Cosmology Course
problems for the written exam
February 6th 2017

1. Show, using the following form of the energy-momentum tensor

$$T^{\mu\nu} = \sum_n \frac{p_n^\mu p_n^\beta}{E_n} \delta^3(\vec{x} - \vec{x}_n)$$

that for a gas of non-relativistic particles of mass m, the energy density (ρ), pressure (p) and the number density (n) are related by $\rho = nm + \frac{3}{2}p$.

2. Show, using the following form of the energy-momentum tensor

$$T^{\mu\nu} = \sum_n \frac{p_n^\mu p_n^\beta}{E_n} \delta^3(\vec{x} - \vec{x}_n)$$

that for a gas of relativistic particles, the energy density (ρ) and pressure (p) are related by $\rho = 3p$.

3. The fundamental equations for cosmology are

   - The Friedmann equation

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2$$

   - The acceleration equation:

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = -8\pi G p$$

   - The energy-momentum conservation (the first law of thermodynamics):

$$\dot{p}R^3 = \frac{d}{dt}\left[R^3(\rho + p)\right]$$

   Show that only two of them are independent.

4. Calculate the non-vanishing components of the curvature tensor $R_{\mu\nu}$ for a static and isotropic metric (as for the Schwarzschild solution).

5. Knowing the non-vanishing components of the curvature tensor $R_{\mu\nu}$:

$$\begin{aligned}
R_{tt} &= -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} \\
R_{rr} &= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{A'}{A} \\
R_{\theta\theta} &= -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} \\
R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta} \\
R_{\mu\nu} &= 0 \quad \text{for} \quad \nu \neq \mu
\end{aligned}$$

   for

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

   find the Schwarzschild solution and determine the Schwarzschild radius.

6. Show that if the energy density is dominated by positive cosmological constant and Universe is spatially flat ($k = 0$) then the expansion is exponential.

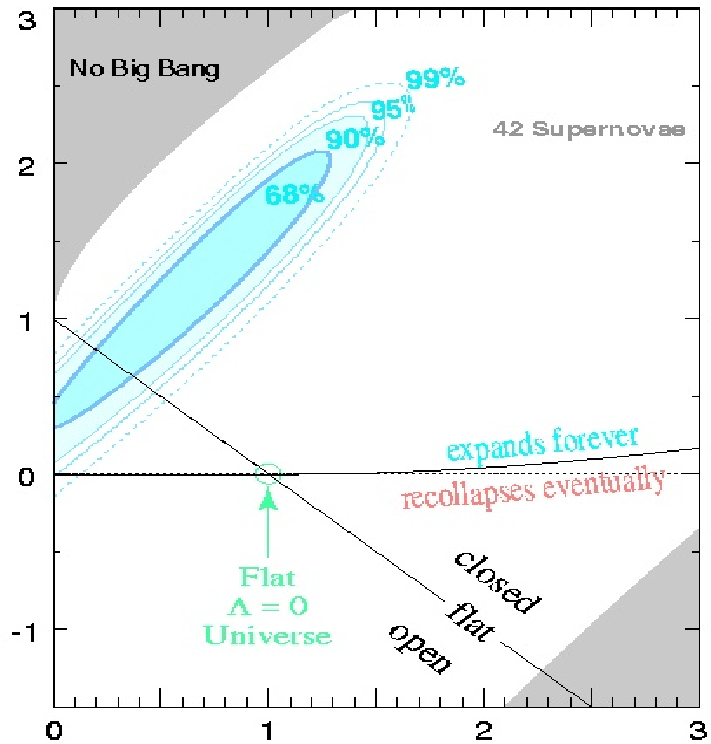

Figure 1: Confidence regions for Ω_m vs. Ω_Λ plane, from SCP.

7. Find the necessary and sufficient conditions for the exponential infation in terms of the equation of state, consider both $k = 0$ and $k \neq 0$.

8. Assuming a single component Universe, find the condition which must be satisfied by the equation of state in order to ensure positive acceleration of the Universe.

9. Find the present Universe age in terms of $\Omega_{\rm rad}$ and H_0 assuming radiation domination and zero curvature ($k = 0$).

10. Find the present Universe age in terms of $\Omega_{\rm m}$ and H_0 assuming matter domination and zero curvature ($k = 0$).

11. Find the present Universe age in terms of Ω_Λ and H_0 assuming that it is flat ($k = 0$) and it contains both matter and cosmological constant.
Hint:
$$\int_0^1 \frac{dx}{(ax^{-1} + bx^2)^{1/2}}\bigg|_{a+b=1} = \frac{1}{3b^{1/2}} \ln\left[\frac{(b + b^{1/2})^2}{b(1-b)}\right]$$

12. Assuming $\Lambda > 0$ and $k = 1$ construct the static Universe containing cosmological constant Λ and non-relativistic matter, i.e. determine Λ_E and R_E, such that $\dot{R}(t) = \ddot{R}(t) = 0$ and discuss its stability. Describe the evolution if

   - $\Lambda < \Lambda_E$
   - $\Lambda > \Lambda_E$

13. Determine the boundary (i.e. find the appropriate condition for $\Omega^0_\Lambda = \Omega^0_\Lambda(\Omega^0_m)$) between regions "expands forever" and "recollapses eventually" in figure 1.

14. Using results obtained for the static Einstein Universe ($k = 1$ and $\Lambda > 0$):

$$R_E = \frac{3}{2}b, \qquad \Lambda_E = \left(\frac{2}{3b}\right)^2, \qquad \text{for} \qquad b \equiv \frac{1}{3}8\pi G\rho_0 R_0^3$$

where ρ_0 is the matter energy density corresponding to the scale factor R_0, determine the boundary of the region "No Big Bang" in the figure 1.

15. Show that in the radiation dominated Universe the deceleration parameter q_0 equals $\Omega^0_{\rm rad}$.

16. In terms of Ω^0_r and Ω^0_m, determine the red-shift for which radiation and matter contributions to the energy density are equal.

17. Assuming that the graviton-SM scattering cross-section scales as

$$\sigma_{\rm grav} \sim \frac{T^2}{M_{Pl}^4}$$

find relation between the present graviton background temperature and the present temperature of CMB. Estimate present graviton contribution to the energy density. Hint: graviton is massless.

18. Assuming that there exist an extra (comparing to the Standard Model) single relativistic species which decouples form the SM at $T_f > 246$ GeV, finds its contribution to the energy density at the BBN.

19. Assuming that the BBN constraints an extra contribution to the energy density ρ^{NP} such that

$$\left.\frac{\rho^{NP}}{\rho^{SM}}\right|_{BBN} < 0.07 \qquad \text{at} \qquad 95\%\text{CL}$$

find the upper limit for the number of extra degrees of freedom that is implied by the constraint.

20. Derive a limit for the number of extra light, stable neutrinos which is implied by the BBN constraint

$$\left.\frac{\rho^{NP}}{\rho^{SM}}\right|_{BBN} < 0.07 \qquad \text{at} \qquad 95\%\text{CL}$$

Assume that those neutrinos decouple at the same temperature as the standard neutrinos.

21. Assume monopoles are produced at $T \simeq 2.5 \cdot 10^{15}$ GeV with $\Omega^{\rm mon}_{\rm in} = 10^{-10}$. For the flat Universe dominated by radiation find the temperature at which $\rho_{\rm rad} = \rho_{\rm mon}$.