

Tree-Level Unitarity in Presence of Warped Geometries

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1. Introduction

- The Randall-Sundrum model.
- The Curvature-Higgs mixing.
- The Lee-Quigg-Thacker bound for the Higgs boson mass.

2. Tree-level unitarity

- $W_L^+ W_L^- \rightarrow G_{KK}, H, \phi \rightarrow W_L^+ W_L^-$
- $f \bar{f} \rightarrow G_{KK}, H, \phi \rightarrow Z_L Z_L$

3. Discussion

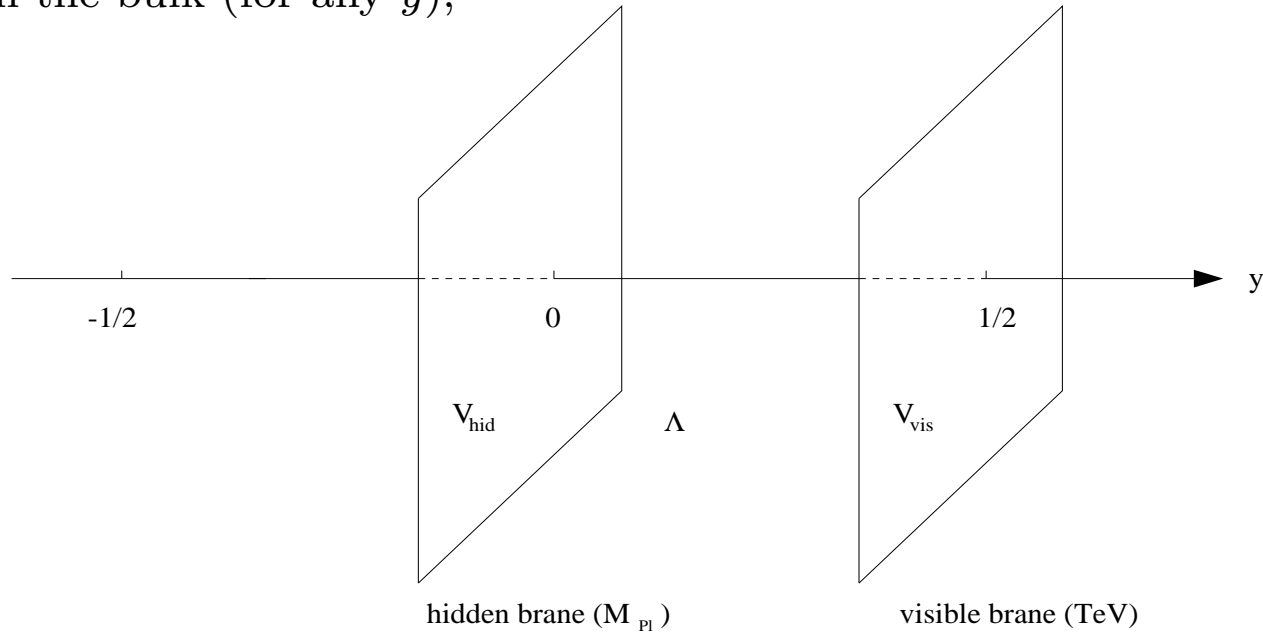
- Determination of the cutoff for the Randall-Sundrum model
- The van Dam-Veltman-Zakharov discontinuity

4. Summary

B.G. and Jack Gunion

The Randall-Sundrum Model

- 3 space, 1 time (x^μ), + 1 extra spatial dimension (y), orbifold: $y \equiv y + 1$, $y \equiv -y$
- Standard Model particles on a “visible” brane (at $y = 1/2$),
- Planck mass scale physics on the “hidden” (at $y = 0$),
- Gravity in the bulk (for any y),



The full 5d action:

$$\begin{aligned}
 S = & - \int d^4x dy \sqrt{-\widehat{g}} \left(\frac{\widehat{R}}{\epsilon^2} + \Lambda \right) \\
 & + \int d^4x \sqrt{-g_{\text{hid}}} (\mathcal{L}_{\text{hid}} - V_{\text{hid}}) + \int d^4x \sqrt{-g_{\text{vis}}} (\mathcal{L}_{\text{vis}} - V_{\text{vis}})
 \end{aligned}$$

The strategy:

- Neglecting \mathcal{L}_{hid} and \mathcal{L}_{vis} we solve the Einstein's equations.

The RS metric

$$\widehat{g}_{\mu\nu}(x, y) = \left(\begin{array}{c|c} e^{-2m_0 b_0 |y|} \eta_{\mu\nu} & 0 \\ \hline 0 & -b_0^2 \end{array} \right)$$

is a solution of the Einstein's equations if:

$$V_{hid} = -V_{vis} = \frac{12m_0}{\epsilon^2} \quad \text{and} \quad \Lambda = -\frac{12m_0^2}{\epsilon^2}$$

- An expansion around the background metric:
 - $\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + \epsilon h_{\mu\nu}(x, y)$,
 - $b_0 \rightarrow b_0 + b(x)$,

$$h_{\mu\nu}(x, y) = \sum_n h_{\mu\nu}^n(x) \frac{\chi^n(y)}{\sqrt{b_0}} \quad \Longrightarrow \quad -\frac{1}{\widehat{\Lambda}_W} \sum_{n \neq 0} h_{\mu\nu}^n T^{\mu\nu} - \frac{\phi_0}{\Lambda_\phi} T_\mu^\mu$$

for

$$\widehat{\Lambda}_W \simeq \sqrt{2} M_{Pl} \Omega_0, \quad \Lambda_\phi = \sqrt{3} \widehat{\Lambda}_W, \quad \Omega_0 = e^{-m_0 b_0/2} \quad \text{and} \quad \phi_0(x) \equiv \sqrt{6} M_{Pl} e^{-m_0(b_0 + b(x))/2}$$

Advantages:

- "Solution" of the hierarchy problem:

All (!) mass parameters of the 5d theory of $\mathcal{O}(M_{Pl})$:

$$\epsilon^{-2} \sim M_{Pl}^3, m_0 \lesssim M_{Pl}, \hat{v} \sim M_{Pl}, 1/b_0 \sim m_0/70$$



$$\text{Effective 4d mass } v_0 = \Omega_0 \hat{v} = e^{-m_0 b_0 / 2 \hat{v}} \sim 1 \text{ TeV}$$

Drawbacks:

- (No stabilization \Leftrightarrow massless radion) \implies Goldberger-Wise model
- Fine tuning of the cosmological constants

The Curvature-Higgs mixing

$$S_\xi = \xi \int d^4x \sqrt{g_{\text{vis}}} R(g_{\text{vis}}) H^\dagger H ,$$

where $R(g_{\text{vis}})$ is the Ricci scalar for the metric induced on the visible brane.

$$\mathcal{L} = -\frac{1}{2} \left\{ 1 + 6\gamma^2 \xi \right\} \phi_0 \square \phi_0 - \frac{1}{2} \phi_0 m_{\phi_0}^2 \phi_0 - \frac{1}{2} h_0 (\square + m_{h_0}^2) h_0 - 6\gamma \xi \phi_0 \square h_0 ,$$

where $\phi_0(x) \equiv \sqrt{6} M_{Pl} e^{-m_0(b_0+b(x))/2}$ and

$$\gamma \equiv v_0 / \Lambda_\phi \quad \text{for} \quad \Lambda_\phi \simeq \sqrt{6} M_{Pl} \Omega_0$$

The mixing angle θ

$$\tan 2\theta \equiv 12\gamma\xi Z \frac{m_{h_0}^2}{m_{\phi_0}^2 - m_{h_0}^2 (Z^2 - 36\xi^2\gamma^2)} \quad \text{for} \quad Z^2 \equiv 1 + 6\xi\gamma^2(1 - 6\xi) \equiv \beta - 36\xi^2\gamma^2 .$$

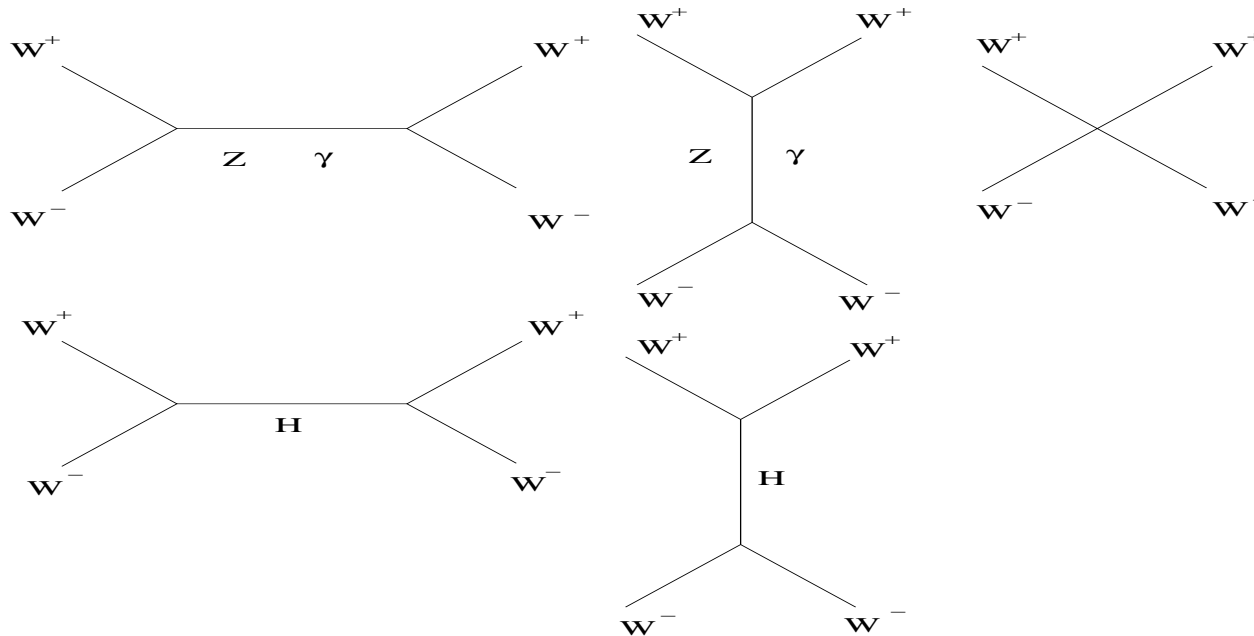
The states that diagonalize the kinetic energy and have canonical normalization are h and ϕ :

$$\begin{aligned} h_0 &= \left(\cos \theta - \frac{6\xi\gamma}{Z} \sin \theta \right) h + \left(\sin \theta + \frac{6\xi\gamma}{Z} \cos \theta \right) \phi \equiv dh + c\phi \\ \phi_0 &= -\cos \theta \frac{\phi}{Z} + \sin \theta \frac{h}{Z} \equiv a\phi + bh . \end{aligned}$$

The Lee-Quigg-Thacker bound for the Higgs boson mass

$$W_L^+ W_L^- \rightarrow W_L^+ W_L^-$$

$$T(s, \cos \theta) = 16\pi \sum_J (2J+1) a_J(s) P_J(\cos \theta) \quad \text{with} \quad a_J(s) = \frac{1}{32\pi} \int_{-1}^1 T(s, \cos \theta) P_J(\cos \theta) d \cos \theta$$



For the SM for $W_L^+ W_L^- \rightarrow W_L^+ W_L^-$ we have

$$a_J = A_J \left(\frac{q}{m_W} \right)^4 + B_J \left(\frac{q}{m_W} \right)^2 + C_J$$

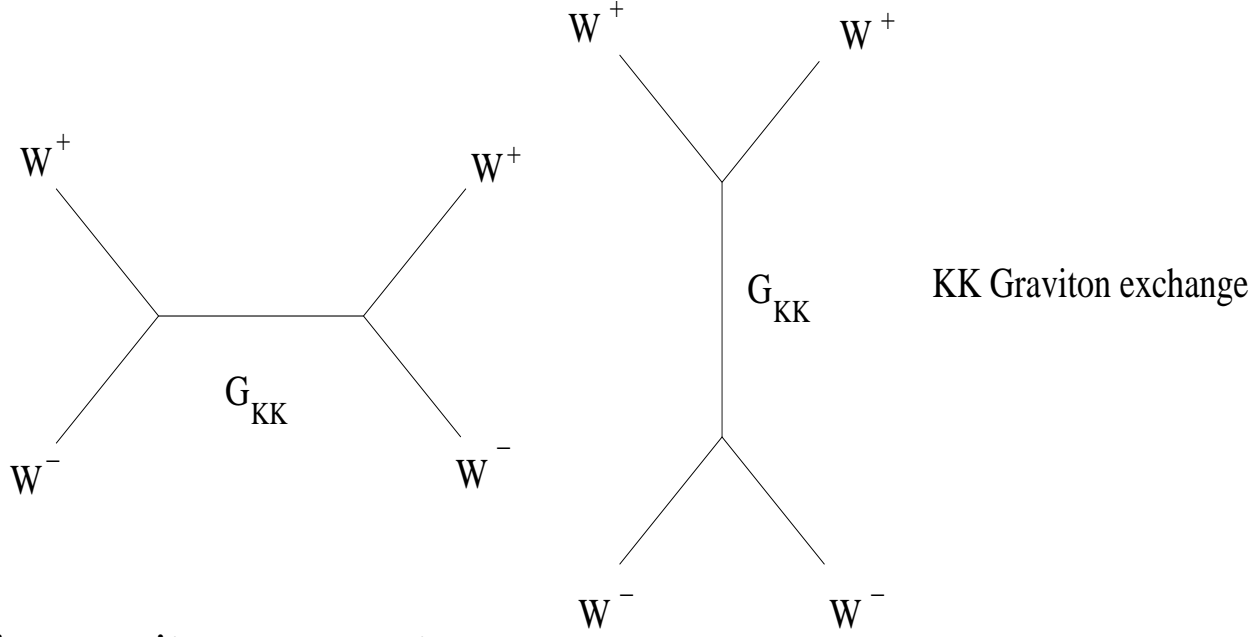
- divergent contributions for $J = 0, 1$ and 2
- A -terms vanish by the virtue of the gauge invariance for $J = 0, 1$ and 2
- for $J = 1$ and 0 , the B -term is cancelled by the Higgs-boson exchange
- eventually a_J turns out to be m_H -dependent constant in the high-energy asymptotic region, that implies the Lee-Quigg-Thacker bound for the Higgs boson mass:

$$\mathbf{Im}(a_J) \geq |a_J|^2 \quad \Rightarrow \quad \mathbf{Re}(a_J) \leq \frac{1}{2}$$

↓

$$m_H \lesssim 870 \text{ GeV}$$

Tree-level unitarity in $W_L^+ W_L^- \rightarrow G_{KK}, H, \phi \rightarrow W_L^+ W_L^-$



- The massive graviton propagator

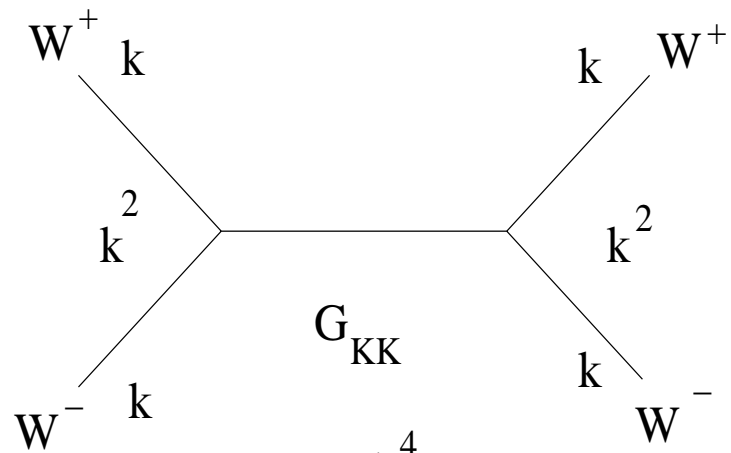
$$D^{\mu\nu,\alpha\beta}(k) = \frac{1}{2} \left(\bar{\eta}^{\mu\alpha} \bar{\eta}^{\nu\beta} + \bar{\eta}^{\mu\beta} \bar{\eta}^{\nu\alpha} - \frac{2}{3} \bar{\eta}^{\mu\nu} \bar{\eta}^{\alpha\beta} \right) \frac{i}{k^2 - m_G^2 + i\epsilon},$$

where $\bar{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{m_G^2}$ for $\eta^{\mu\nu}$ being the Minkowski metric.

- The graviton couples to the energy-momentum tensor $T_{\mu\nu}$, so the amplitude reads

$$T_{\mu\nu} D^{\mu\nu,\alpha\beta} T_{\alpha\beta} \quad \text{for} \quad T_{\mu\nu} \ni k_\mu^2, \dots$$

- $\epsilon_\mu^{W_L}(k) = \frac{k_\mu}{m_W} + \mathcal{O}\left(\frac{m_W}{E}\right)$



KK Graviton exchange

$$G_{\text{KK}} \frac{k^4}{k^2} \Downarrow a_J \propto k^{10}$$

$$k_\mu T^{\mu\nu} = 0$$

$$\langle 0 | T^{\mu\nu} | W_L^+ W_L^- \rangle =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6}[(1 - 2\beta_W)d_{0,0}^0 + 2(\beta_W - 2)d_{0,0}^2]s & 0 & -\frac{1}{\sqrt{6}}(s + 4m_W^2)d_{1,0}^2 \\ 0 & 0 & -\frac{1}{2}s d_{0,0}^0 & 0 \\ 0 & -\frac{1}{\sqrt{6}}(s + 4m_W^2)d_{1,0}^2 & 0 & -\frac{1}{6}[(1 + \beta_W)d_{0,0}^0 + 2(\beta_W - 2)d_{0,0}^2]s, \end{pmatrix}$$

in the reference frame in which off-shell graviton is at rest. The scattering angle is measured relatively to the direction of motion W^- , $d_{\mu\mu'}^J(\cos\theta) = d_{\mu\mu'}^J$ stands for the Wigner d function and $\beta_W \equiv 1 - 4m_W^2/s$.

↓

$$a_J \propto k^2$$

Note that the RS model is an effective theory (dim 5 operators: $\propto \frac{1}{\Lambda_W} h_{\mu\nu} T^{\mu\nu}$) having a cutoff $\mathcal{O}(1\text{TeV})$, therefore the amplitude should satisfy the unitarity conditions up to $\sqrt{s} \simeq 1 \text{ TeV}$.

$$\begin{aligned}
a_2 &= -\frac{1}{192\pi\hat{\Lambda}_W^2} \left\{ \left[91 + 30 \log \left(\frac{m_G^2}{s} \right) \right] s + \left[241 + 210 \log \left(\frac{m_G^2}{s} \right) \right] m_G^2 + 32g^2v^2 \right\} + \mathcal{O}(s^{-1}) \\
a_1 &= -\frac{1}{384\pi\hat{\Lambda}_W^2} \left\{ \left[73 + 36 \log \left(\frac{m_G^2}{s} \right) \right] s + 36 \left[1 + 3 \log \left(\frac{m_G^2}{s} \right) \right] m_G^2 + 37g^2v^2 \right\} + \\
&\quad + \frac{1}{32\pi} \left[\frac{s}{v^2} + \frac{1}{2 \cos^2 \theta_W} (12 \cos^2 \theta_W - 1)g^2 \right] - \frac{1}{32\pi} R^2 \left(\frac{s}{v^2} - g^2 \right) + \mathcal{O}(s^{-1}) \\
a_0 &= -\frac{1}{384\pi\Lambda_W^2} \left\{ \left[11 + 12 \log \left(\frac{m_G^2}{s} \right) \right] s - \left[10 - 12 \log \left(\frac{m_G^2}{s} \right) \right] m_G^2 + 19g^2v^2 \right\} + \\
&\quad + \frac{1}{32\pi} \frac{s}{v^2} - \frac{1}{32\pi} \left[R^2 \left(\frac{s}{v^2} - g^2 \right) + 4 \frac{\bar{m}_{\text{scal}}^2}{v^2} \right] + \mathcal{O}(s^{-1})
\end{aligned}$$

where $\bar{m}_{\text{scal}}^2 = g_{vvh}^2 m_h^2 + g_{vv\phi}^2 m_\phi^2$ and $R^2 \equiv g_{vvh}^2 + g_{vv\phi}^2$ satisfies the following sum rule

$$\begin{aligned}
R^2 &= 1 + \left[\frac{\gamma(1-6\xi)}{Z} \right]^2 \quad \text{for } \gamma \equiv \frac{v}{\Lambda_\phi} \text{ and } Z^2 \equiv 1 + 6\xi\gamma^2(1-6\xi) \\
a_0 &= \frac{1}{32\pi} \left[(1-R^2) \frac{s}{v^2} + g^2 R^2 - 4 \frac{\bar{m}_{\text{scal}}^2}{v^2} + \text{graviton contributions} \right],
\end{aligned}$$

The very first term is responsible for the large violation of unitarity

$$f \equiv (1-R^2) \frac{s}{v^2} = - \left(\frac{1-6\xi}{Z} \right)^2 \frac{s}{\Lambda_\phi^2}$$

$$\underline{f\bar{f} \rightarrow G_{KK}, H, \phi \rightarrow Z_L Z_L}$$

$(h_{\bar{f}}, h_f)$	SM: t,u-channels	$(h - \phi)$ s-channel	G s-channel
(h, h)	$-2h \frac{m_f s^{1/2}}{v^2} + \mathcal{O}(s^0)$	$2hR^2 \frac{m_f s^{1/2}}{v^2} + \mathcal{O}(s^0)$	$-h \frac{8}{3} \frac{m_f s^{1/2}}{\hat{\Lambda}_W^2} (d_{0,0}^2 + \frac{1}{2}d_{0,0}^0) + \mathcal{O}(s^{-1/2})$
$(h, -h)$	0	0	$\sqrt{\frac{2}{3}} \frac{s}{\hat{\Lambda}_W^2} d_{1,0}^2 + \mathcal{O}(s^0)$

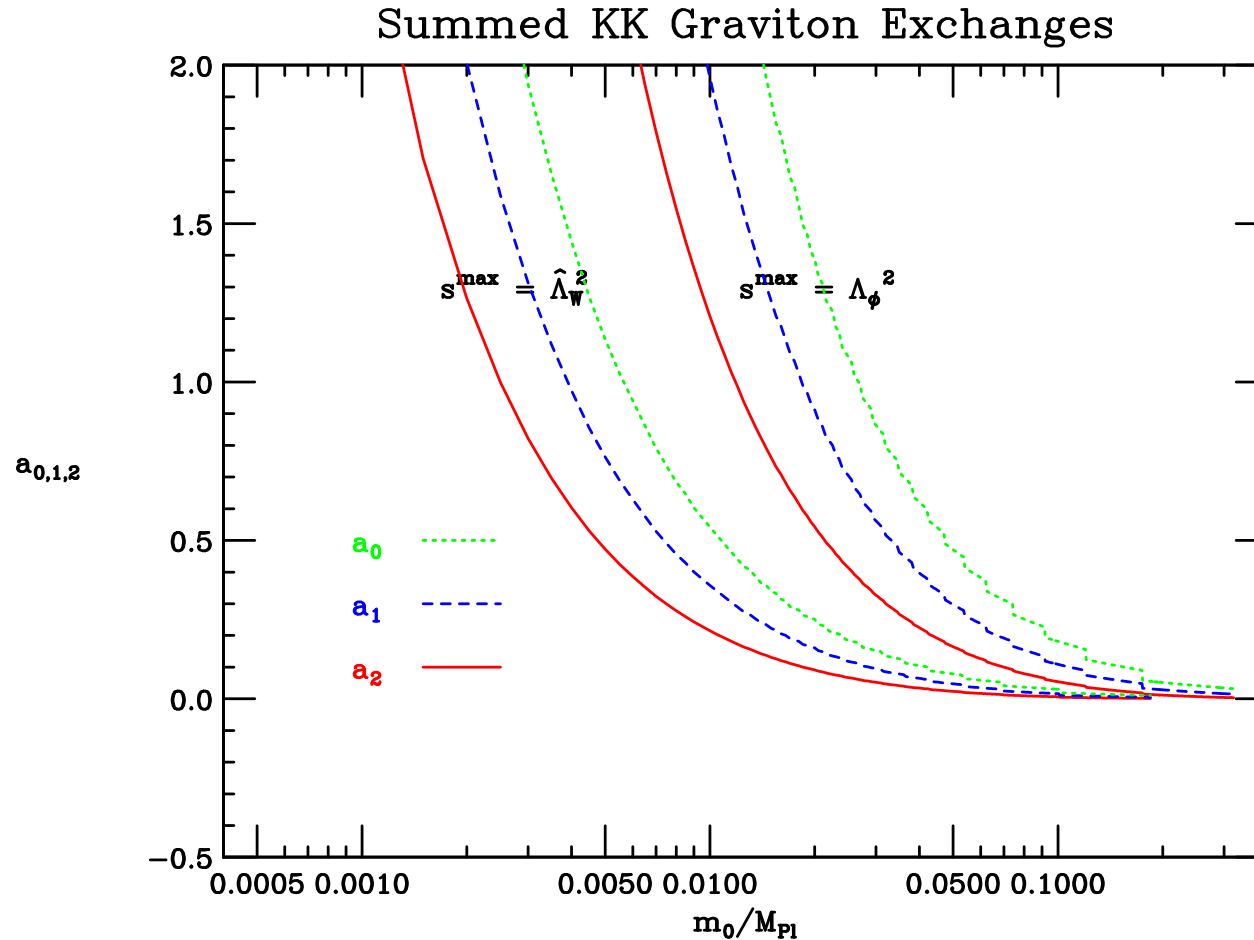
Determination of the cutoff for the Randall-Sundrum model

$$\begin{aligned}\widehat{\Lambda}_W &\simeq \sqrt{2}M_{Pl}\Omega_0, \\ \Lambda_\phi &= \sqrt{6}M_{Pl}\Omega_0 = \sqrt{3}\widehat{\Lambda}_W \\ m_n &= m_0x_n\Omega_0,\end{aligned}$$

where $\Omega_0 M_{Pl} = e^{-m_0 b_0/2} M_{Pl}$ should be of order a TeV to solve the hierarchy problem. The x_n are the zeroes of the Bessel function J_1 ($x_1 \sim 3.8$, $x_2 \sim 7.0$). A useful relation following from the above equations is:

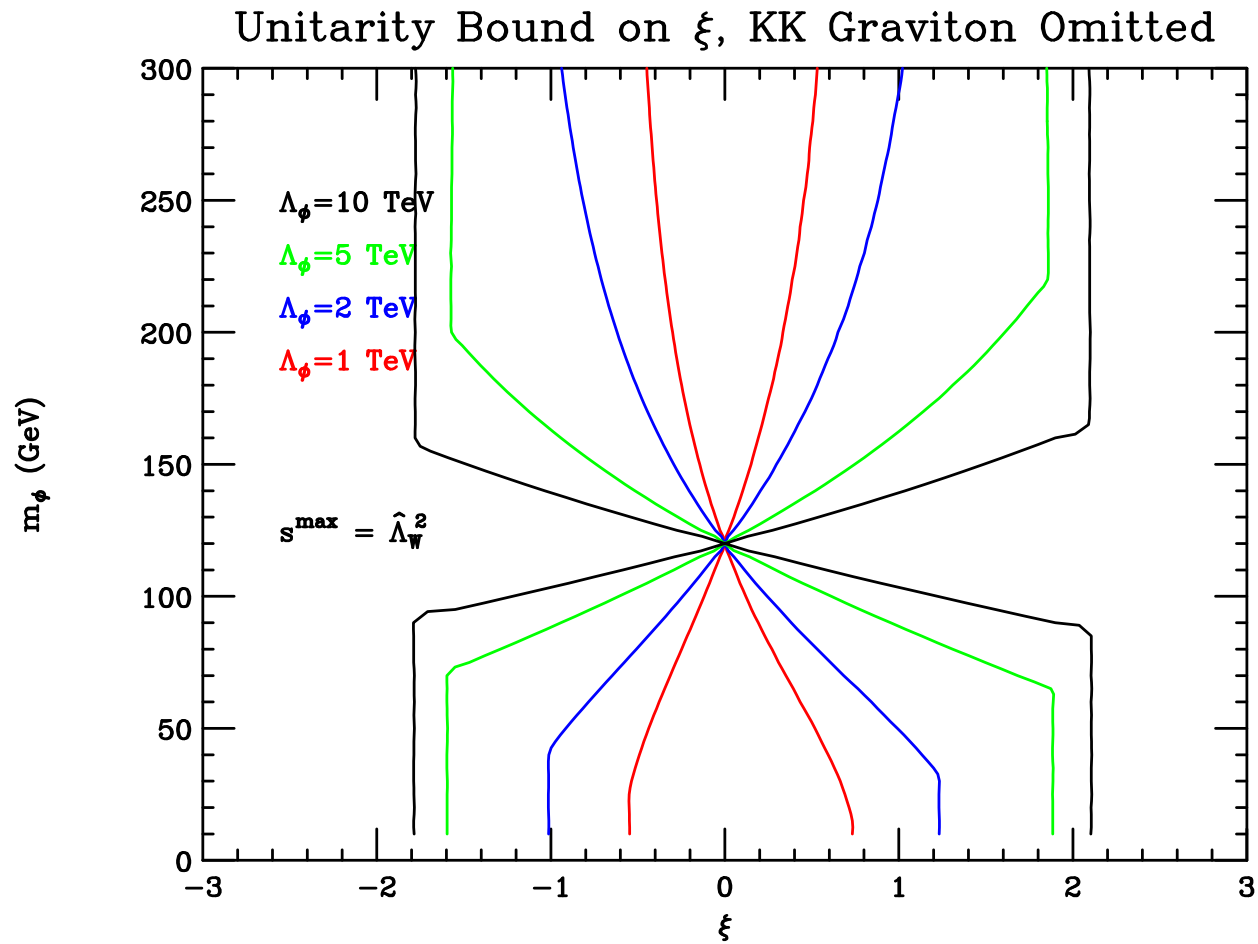
$$m_n = x_n \frac{m_0}{M_{Pl}} \frac{\Lambda_\phi}{\sqrt{6}} \quad \text{with} \quad 0.01 \lesssim \frac{m_0}{M_{Pl}} \lesssim 0.1$$

The RS model, no curvature-Higgs mixing



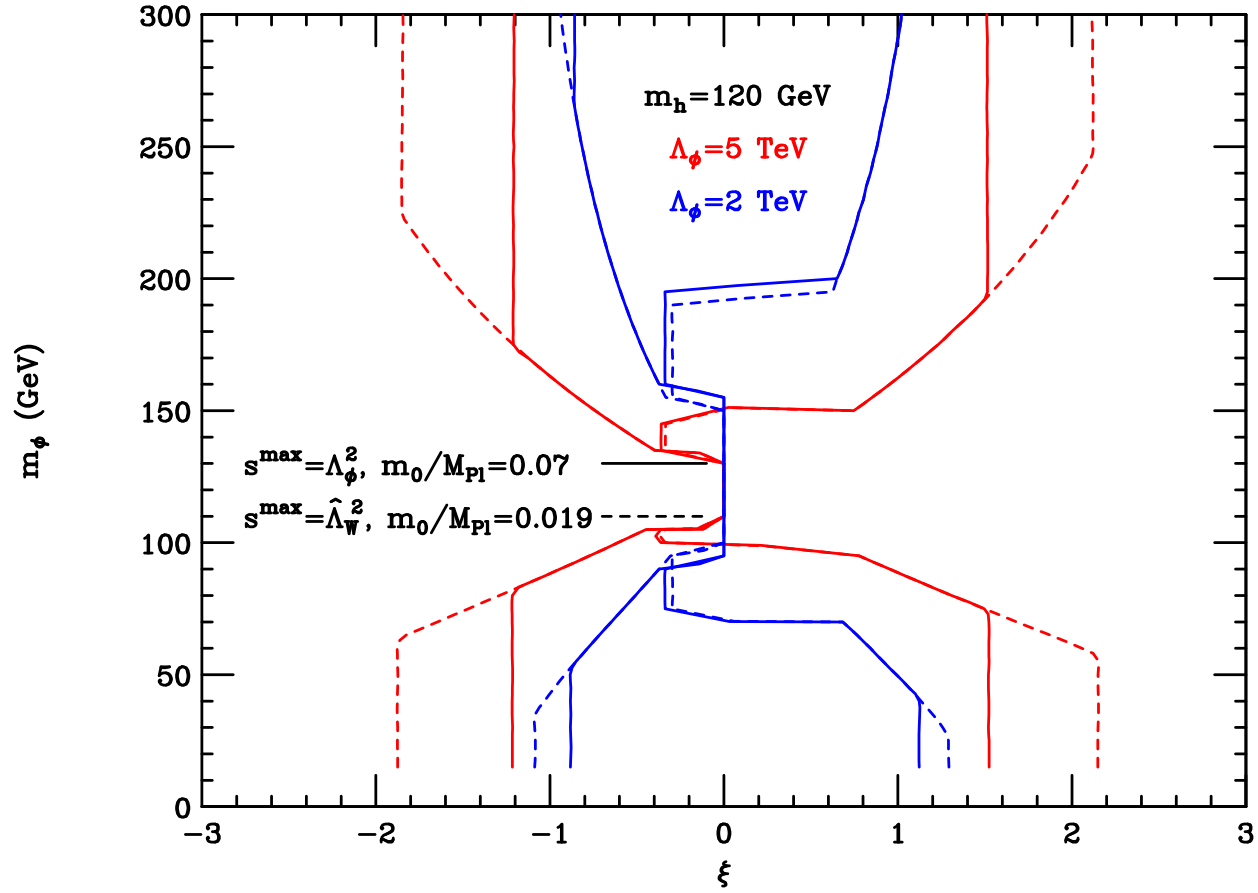
The amplitudes $a_{0,1,2}(s = \Lambda_\phi^2)$ and $a_{0,1,2}(s = \hat{\Lambda}_W^2)$ are plotted as functions of m_0/M_{Pl} , after summing:
 $a_J(s) = \sum_{n, 2m_n < \sqrt{s}} a_J(m_G = m_n, s)$. The plotted values of a_i terminate when m_0/M_{Pl} is such that
 $2m_1$ exceeds $\Lambda_\phi = \sqrt{3}\hat{\Lambda}_W$ or $\hat{\Lambda}_W$, for the two respective s values above.

The curvature-Higgs mixing model



The unitarity limits on ξ for $m_h = 120$ GeV when KK gravitons are omitted.

Unitarity Constraints on ξ with summed KK exchanges



The unitarity limits on ξ after summing over all KK excitations with $2m_n < \sqrt{s}$.

The van Dam-Veltman-Zakharov discontinuity

$$\text{massless graviton : } D^{\mu\nu,\alpha\beta}(k) = \frac{1}{2} \left(\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta} \right) \frac{i}{k^2 + i\epsilon}$$

$$\text{massive graviton : } D^{\mu\nu,\alpha\beta}(k) = \frac{1}{2} \left(\bar{\eta}^{\mu\alpha} \bar{\eta}^{\nu\beta} + \bar{\eta}^{\mu\beta} \bar{\eta}^{\nu\alpha} - \frac{2}{3} \bar{\eta}^{\mu\nu} \bar{\eta}^{\alpha\beta} \right) \frac{i}{k^2 - m_G^2 + i\epsilon}$$

$$\text{where } \bar{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{m_G^2}$$

The s-channel graviton contributions to the amplitude for $f\bar{f} \rightarrow Z_L Z_L$ calculated both for $m_G \neq 0$ and for $m_G = 0$ for possible initial state helicities ($h = \pm \frac{1}{2}$).

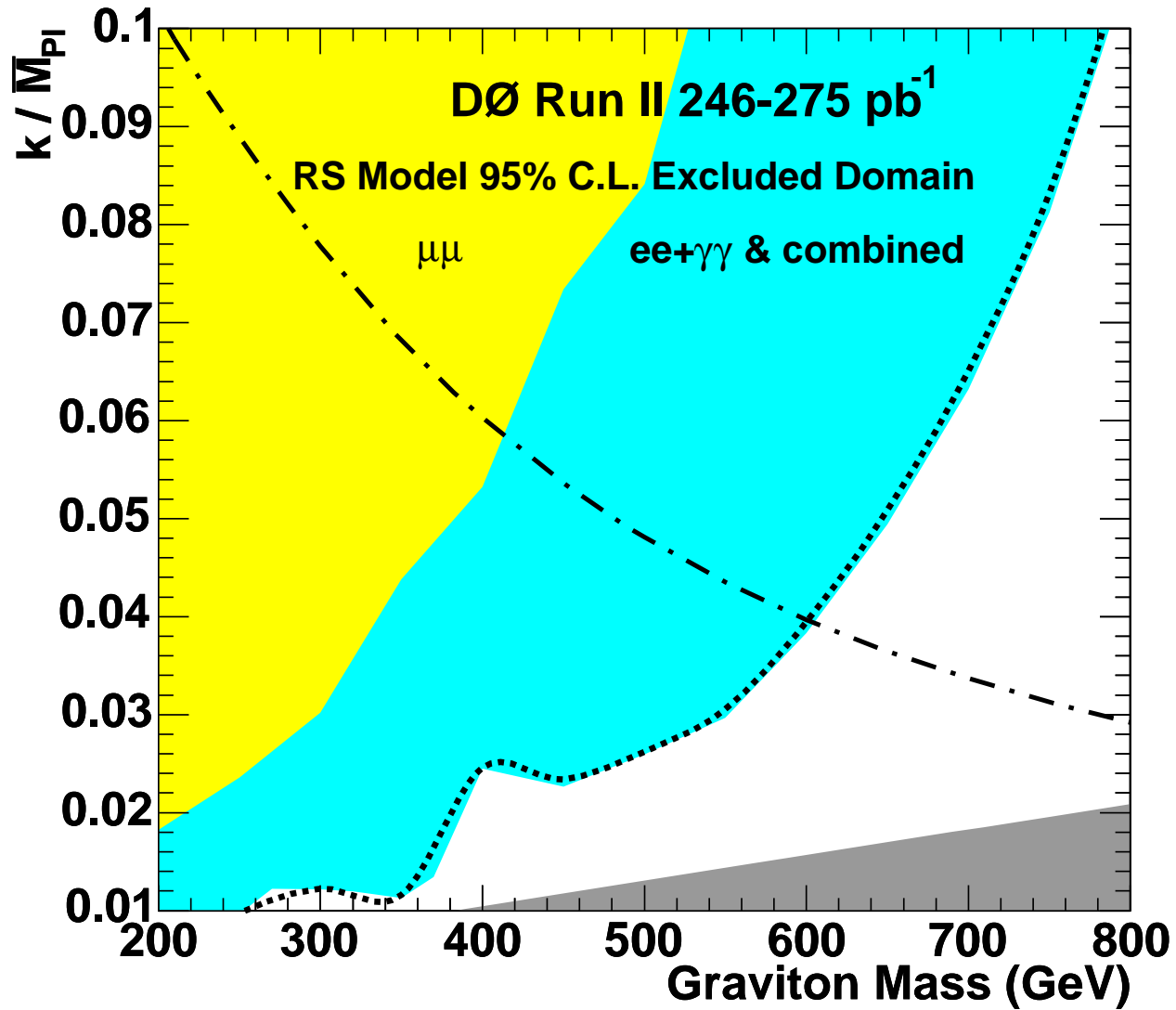
$(h_{\bar{f}}, h_f)$	$m_G \neq 0$	$m_G = 0$
(h, h)	$-h \frac{8}{3} \beta_f^{1/2} \frac{s+4m_Z^2}{s-m_G^2} \frac{m_f s^{1/2}}{\hat{\Lambda}_W^2} d_{0,0}^2$	$-h \frac{8}{3} \beta_f^{1/2} \frac{s+4m_Z^2}{s} \frac{m_f s^{1/2}}{\hat{\Lambda}_W^2} \left(d_{0,0}^2 + \frac{1}{2} \frac{s-2m_Z^2}{s+4m_Z^2} d_{0,0}^0 \right)$
$(h, -h)$	$\sqrt{\frac{2}{3}} \beta_f^{1/2} \frac{s+4m_Z^2}{s-m_G^2} \frac{s}{\hat{\Lambda}_W^2} d_{1,0}^2$	$\sqrt{\frac{2}{3}} \beta_f^{1/2} \frac{s+4m_Z^2}{s} \frac{s}{\hat{\Lambda}_W^2} d_{1,0}^2$

Comments:

- The amplitudes found for massless and massive gravitons are different.
- Even in the limit $m_G \rightarrow 0$ the massive amplitude doesn't coincide with the massless result. This is an illustration of the celebrated van Dam-Veltman-Zakharov discontinuity; regardless how small is the graviton mass, its very presence has physical consequences.
- Note however, that in all cases considered here, for $J = 2$, the limit $m_G \rightarrow 0$ of a massive amplitude does reproduce the massless calculation, in particular there is no discontinuity for opposite helicities. This result is consistent with the fact that the massive graviton does not contain $J = 0$ component, while the massless one does.
- Since the $J = 0$ component can not contribute to opposite helicity amplitudes, no wonder in those cases there is no discontinuity.
- The presence of $J = 0$ component of the massless graviton leads also to different high-energy behavior of the amplitudes and therefore may influence the unitarity constraints differently.

Summary

- The graviton exchange leads to divergent partial wave amplitudes for $V_L V_L \rightarrow V_L V_L$, $a_J \propto \frac{s}{\Lambda_W^2}$, and therefore can substantially modify their high-energy behavior.
- The tree-level unitarity requirement can be either adopted to determine the cutoff in the Randall-Sundrum model or to impose constraints on the free parameter $\frac{m_0}{M_{Pl}}$.
- The results obtained here for the graviton exchange are applicable to models that have massive gravitons which couples as $\frac{1}{\Lambda} \sum_{n \neq 0} h_{\mu\nu}^n T^{\mu\nu}$.
- In the curvature-Higgs mixing scenario, the presence of the radion-Higgs mixing spoils the cancellation of $a_{1,0} \propto s$ by the Higgs-boson exchange. Therefore the requirement of proper high-energy behavior severely constraints the allowed region for the mixing parameter ξ .
- The van Dam-Veltman-Zakharov discontinuity was observed in the process $f\bar{f} \rightarrow Z_L Z_L$.



95% C.L. exclusion limits on the RS model parameters M_1 and k/M_{Pl} . The area below the dashed-dotted line is excluded from the precision electroweak data. The dark shaded area in the lower right-hand corner corresponds to $\Lambda_\phi > 10$ TeV, which requires a significant amount of fine-tuning.

Scalar scattering

For completeness we also consider processes of scattering involving scalars only. Here however, there is no need to calculate SM-like diagrams involving only scalars since they don't contain any contributions growing with energy, so there is no chance for a compensation of $\gamma^2 = v^2/\hat{\Lambda}_W^2$.

First we need to derive Ghh , $G\phi\phi$ and $G\phi h$ vertices for off-shell external particles. Therefore one has to expand the curvature-Higgs mixing term around the R-S background metrics:

$$\begin{aligned}
 S_\xi &= \xi \int d^4x \sqrt{g_{\text{vis}}} R(g_{\text{vis}}) \hat{H}^\dagger \hat{H} = \\
 &6\xi \int d^4x \Omega \left\{ -\square \Omega + \epsilon \left[h_{\mu\nu} \partial^\mu \partial^\nu \Omega - \frac{1}{2} (\partial^\nu h_{\mu}^{\mu}) \partial_\nu \Omega + (\partial^\mu h_{\mu\nu}) \partial^\nu \Omega + \right. \right. \\
 &\quad \left. \left. \frac{1}{6} \Omega (\partial^\mu \partial^\nu - \eta^{\mu\nu} \square) h_{\mu\nu} \right] + \mathcal{O}(\epsilon^2) \right\} H_0^\dagger H_0,
 \end{aligned}$$

where $\epsilon \equiv (M_{\text{P}15})^{3/2}$.

Among all possible final and initial states made of ϕ and h , only the elastic Higgs-boson scattering have a chance to provide a result which is not suppressed by $\gamma^2 = v^2/\hat{\Lambda}_W^2$. Therefore below we focus on the graviton-exchange contributions to the process $hh \rightarrow hh$. We obtain the following contributions to the amplitude from the s- t- and u-channel:

$$a^{(s)}(\cos\theta) = -\frac{(A_{hh} - C_{hh})^2}{12\pi} \frac{s - 4m_h^2}{s - m_G^2} \frac{s}{\hat{\Lambda}_W^2}$$

$$a^{(t)}(\cos \theta) = \frac{(A_{hh} - C_{hh})^2}{12\pi} \frac{16(1 + \cos \theta)^2 m_h^4 - 8(5 + 6 \cos \theta + \cos^2 \theta) m_h^2 s + (13 + 10 \cos \theta + \cos^2 \theta)}{\hat{\Lambda}_W^2 [2m_G^2 + (1 - \cos \theta)(s - 4m_h^2)]}$$

$$a^{(u)}(\cos \theta) = a^{(t)}(-\cos \theta)$$

where

$$A_{hh} = -6\xi\gamma b(\gamma b + 2d), \quad C_{hh} = d^2$$

Integrating with the Legendre polynomials we obtain the following coefficients for the $J = 2, 1$ and 0 partial waves.

$$a_2 = -\frac{(A_{hh} - C_{hh})^2}{96\pi} \left[181 + 60 \log \left(\frac{m_G^2}{s} \right) \right] \frac{s}{\hat{\Lambda}_W^2} + \mathcal{O}(s^0)$$

$$a_1 = 0$$

$$a_0 = -\frac{(A_{hh} - C_{hh})^2}{96\pi} \left[11 + 12 \log \left(\frac{m_G^2}{s} \right) \right] \frac{s}{\hat{\Lambda}_W^2} + \mathcal{O}(s^0),$$

Expanding in powers of γ we get

$$(A_{hh} - C_{hh})^2 = 1 - \frac{72\gamma^2 \xi m_h^4}{(m_h^2 - m_\phi^2)^2} + \mathcal{O}(\gamma^4)$$

Note that for the scalar–scalar scattering, the Higgs and radion exchange could lead to terms growing as s , however those cancel and the result is of the order of $\mathcal{O}(s^0)$.

It turns out the $a_2 \simeq 10^{-2} \div 10^{-3}$ for reasonable set of parameters.