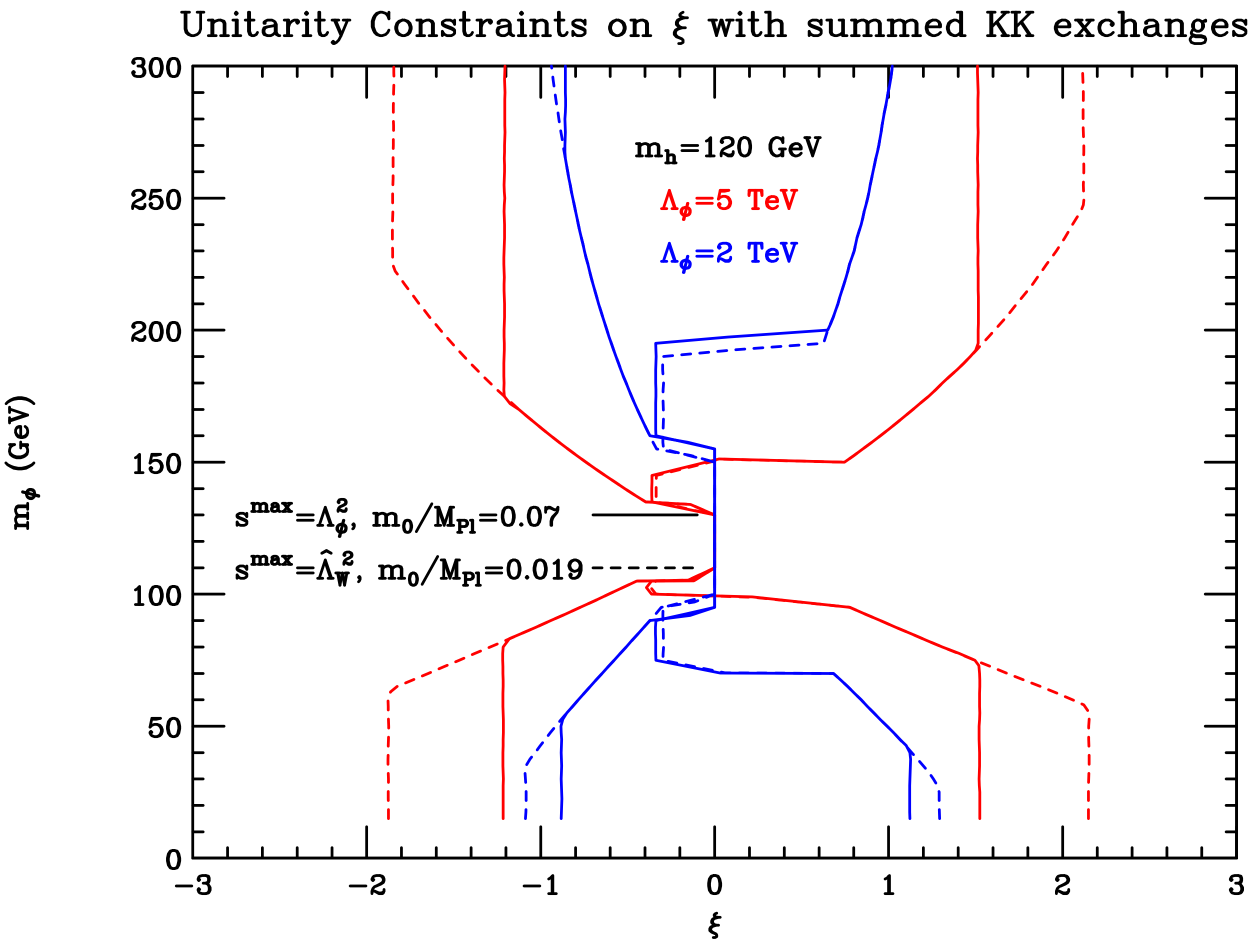
$$m_n = x_n \frac{m_0}{M_{Pl}} \frac{\Lambda_\phi}{\sqrt{6}} \quad \text{with} \quad 0.01 \lesssim \frac{m_0}{M_{Pl}} \lesssim 0.1$$



The amplitudes $a_{0,1,2}(s=\Lambda_\phi^2)$ and $a_{0,1,2}(s=\widehat{\Lambda}_W^2)$ are plotted as functions of m_0/M_{Pl}, after summing: $a_J(s)=\sum_{n,2m_n<\sqrt{s}} a_J(m_G=m_n,s)$. The plotted values of a_i terminate when m_0/M_{Pl} is such that $2m_1$ exceeds $\Lambda_\phi=\sqrt{3}\widehat{\Lambda}_W$ or $\widehat{\Lambda}_W$, for the two respective s values above.



The unitarity limits on ξ for $m_h = 120$ GeV when KK gravitons are omitted.



The unitarity limits on ξ after summing over all KK excitations with $2m_n < \sqrt{s}$.

The van Dam-Veltman-Zakharov discontinuity

$$\text{massless graviton :} \qquad D^{\mu\nu,\alpha\beta}(k) = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)\frac{i}{k^2 + i\epsilon}$$

$$\text{massive graviton :} \qquad D^{\mu\nu,\alpha\beta}(k) = \frac{1}{2}\left(\bar{\eta}^{\mu\alpha}\bar{\eta}^{\nu\beta} + \bar{\eta}^{\mu\beta}\bar{\eta}^{\nu\alpha} - \frac{2}{3}\bar{\eta}^{\mu\nu}\bar{\eta}^{\alpha\beta}\right)\frac{i}{k^2 - m_G^2 + i\epsilon}$$

where $\bar{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{m_G^2}$

The s-channel graviton contributions to the amplitude for $f\bar{f} \to Z_L Z_L$ calculated both for $m_G \neq 0$ and for $m_G = 0$ for possible initial state helicities ($h = \pm\frac{1}{2}$).

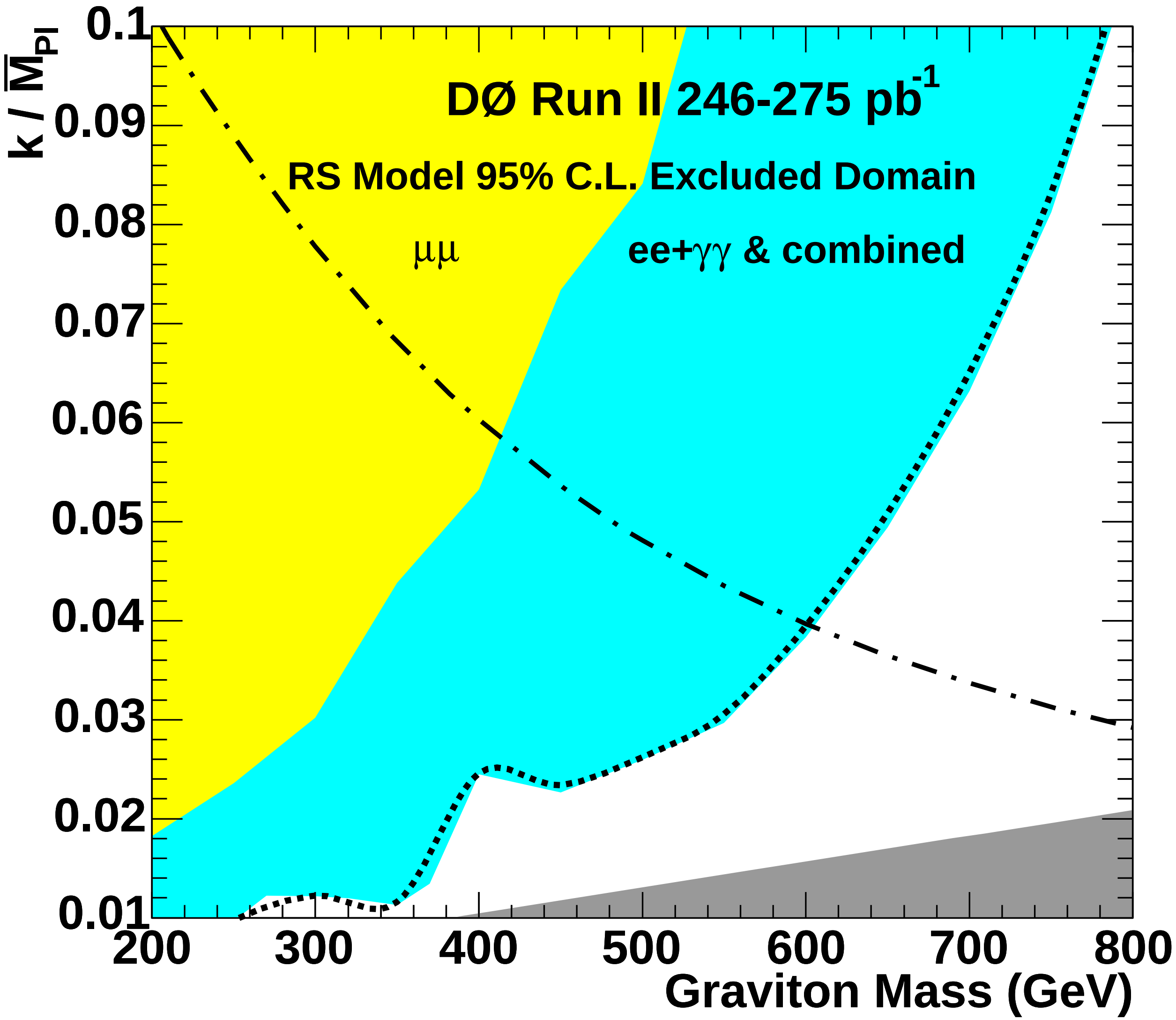
$(h_{\bar{f}},\ h_f)$	$m_G \neq 0$	$m_G = 0$
$(h,\ h)$	$-h\frac{8}{3}\beta_f^{1/2}\frac{s+4m_Z^2}{s-m_G^2}\frac{m_f s^{1/2}}{\hat{\Lambda}_W^2}d^2_{0,0}$	$-h\frac{8}{3}\beta_f^{1/2}\frac{s+4m_Z^2}{s}\frac{m_f s^{1/2}}{\hat{\Lambda}_W^2}\left(d^2_{0,0} + \frac{1}{2}\frac{s-2m_Z^2}{s+4m_Z^2}d^0_{0,0}\right)$
$(h,\ -h)$	$\sqrt{\frac{2}{3}}\beta_f^{1/2}\frac{s+4m_Z^2}{s-m_G^2}\frac{s}{\hat{\Lambda}_W^2}d^2_{1,0}$	$\sqrt{\frac{2}{3}}\beta_f^{1/2}\frac{s+4m_Z^2}{s}\frac{s}{\hat{\Lambda}_W^2}d^2_{1,0}$

Comments:

- The amplitudes found for massless and massive gravitons are different.
- Even in the limit $m_G \to 0$ the massive amplitude doesn't coincide with the massless result. This is an illustration of the celebrated van Dam-Veltman-Zakharov discontinuity; regardless how small is the graviton mass, its very presence has physical consequences.
- Note however, that in all cases considered here, for $J = 2$, the limit $m_G \to 0$ of a massive amplitude does reproduce the massless calculation, in particular there is no discontinuity for opposite helicities. This result is consistent with the fact that the massive graviton does not contain $J = 0$ component, while the massless one does.
- Since the $J = 0$ component can not contribute to opposite helicity amplitudes, no wonder in those cases there is no discontinuity.
- The presence of $J = 0$ component of the massless graviton leads also to different high-energy behavior of the amplitudes and therefore may influence the unitarity constraints differently.

Summary

- The graviton exchange leads to divergent partial wave amplitudes for $V_L V_L \to V_L V_L$, $a_J \propto \frac{s}{\widehat{\Lambda}_W^2}$, and therefore can substantially modify their high-energy behavior.
- The tree-level unitarity requirement can be either adopted to determine the cutoff in the Randall-Sundrum model or to impose constraints on the free parameter $\frac{m_0}{M_{Pl}}$.
- The results obtained here for the graviton exchange are applicable to models that have massive gravitons which couples as $\frac{1}{\Lambda}\Sigma_{n\neq 0} h^n_{\mu\nu} T^{\mu\nu}$.
- In the curvature-Higgs mixing scenario, the presence of the radion-Higgs mixing spoils the cancellation of $a_{1,0} \propto s$ by the Higgs-boson exchange. Therefore the requirement of proper high-energy behavior severely constraints the allowed region for the mixing parameter ξ.
- The van Dam-Veltman-Zakharov discontinuity was observed in the process $f\bar{f} \to Z_L Z_L$.



95% C.L. exclusion limits on the RS model parameters M_1 and k/M_{Pl}. The area below the dashed-dotted line is excluded from the precision electroweak data. The dark shaded area in the lower right-hand corner corresponds to $\Lambda_\phi > 10$ TeV, which requires a significant amount of fine-tuning.

Scalar scattering

For completeness we also consider processes of scattering involving scalars only. Here however, there is no need to calculate SM-like diagrams involving only scalars since they don't contain any contributions growing with energy, so there is no chance for a compensation of $\gamma^2 = v^2/\hat{\Lambda}_W^2$.

First we need to derive Ghh, $G\phi\phi$ and $G\phi h$ vertices for off-shell external particles. Therefore one has to expand the curvature-Higgs mixing term around the R-S background metrics:

$$\begin{aligned} S_\xi &= \xi \int d^4x \sqrt{g_{\rm vis}} R(g_{\rm vis}) \hat{H}^\dagger \hat{H} = \\ & 6\xi \int d^4x\, \Omega \Big\{ -\Box\, \Omega + \epsilon \Big[h_{\mu\nu} \partial^\mu \partial^\nu \Omega - \frac{1}{2} (\partial^\nu h^\mu_\mu) \partial_\nu \Omega + (\partial^\mu h_{\mu\nu}) \partial^\nu \Omega + \\ & \frac{1}{6} \Omega (\partial^\mu \partial^\nu - \eta^{\mu\nu} \Box\,) h_{\mu\nu} \Big] + \mathcal{O}(\epsilon^2) \Big\} H_0^\dagger H_0 \,, \end{aligned}$$

where $\epsilon \equiv (M_{\rm Pl5})^{3/2}$.

Among all possible final and initial states made of ϕ and h, only the elastic Higgs-boson scattering have a chance to provide a result which is not suppressed by $\gamma^2 = v^2/\hat{\Lambda}_W^2$. Therefore below we focus on the graviton-exchange contributions to the process $hh \to hh$. We obtain the following contributions to the amplitude from the s- t- and u-channel:

$$a^{(s)}(\cos\theta) = -\frac{(A_{hh} - C_{hh})^2}{12\pi} \frac{s - 4m_h^2}{s - m_G^2} \frac{s}{\hat{\Lambda}_W^2}$$

$$a^{(t)}(\cos\theta) = \frac{(A_{hh}-C_{hh})^2}{12\pi}\frac{16(1+\cos\theta)^2 m_h^4 - 8(5+6\cos\theta+\cos^2\theta)m_h^2 s + (13+10\cos\theta+\cos^2\theta)}{\hat{\Lambda}_W^2[2m_G^2+(1-\cos\theta)(s-4m_h^2)]}$$
$$a^{(u)}(\cos\theta) = a^{(t)}(-\cos\theta)$$

where

$$A_{hh} = -6\xi\gamma b(\gamma b + 2d), \qquad C_{hh} = d^2$$

Integrating with the Legendre polynomials we obtain the following coefficients for the $J = 2, 1$ and 0 partial waves.

$$a_2 = -\frac{(A_{hh}-C_{hh})^2}{96\pi}\left[181 + 60\log\left(\frac{m_G^2}{s}\right)\right]\frac{s}{\hat{\Lambda}_W^2} + \mathcal{O}(s^0)$$
$$a_1 = 0$$
$$a_0 = -\frac{(A_{hh}-C_{hh})^2}{96\pi}\left[11 + 12\log\left(\frac{m_G^2}{s}\right)\right]\frac{s}{\hat{\Lambda}_W^2} + \mathcal{O}(s^0)\,,$$

Expanding in powers of γ we get

$$(A_{hh}-C_{hh})^2 = 1 - \frac{72\gamma^2\xi m_h^4}{(m_h^2-m_\phi^2)^2} + \mathcal{O}(\gamma^4)$$

Note that for the scalar–scalar scattering, the Higgs and radion exchange could lead to terms growing as s, however those cancel and the result is of the order of $\mathcal{O}(s^0)$.

It turns out the $a_2 \simeq 10^{-2} \div 10^{-3}$ for reasonable set of parameters.