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# Majorana Fermions and CP Violation from 5-dimensional Theories

*Bohdan GRZADKOWSKI*

*Institute of Theoretical Physics, Warsaw University*
*Hoża 69, PL-00-681 Warsaw, POLAND*

1. Motivations:
   i) The hierarchy problem.
   ii) CP is violated spontaneously in 5D QED compactified on a circle.
   iii) How to generalize the above mechanism for an orbifold?
2. General properties of 5-dimensional gauge theories:
   i) periodicity, orbifolding and the consistency conditions,
   ii) options for CP violation: explicit versus spontaneous,
   iii) **the general solutions for the consistent boundary conditions.**
3. Properties of zero modes: Majorana fermions, gauge symmetry.
4. Conclusions


- B.G. and Jóse Wudka, "Majorana Fermions and CP Violation from 5-dimensional Theories: a Systematic Approach", arXiv:hep-ph/0501238
- B.G. and Jóse Wudka, "CP violation from 5-dimensional QED", Phys. Rev. Lett. **93**, 211603 (2004), arXiv:hep-ph/0401232

## 1. Motivations: i) The hierarchy problem

- Tree-level problem: why scalar masses are so different, e.g. doublet-triplet splitting problem in the GUT SU(5) model.
- The loop hierarchy problem: stabilization of the lightest scalar mass in the perturbation expansion (quadratic divergences).

Solutions to the hierarchy problem:

- SUSY: provides a mechanism to stabilize the scalar mass in the perturbation expansion, no quadratic divergences.
- Extra dimensions:
  - Higgsless models:
    - orbifold (e.g. $S^1/Z_2$): gauge symmetry breaking by periodicity or orbifold reflection twist operators (the Scherk-Schwarz mechanism),
    - interval: gauge symmetry breaking by boundary conditions.
  - The Higgs boson as an extra component of a higher-dimensional gauge field:
    - the tree level mass forbidden by the gauge symmetry, so no tree-level hierarchy problem,
    - scalar mass calculable and <u>finite at the loop level</u>, no divergences, so no loop hierarchy problem.

## 1. Motivations: ii) CP violation in 5D QED on a circle

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{MN}^2 + \sum_i \bar{\psi}_i \left(i\gamma^M D_M - M_i\right)\psi_i + \mathcal{L}_{gf}\,,$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$, $D_M = \partial_M + ie_5 q_i A_M$, $q_i$ denotes the charge of $\psi_i$, $\gamma_M = (\gamma\mu, \gamma_4 = i\gamma_5)$ for $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\mathcal{L}_{gf}$ stands for a gauge-fixing term.

Boundary conditions:

$$\begin{aligned} A_M(x^\mu, y+L) &= A_M(x^\mu, y), \\ \psi(x^\mu, y+L) &= e^{i\alpha}\psi(x^\mu, y)\,, \qquad \text{where} \qquad L = 2\pi R\,. \end{aligned}$$

KK expansion:

$$\begin{aligned} A_M(x,y) &= \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty} A_{M\,n}(x)\,e^{i\omega_n y} \qquad \text{for} \qquad \omega_n = 2\pi n/L \\ \psi(x,y) &= \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty} \psi_n(x)\,e^{i\bar{\omega}_n y} \qquad \text{for} \qquad \bar{\omega}_n = \omega_n + \alpha/L\,. \end{aligned}$$

- $A_{4\,n}$ become 4D scalars & $A_{4\,n} \xrightarrow{\text{CP}} -A_{4\,n}$.
- $\langle A_{4\,n}\rangle \neq 0 \quad \Longrightarrow \quad$ possibility of spontaneous violation of CP.

Symmetries:

- The Lagrangian and the BC are invariant under the gauge U(1) symmetry

$$\psi(x,y) \xrightarrow{\mathrm{U}(1)} e^{-ie_5 q\Lambda(x,y)}\psi(x,y),$$
$$A_M(x,y) \xrightarrow{\mathrm{U}(1)} A_M(x,y) + \partial_M \Lambda(x,y)\,.$$

- Discrete transformations

| | 4D | | | 5D | | |
|---|---|---|---|---|---|---|
| | $x^\mu$ | $\psi$ | $A_\mu$ | $x^M$ | $\psi$ | $A_M$ |
| P | $x_\mu$ | $\gamma_0\psi$ | $A^\mu$ | $x^0, -x^i, x^4$ | $\gamma_0\gamma_4\psi$ | $A_0, -A_i, A_4$ |
| C | $x^\mu$ | $(\gamma_0\gamma_2)(\bar{\psi})^T$ | $-A_\mu$ | $x^M$ | $(\gamma_1\gamma_3)(\bar{\psi})^T$ | $-A_M$ |

where $\gamma_4 = i\gamma_5$ with $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

The action is symmetric under the 5D CP transformations:

$$x^{0,4} \xrightarrow{\mathrm{CP}} +x^{0,4}\,, \qquad x^{1,2,3} \xrightarrow{\mathrm{CP}} -x^{1,2,3}$$
$$A^{0,4} \xrightarrow{\mathrm{CP}} -A^{0,4}\,, \qquad A^{1,2,3} \xrightarrow{\mathrm{CP}} +A^{1,2,3}$$
$$\psi \xrightarrow{\mathrm{CP}} \eta\gamma^0\gamma^2\psi^\star\,, \qquad |\eta| = 1\,,$$

## Fermionic kinetic terms

$$\int_0^L \mathcal{L}_{QED}\, dy \quad \Longrightarrow \quad \mathcal{L}_\psi = \sum_n \bar{\psi}_n \left[ i\gamma^\mu \partial_\mu - M + i\gamma_5 \mu_n \right] \psi_n \,,$$

where $\mu_n \equiv \left[2\pi n + (\alpha + eqLa)\right]/L$, with $e \equiv e_5/\sqrt{L}$ the 4D gauge coupling.

A chiral rotation to diagonalize the fermion mass:

$$\psi_n \to e^{i\gamma_5 \theta_n} \psi_n \,, \qquad \tan(2\theta_n) = \frac{\mu_n}{M}; \quad |\theta_n| \leq \pi/4$$

The physical fermion masses: $m_n = \sqrt{M^2 + \mu_n^2}$.

## Interactions

$$\mathcal{L}_{\varphi\psi} = -eq\varphi \sum_n \bar{\psi}_n \Gamma_n \psi_n \,, \qquad \Gamma_n \equiv \sin(2\theta_n) - i\gamma_5 \cos(2\theta_n) \,,$$

where $\varphi \equiv A_{4\,0}$.

*$\varphi(x) \equiv A_{4\,0}(x)$ is a new, physical, tree-level massless, 4D degree of freedom, Yukawa couplings of which appear to be CP-violating.*

$$\begin{aligned} A_M(x,y) &\xrightarrow{\mathrm{U}(1)} A_M(x,y) + \partial_M \Lambda(x,y) \\ \Lambda(x,y) &= \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{+\infty} \Lambda_n(x) e^{i\omega_n y} \end{aligned}$$

$$\Downarrow$$

$$A_{4\,k} \xrightarrow{\mathrm{U}(1)} A_{4\,k} + i\omega_k \Lambda_k \quad \text{with} \quad \omega_k = \frac{2\pi k}{L}$$

$$\Downarrow$$

$$\varphi \xrightarrow{\mathrm{U}(1)} \varphi$$

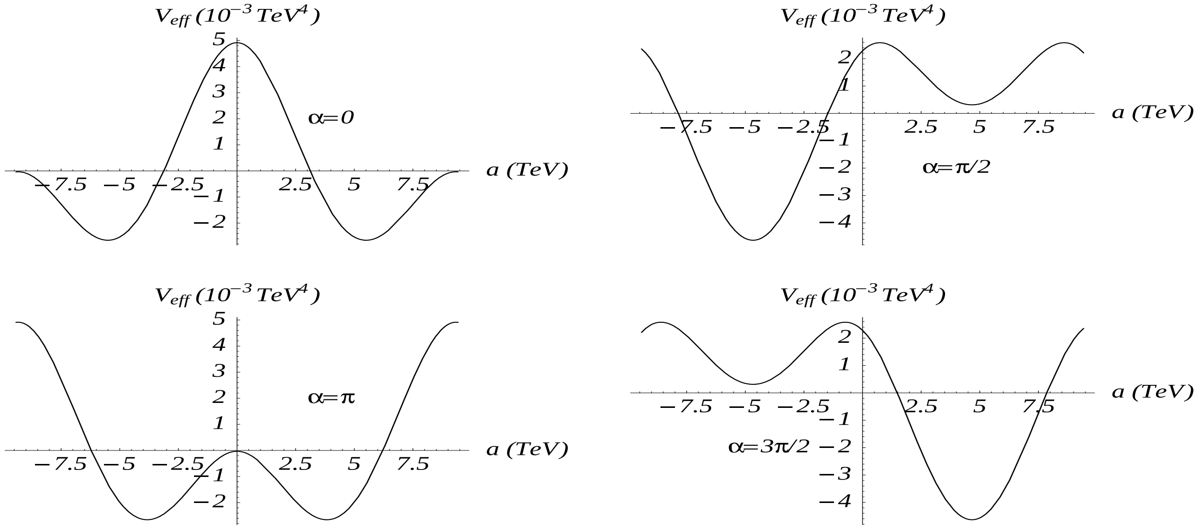
$$\Downarrow$$

The gauge symmetry does not protect $m_\varphi = 0$ !!!

$$\Downarrow$$

$\varphi$ will receive calculable finite mass at higher orders.

# The effective potential




- Dimensional regularization for the $d^4p$ integral.
- Summation over an infinite tower of KK modes.

After dropping an irrelevant constant contribution:

$$V_{\rm eff}(a) = \frac{1}{2\pi^2 L^4} \sum_i \left[ x_i^2 Li_3(r_i e^{-x_i}) + 3x_i Li_4(r_i e^{-x_i}) + 3Li_5(r_i e^{-x_i}) + \text{H.c.} \right] ,$$

$$x_i \equiv LM_i, \qquad E_i = (\alpha_i + eq_i La)/L, \qquad r_i = \exp(iLE_i), \qquad Li_n(x) = \sum_{s=1}^{\infty} \frac{x^s}{s^n} .$$

**1. Motivations: iii) How to generalize the mechanism of CP violation for an orbifold?**

$$A_{4\,0} \qquad \text{must exist !}$$

$$\Downarrow$$

$$A_4(y+L) = A_4(y) \qquad \text{and} \qquad A_4(-y) = A_4(y)$$

$$\Downarrow$$

$$A_\mu(-y) = -A_\mu(y) \qquad \text{since} \qquad F_{4\mu} = \partial_4 A_\mu - \partial_\mu A_4$$

$$A_\mu(-y) = -A_\mu(y) \qquad \text{and} \qquad A_4(-y) = A_4(y)$$

Assume the standard fermionic orbifold transformation:

$$\psi(y) \rightarrow \psi(-y) = e^{i\beta}\gamma_5\psi(y)$$

Then

$$\bar{\psi}\gamma^N[\partial_N + ig_5 q A_N]\psi \rightarrow \bar{\psi}\gamma^N[\partial_N + ig_5(-q)A_N]\psi$$

We need to switch the sign of the charge $q$, so it suggests to adopt the charge conjugation:

The periodicity BC

$$A_\mu(y+L) = A_\mu(y)\,, A_4(y+L) = A_4(y)\,, \psi(y+L) = e^{i\alpha}\psi(y)$$

The orbifold BC

$$A_\mu(-y) = -A_\mu(y)\,, A_4(-y) = A_4(y)\,, \psi(-y) = e^{i\beta}\gamma_5\psi^C(y)$$

- $\mathcal{L}(\psi, A_N)$ is invariant.
- The consistency conditions are satisfied.
- Majorana zero modes seem to be conceivable.
- The gauge field zero modes are disallowed by the BC, no 4D QED.

**2. i) Periodicity, orbifolding and the consistency conditions**

$$\mathcal{L} = -\frac{1}{4}\sum_a \frac{1}{g_a^2}\, F^a_{MN} F^{a\ MN} + \bar{\Psi}(i\gamma^N D_N - M)\Psi\,,$$

where $D_N = \partial_N + ig_5 A_N$, $A_N = A^a_N T^a$.

- $x^\mu$, $\mu = 0, \ldots, 3$ are the $\mathbb{M}^4$ coordinates, $y$ the extra component: $S_1/Z_2$, $0 \leq y \leq L$, $y$ identified with $-y$ and $M = 0, 1, 2, 3, 4$.
- Topology: is $\mathbb{M}^4 \times (S_1/Z_2)$.
- Flat matric $g_{NM} = \mathrm{diag}(1, -1, -1, -1, -1)$ with the last entry associated with $S_1/Z_2$.

Periodicity

$$\Psi(y+L) = \Gamma\Psi(y) + \Upsilon^*\Psi^c(y)$$
$$A_N(y+L) = \begin{cases} +U_1^\dagger A_N(y) U_1 & (P1) \\ -U_2^\dagger A_N^T(y) U_2 & (P2) \end{cases},$$

where $U_{1,2}$ are global elements of the gauge group.

$$\chi \equiv \begin{pmatrix} \Psi^c \\ \Psi \end{pmatrix}, \qquad \mathcal{A} \equiv \begin{pmatrix} \Gamma & -\Upsilon^* \\ \Upsilon & \Gamma^* \end{pmatrix}$$

in terms of which the fermionic periodicity conditions are simply

$$\chi(y+L) = \mathcal{A}^*\chi(y)\,.$$

Requiring invariance of the kinetic term $\bar\Psi i\gamma^N D_N\Psi$ gives the following conditions on the acceptable BC:

$$\mathcal{A}^\dagger\mathcal{A} = \mathbb{1}, \qquad \begin{array}{ll} P1: & [\tau_a, \mathcal{U}_1\mathcal{A}] = 0 \\ P2: & [\tau_a, \mathcal{U}_2\mathcal{A}] = 0 \end{array} \qquad \text{with}$$

$$\tau^a \equiv \begin{pmatrix} T_a & 0 \\ 0 & -T_a^* \end{pmatrix}, \quad \mathcal{U}_1 \equiv \begin{pmatrix} U_1 & 0 \\ 0 & U_1^* \end{pmatrix}, \quad \mathcal{U}_2 \equiv \begin{pmatrix} 0 & U_2^* \\ U_2 & 0 \end{pmatrix}$$

Orbifold Parity

$$\begin{aligned}\chi(-y) &= \gamma_5 \mathcal{B}^* \chi(y) \\ A_N(-y) &= \begin{cases} (-1)^{s_N} \tilde{U}_1^\dagger A_N(y) \tilde{U}_1 & (R1) \\ (-1)^{1-s_N} \tilde{U}_2^\dagger A_N^T(y) \tilde{U}_2 & (R2) \end{cases} ,\end{aligned}$$

where $s_N = \delta_{N,4}$, $\tilde{U}_{1,2}$ are global gauge transformations and

$$\mathcal{B} \equiv \begin{pmatrix} -\tilde{\Gamma} & \tilde{\Upsilon}^* \\ \tilde{\Upsilon} & \tilde{\Gamma}^* \end{pmatrix} .$$

Requiring now the invariance of $\mathcal{L}$ under the twist implies

$$\mathcal{B}^\dagger \mathcal{B} = 1 \,, \qquad \begin{aligned} R1: &\quad [\tau_a, \tilde{\mathcal{U}}_1 \mathcal{B}] = 0 \\ R2: &\quad [\tau_a, \tilde{\mathcal{U}}_2 \mathcal{B}] = 0 \qquad \text{with} \end{aligned}$$

$$\tilde{\mathcal{U}}_1 \equiv \begin{pmatrix} \tilde{U}_1 & 0 \\ 0 & \tilde{U}_1^* \end{pmatrix} \qquad \tilde{\mathcal{U}}_2 \equiv \begin{pmatrix} 0 & \tilde{U}_2^* \\ \tilde{U}_2 & 0 \end{pmatrix} .$$

Consistency Conditions

The periodicity and reflection transformations are not independent since $-y = [-(y+L)] + L$ and $-(-y) = y$, therefore

$$\begin{aligned} \mathcal{B} &= \mathcal{A}\mathcal{B}\mathcal{A}\,, \\ \mathcal{B}^2 &= \mathbb{1} \end{aligned} \tag{1}$$

for the fermions. For the $Pi - Rj$ BC ($i, j = 1, 2$) the corresponding constraints on the gauge bosons give (no sum over $i$ and $j$)

$$\begin{aligned} &[\tau_a, \tilde{\mathcal{V}}_j \mathcal{V}_i \tilde{\mathcal{V}}_j \mathcal{V}_i^\dagger] = 0\,, \\ &[\tau_a, \tilde{\mathcal{V}}_i^2] = 0\,, \end{aligned} \tag{2}$$

where $\mathcal{V}_1 = \mathcal{U}_1,\ \tilde{\mathcal{V}}_1 = \tilde{\mathcal{U}}_1$ and $\mathcal{V}_2 = \mathcal{U}_2^*,\ \tilde{\mathcal{V}}_2 = \tilde{\mathcal{U}}_2^*$.

These conditions imply that $\tilde{\mathcal{V}}_j \mathcal{V}_i \tilde{\mathcal{V}}_j \mathcal{V}_i^\dagger$ and $\tilde{\mathcal{V}}_i^2$ belong to the center of the group. If the representation generated by $\{\tau_a\}$ is split into its irreducible components, the projection of these matrices onto each irreducible subspace must be proportional to the unit matrix as a consequence of the Schur's lemma.

**2. ii) Options for CP violation: explicit versus spontaneous**

Under CP we obtain

$$\chi \xrightarrow{CP} \gamma_0\gamma_4\mathcal{D}\chi \equiv \gamma_0\gamma_4 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \chi$$

while the gauge fields transform as

$$A_i \xrightarrow{CP} + A_i^T\,, \qquad A_{0,4} \xrightarrow{CP} - A_{0,4}^T\,,$$

where $i = 1, 2, 3$.

In order to conserve CP we need to satisfy:

- $\Gamma, \Upsilon, \tilde{\Gamma}, \tilde{\Upsilon}$ real,
- $\langle A_4 \rangle = 0$.

**2. iii) The general solutions for the consistent boundary conditions.**

For the real or pseudoreal representation generators satisfy

$$T_a^{(\bar{r})} = \left[-T_a^{(r)}\right]^* = ST_a^{(r)}S^\dagger ,$$

for some unitary matrix $S_u$ that is (anti)symmetric for (pseudo)real representations.

The following comments originate from the general solution for the consistent boundary conditions:

- The matrices $\Gamma$, $\Upsilon$, $\tilde{\Gamma}$ and $\tilde{\Upsilon}$ do not mix $\psi_r$ and $\psi_s$ unless $r$ is equivalent to $s$ or $\bar{s}$.
- For the BC P1 and R1 (P2 and R2), in the subspace spanned by all multiplets in the same *complex* irreducible representation $r$, the matrices $\Upsilon$ and $\tilde{\Upsilon}$ ($\Gamma$ and $\tilde{\Gamma}$) vanish. In contrast, $\Gamma$ and $\tilde{\Gamma}$ ($\Upsilon$ and $\tilde{\Upsilon}$) are direct products of unitary rotation in flavor indices and global gauge transformation in gauge indices.
- In the subspace spanned by all multiplets carrying the same *(pseudo) real* irreducible representation $r$ in general $\Upsilon$ ($\tilde{\Upsilon}$) and $\Gamma$ ($\tilde{\Gamma}$) are non-zero.

## 3. Properties of zero modes

$$(\mathbb{1} - \mathcal{A}^*)\chi_0 = 0 \qquad (\mathbb{1} - \gamma_5 \mathcal{B}^*)\chi_0 = 0\,.$$

where

$$\chi(x,y) = \sum \chi_n(x) v_n(y) \qquad \chi_0 = \begin{pmatrix} \zeta^c \\ \zeta \end{pmatrix}$$

$$\begin{aligned} (\mathbb{1} - \Gamma)\,\zeta_L &= \Upsilon^*\,(\zeta_R)^c & \left(\mathbb{1} + \tilde{\Gamma}\right)\zeta_L &= -\tilde{\Upsilon}^*\,(\zeta_R)^c \\ \Upsilon \zeta_L &= (\Gamma^* - \mathbb{1})\,(\zeta_R)^c & \tilde{\Upsilon}\zeta_L &= \left(\tilde{\Gamma}^* - \mathbb{1}\right)(\zeta_R)^c\ . \end{aligned}$$

The standard strategy:

$$\Upsilon = \tilde{\Upsilon} = 0 \qquad \Longrightarrow \qquad \begin{cases} \Gamma = \mathbb{1}, \tilde{\Gamma} = +\mathbb{1} & \zeta_L = 0 \\ \Gamma = \mathbb{1}, \tilde{\Gamma} = -\mathbb{1} & \zeta_R = 0 \end{cases}$$

The generalized Majorana 4D condition

$$\zeta = NC_4(\bar{\zeta})^T \,,$$

where $C_4$ is the 4D charge conjugation operator ($C_4 = \gamma_0\gamma_2$ while the 5D one is $C_5 = \gamma_1\gamma_3$. Note that $\gamma_5 C_5 = -iC_4$.), and $N$ acts on flavor and gauge indices. Then

$$\zeta = \begin{pmatrix} N\sigma_2\varphi^* \\ \varphi \end{pmatrix} ,$$

where $\varphi$ denotes a 2-component spinor. Consistency of this expression requires $NN^* = \mathbb{1}$.

For the Majorana spinor $\zeta$, the zero-mode conditions become

$$\begin{aligned} &(\mathbb{1} - \Gamma)\varphi + i\Upsilon^*\sigma_2\varphi^* = 0 \,, \quad (N^*\Gamma - \Gamma^* N^*)\varphi = 0 \,, \quad (N^*\Upsilon^* - \Upsilon N)\varphi^* = 0 \\ &(N + i\tilde{\Upsilon}^*)\sigma_2\varphi^* - \tilde{\Gamma}\varphi = 0 \,, \quad (N^*\tilde{\Gamma} + \tilde{\Gamma}^* N^*)\varphi = 0 \,, \quad (N^*\tilde{\Upsilon}^* + \tilde{\Upsilon} N)\varphi^* = 0 \,. \end{aligned}$$

It is useful to illustrate the above conditions by certain special cases:

- If $\tilde{\Gamma} = 0$ then there is always a Majorana zero mode with $N = -i\tilde{\Upsilon}^\dagger$. This case is illustrated by the BC $(P1 - R2)$ for a single Abelian fermion with $N = -i$.
- If $\tilde{\Upsilon} = 0$, $\Gamma^* \neq \mathbb{1}$ and $\Upsilon$ is invertible (so charge conjugated field appears in the periodicity BC) then again there exists a Majorana zero mode for $N = -i\Upsilon^{-1}(\mathbb{1} - \Gamma^*)\tilde{\Gamma}^*$. For an Abelian model this case requires more than one flavor.

The gauge invariance of the zero-mode sector

For $y$-independent $\Omega$ and for non-complex representations the BC are preserved by gauge transformations that obey

$$P1: \ [U_1, \Omega] = 0 \qquad R1: \ [\tilde{U}_1, \Omega] = 0$$
$$P2: \ [S^\dagger \cdot U_2, \Omega] = 0 \qquad R2: \ [S^\dagger \cdot \tilde{U}_2, \Omega] = 0\,,$$

Let us consider SU(2) gauge theory with a single doublet of fermions, adopt the $P1 - R1$ BC and choose

$$\begin{array}{lll} \Gamma = \sigma_3 & \Upsilon = 0 & U_1 = i\sigma_3 \\ \tilde{\Gamma} = \mathbb{1} & \tilde{\Upsilon} = 0 & \tilde{U}_1 = \mathbb{1}\,, \end{array} \tag{3}$$

$$\begin{aligned} \omega_1(y) = & \ -\omega_1(y+L) = & +\omega_1(-y) \\ \omega_2(y) = & \ -\omega_2(y+L) = & +\omega_2(-y) \\ \omega_3(y) = & \ +\omega_3(y+L) = & +\omega_3(-y) \end{aligned}$$

*Zero-mode for $\omega_{1,2}$ is forbidden, whereas zero-mode for $\omega_3$ is allowed: $SU(2) \to U(1)$.*

The zero-mode fermion obey

$$\zeta_L = 0 \qquad (\sigma_3 - \mathbb{1})\,(\zeta_R)^c = 0\,.$$

The zero-mode sector contains only the $A_3^\mu$ massless gauge bosons and a right-handed, charged (and therefore massless), fermion.

## 4. Conclusions

- We have considered a generic gauge theory in a 5-dimensional space compactified on $\mathbb{M}^4 \times (S_1/Z_2)$, and studied the effects of a generalized set of boundary conditions that allow for mixing between particles and anti-particles after a translation by the size of the extra dimension or after an orbifold reflection.

- The general form of the periodicity and orbifold conditions that are allowed by consistency requirements was found.

- General conditions for the presence of massless Kaluza-Klein modes were formulated. Gauge symmetry of the zero-mode sector was determined. It was shown that if the orbifold twist operation transforms particles into anti-particles then the zero-mode fermions are 4-dimensional Majorana fermions.

- We have determined the conditions under which CP would be violated (explicitly) by the BC as well as spontaneously, through a possible vacuum expectation value of the fifth component of the gauge fields. It turns out that two Abelian fermions with (P1-R2) boundary conditions lead to spontaneous CP violation through $\langle A_4 \rangle \neq 0$ (the Hosotani mechanism). The U(1) gauge symmetry of the zero-mode sector is broken in that case.