Determine the explicite form of $\Lambda^\alpha{}_\beta$ for $\vec{v} = (v, 0, 0)$ 1

Solution

$$\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2}, \quad \Lambda^j{}_0 = \Lambda^0{}_j = -\gamma v_i, \quad \Lambda^0{}_0 = \gamma$$

$$v_i = v\,\delta_{i1}$$

$$\Lambda^i{}_j = \delta_{ij} + \cancel{v^2}\,\delta_{i1}\,\delta_{j1}\frac{\gamma-1}{\cancel{v^2}} = \delta_{ij} + \delta_{i1}\delta_{j1}(\gamma - 1)$$

$$\Lambda^j{}_0 = \Lambda^0{}_j = \gamma v\,\delta_{j1}$$

$$\Lambda = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \cancel{1} + \gamma - \cancel{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that the LT is the only non-singular coordinate transformation $x \to x'$ that leaves $d\tau^2$ invariant (non-singular means that $x(x')$ and $x'(x)$ are well behaved differentiable functions).

Solution:

A general coordinate transformation $x \to x'$ will change $d\tau^2$ into $d\tau'^2$:

$$d\tau'^2 = \eta_{\alpha\beta}\, dx'^\alpha dx'^\beta = \eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} dx^\gamma dx^\delta$$

Since we require $d\tau'^2 = d\tau^2$ so

$$(*) \quad \eta_{\gamma\delta} = \eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} \qquad \Big| \frac{\partial}{\partial x^\varepsilon}$$

differential equation for $x' = x'(x)$

$$0 = \eta_{\alpha\beta} \frac{\partial^2 x'^\alpha}{\partial x^\varepsilon \partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} + \eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial^2 x'^\beta}{\partial x^\varepsilon \partial x^\delta}$$

$$+ (\gamma \leftrightarrow \varepsilon) - (\varepsilon \leftrightarrow \delta)$$

$$\Downarrow$$

$$0 = \eta_{\alpha\beta} \Big[\frac{\partial^2 x'^\alpha}{\partial x^\varepsilon \partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} + \cancel{\frac{\partial^2 x'^\beta}{\partial x^\varepsilon \partial x^\delta} \frac{\partial x'^\alpha}{\partial x^\gamma}} +$$

$$+ \frac{\partial^2 x'^\alpha}{\partial x^\gamma \partial x^\varepsilon} \frac{\partial x'^\beta}{\partial x^\delta} + \cancel{\frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\delta} \frac{\partial x'^\alpha}{\partial x^\varepsilon}} +$$

$$- \cancel{\frac{\partial^2 x'^\alpha}{\partial x^\delta \partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\varepsilon}} - \cancel{\frac{\partial^2 x'^\beta}{\partial x^\delta \partial x^\varepsilon} \frac{\partial x'^\alpha}{\partial x^\gamma}} \Big]$$

$$0 = \eta_{\alpha\beta} \frac{\partial^2 x'^{\alpha}}{\partial x^{\varepsilon} \partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}}$$

Since $\eta_{\alpha\beta}$ and $\frac{\partial x'^{\beta}}{\partial x^{\delta}}$ are non-singular, therefore we get

$$\frac{\partial^2 x'^{\alpha}}{\partial x^{\varepsilon} \partial x^{\gamma}} = 0$$

$$\Downarrow$$

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + c^{\alpha}$$

but we have (*)

$$\eta_{\gamma\delta} = \eta_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}}$$

$$\Downarrow$$

$$\eta_{\gamma\delta} = \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \qquad \square$$

Show that the electromagnetic 4-force acting on a charged particle reads

$$f^{\alpha} = e\,\eta_{\beta\gamma}F^{\alpha\beta}\frac{dx^{\gamma}}{d\tau} \qquad (*)$$

Solution

- We show that f^{α} reduces to the known 3-force in the rest frame, e.g. for $\bar{v}=0$

$$f^{0} = e\,\eta_{\beta 0}F^{0\beta}\underbrace{\frac{dx^{0}}{d\tau}}_{=1} = eF^{0}{}_{0} = 0$$

$$f^{i} = e\,\eta_{\beta 0}F^{i\beta}\frac{dx^{0}}{d\tau} = e\,\eta_{\beta 0}F^{i\beta} = e\underbrace{F^{i0}}_{E^{i}} =$$

$$= -eE_{i}$$

$$\frac{dp^{\alpha}}{d\tau} = f^{\alpha} \quad\Rightarrow\quad \frac{dp^{i}}{dt} = eE_{i}$$

$$\bar{p} = (p^{1}, p^{2}, p^{3})$$

$$\bar{E} = (E^{1}, E^{2}, E^{3})$$

$$\Downarrow$$

$$\frac{d\bar{p}}{dt} = e\bar{E}$$

- Since f^{α} is a 4-vector, the formula $(*)$ must be correct

17 lutego 2011
18:17

Show that $\frac{dp^\alpha}{d\tau} = f^\alpha$ and $f^\alpha = e\eta_{\beta\gamma} F^{\alpha\beta} \frac{dx^\gamma}{d\tau}$

implies

$$\frac{d\bar{p}}{dt} = e(\bar{E} + \bar{v} \times \bar{B})$$

Solution

$$\frac{dp^\alpha}{d\tau} = \frac{dp^\alpha}{dt}\cancel{\frac{dt}{d\tau}} = e\eta_{\beta\gamma} F^{\alpha\beta} \frac{dx^\gamma}{dt}\cancel{\frac{dt}{d\tau}}$$

$\alpha = i$

$$\frac{dp^i}{dt} = e\eta_{\beta\gamma} F^{i\beta} \frac{dx^\gamma}{dt} = e\left(\eta_{0\gamma} F^{i0} \frac{dx^\gamma}{dt} + \eta_{j\gamma} F^{ij} \frac{dx^\gamma}{dt}\right) =$$

$$F^{i0} = -F^{0i} = E^i$$

$$F^{ij} = -\epsilon_{ijl} B^l$$

$$= e\Big(\underbrace{\frac{dx^0}{dt}}_{1} \underbrace{F^{i0}}_{E^i} + \underbrace{\eta_{jk}}_{(-)\delta_{jk}} \underbrace{F^{ij}}_{(-)\epsilon_{ijl} B^l} v^k\Big) = e\left(E^i + \epsilon_{ikl} B^l v^k\right) =$$

$$= e\left(E_i + (\bar{v} \times \bar{B})_i\right) = e(\bar{E} + \bar{v} \times \bar{B})_i$$

$$\frac{dp^0}{dt} = e\eta_{\beta\gamma} F^{0\beta} \frac{dx^\gamma}{dt} = e\eta_{ij} F^{0i} \frac{dx^j}{dt} =$$

$$= -e F^{0i} \frac{dx^i}{dt} =$$

$$= e E^i v^i = e\bar{E} \cdot \bar{v}$$

3 marca 2011
09:24

Show that $\epsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0 \Leftrightarrow \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

Solution:

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \quad \Big| \, \epsilon_\mu{}^{\nu'\rho'\sigma'}$$

identities:

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -24$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho}{}^{\sigma'} = -6\eta^{\sigma\sigma'}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}{}^{\rho'\sigma'} = -2\left(\eta^{\rho\rho'}\eta^{\sigma\sigma'} - \eta^{\rho\sigma'}\eta^{\sigma\rho'}\right)$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu{}^{\nu'\rho'\sigma'} = -\boxed{\det\left(\eta^{\alpha\alpha'}\right)} \quad \alpha = \nu, \rho, \sigma \quad \alpha' = \nu', \rho', \sigma'$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'} = -\det\left(\eta^{\alpha\alpha'}\right) \quad \alpha = \mu, \nu, \rho, \sigma \quad \alpha' = \mu', \nu', \rho', \sigma'$$

$$= \begin{vmatrix} \eta^{\nu\nu'} & \eta^{\nu\rho'} & \eta^{\nu\sigma'} \\ \eta^{\rho\nu'} & \eta^{\rho\rho'} & \eta^{\rho\sigma'} \\ \eta^{\sigma\nu'} & \eta^{\sigma\rho'} & \eta^{\sigma\sigma'} \end{vmatrix} = \eta^{\nu\nu'}\left(\eta^{\rho\rho'}\eta^{\sigma\sigma'} - \eta^{\sigma\rho'}\eta^{\rho\sigma'}\right) +$$

$$- \eta^{\rho\nu'}\left(\eta^{\nu\rho'}\eta^{\sigma\sigma'} - \eta^{\sigma\rho'}\eta^{\nu\sigma'}\right) +$$

$$+ \eta^{\sigma\nu'}\left(\eta^{\nu\rho'}\eta^{\rho\sigma'} - \eta^{\rho\rho'}\eta^{\nu\sigma'}\right)$$

$$\eta^{\nu\nu'}\left(\eta^{\rho\rho'}\eta^{\sigma\sigma'} - \eta^{\sigma\rho'}\eta^{\rho\sigma'}\right)\partial_\nu F_{\rho\sigma} +$$

$$- \eta^{\rho\nu'}\left(\eta^{\nu\rho'}\eta^{\sigma\sigma'} - \eta^{\sigma\rho'}\eta^{\nu\sigma'}\right)\partial_\nu F_{\rho\sigma} +$$

$$+ \eta^{\sigma\nu'}\left(\eta^{\nu\rho'}\eta^{\rho\sigma'} - \eta^{\rho\rho'}\eta^{\nu\sigma'}\right)\partial_\nu F_{\rho\sigma} = 0$$

$$\partial^{\nu'} F^{\rho'\sigma'} - \cancel{\partial^{\rho'} F^{\nu'\sigma'}} + 2\partial^{\sigma'} F^{\nu'\rho'} + 2\partial^{\rho'} F^{\sigma'\nu'} - \cancel{\partial^{\sigma'} F^{\rho'\nu'}} = 0$$

$$\partial^{\nu'} 2F^{\rho'\sigma'} - \cancel{\partial^{\rho'} F^{\nu'\sigma'}} + 2\partial^{\sigma'} F^{\nu'\rho'} + 2\partial^{\rho'} F^{\sigma'\nu'} - \cancel{\partial^{\sigma'} F^{\rho'\nu'}} = 0$$

$$\partial^{\nu'} F^{\rho'\sigma'} + \partial^{\rho'} F^{\sigma'\nu'} + \partial^{\sigma'} F^{\nu'\rho'} = 0 \quad \Rightarrow \checkmark$$

$$\partial^{\nu} F^{\rho\sigma} + \partial^{\rho} F^{\sigma\nu} + \partial^{\sigma} F^{\nu\rho} = 0 \quad \Big| \, \epsilon_{\gamma\nu\rho\sigma}$$

$$\epsilon_{\gamma\nu\rho\sigma} \partial^{\nu} F^{\rho\sigma} + \epsilon_{\gamma\nu\rho\sigma} \partial^{\rho} F^{\sigma\nu} + \epsilon_{\gamma\nu\rho\sigma} \partial^{\sigma} F^{\nu\rho} = 0$$

$$\underbrace{\epsilon_{\gamma\sigma\nu\rho}} \partial^{\nu} F^{\rho\sigma}$$

$$+ \epsilon_{\gamma\nu\rho\sigma}$$

$$\Leftarrow \checkmark \qquad \square$$

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16:57

Show that for $A_\mu \to A'_\mu = \omega A_\mu \omega^{-1} + \omega \partial_\mu \omega$

$F_{\mu\nu}$ transform as follows

$$F_{\mu\nu} \to \omega F_{\mu\nu} \omega^{-1}$$

Solution

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$D_\mu = \partial_\mu + A_\mu$$

$$[D_\mu, D_\nu]\varphi = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]\varphi =$$

$$= \Big(\underbrace{[\partial_\mu, \partial_\nu]}_{=0} + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + [A_\mu, A_\nu]\Big)\varphi =$$

$$[\partial_\mu, A_\nu]\varphi = (\partial_\mu A_\nu - A_\nu \partial_\mu)\varphi =$$

$$= (\partial_\mu A_\nu)\varphi + A_\nu(\partial_\mu \varphi) - A_\nu \partial_\mu \varphi = (\partial_\mu A_\nu)\varphi$$

$$\Downarrow$$

$$[\partial_\mu, A_\nu] = \partial_\mu A_\nu$$

$$\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)\varphi = F_{\mu\nu}\varphi$$

$$\Downarrow$$

$$F_{\mu\nu} = [D_\mu, D_\nu]$$

$$D_\mu \varphi \to (D_\mu \varphi)' = \omega (D_\mu \varphi)$$

$$D'_\mu \varphi' = D'_\mu \omega \varphi = \omega D_\mu \varphi$$

$$\Downarrow$$

$$D_\mu' = \omega D_\mu \omega^{-1}$$

$$\Downarrow$$

$$F_{\mu\nu} \to F_{\mu\nu}' = [D_\mu', D_\nu'] = \omega [D_\mu, D_\nu] \omega^{-1} =$$

$$= \omega F_{\mu\nu} \omega^{-1}$$

$\square$

3 marca 2011
17:32

Show that Hermiticity of $\hat{H} = -i\alpha_i \partial_i$ requires $\alpha_i = \alpha_i^+$.

Solution

$$\left(\int \varphi^+ \hat{H} \varphi\right)^* = \int \varphi^+ \hat{H} \varphi$$

$$\left(-i\int \varphi^+ \alpha_i \partial_i \varphi\right)^* = \left(-i(-)\int (\partial_i \varphi^+)\alpha_i \varphi\right)^* + \text{boundary term}$$

$$= -i \left(\int \partial_i \varphi_a^* (\alpha_i)_{ab} \varphi_b\right)^* = -i \int \varphi_b^* (\alpha_i)_{ab}^* \partial_i \varphi_a =$$

$$= \int \varphi_b^* \, -i(\alpha_i^+)_{ba} \partial_i \varphi_a = \int \varphi^+ (-i\alpha_i^+ \partial_i) \varphi =$$

Hermiticity

$$= \int \varphi^+ (-i\alpha_i \partial_i) \varphi$$

$$\Downarrow$$

$$\alpha_i = \alpha_i^+$$

Show that symmetric energy-momentum tensor for electrodynamics reads

$$T^{\mu\nu} = F^{\mu\sigma} F_\sigma{}^\nu - \frac{1}{4}\eta^{\mu\nu} F_{\sigma\gamma} F^{\sigma\gamma}$$

Solution

For the canonical energy-momentum tensor we have

$$T_{can}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \partial^\nu \phi^i - \eta^{\mu\nu}\mathcal{L}$$

$$\phi^i \to A_\alpha \quad , \quad \mathcal{L} \to -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = 2F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} =$$

$$= 2F^{\alpha\beta} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) =$$

$$= 2F^{\alpha\beta}\left(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu\right) = 2F^{\mu\nu} - 2F^{\nu\mu} =$$
$$= 4F^{\mu\nu}$$

So

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} = -F^{\mu\alpha} \quad \text{and}$$

$$T_{can}^{\mu\nu} = -F^{\mu\alpha}\partial^\nu A_\alpha + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

drawbacks: - gauge variance
- $T_{\mu\nu} \neq T_{\nu\mu}$

$$T^{\mu\nu} \to T_{can}^{\mu\nu} + \partial_\lambda \Omega^{\mu\lambda\nu}$$

if $\Omega^{\mu\lambda\nu} = -\Omega^{\lambda\mu\nu}$ then $\partial_\mu \partial_\lambda \Omega^{\mu\lambda\nu} = 0$

so that

$$\partial_\mu T^{\mu\nu} = 0$$

Choose $\Omega^{\mu\lambda\nu} = F^{\mu\lambda} A^\nu$, then

$$\partial_\lambda \Omega^{\mu\lambda\nu} = \partial_\lambda\left(F^{\mu\lambda}A^\nu\right) = -\left(\partial_\lambda F^{\lambda\mu}\right)A^\nu + F^{\mu\lambda}\partial_\lambda A^\nu$$

Maxwell equations for vacuum $\underbrace{\qquad}_{=0}$

So $\quad T^{\mu\nu} = -F^{\mu\alpha}\partial^\nu A_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^{\mu\alpha}\partial_\alpha A^\nu =$

$$= -F^{\mu\alpha}\underbrace{\left(\partial^\nu A_\alpha - \partial_\alpha A^\nu\right)}_{F^\nu{}_\alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

So $\quad T^{\mu\nu} = -F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$

$F^\mu{}_\gamma F^{\gamma\alpha}$

Derive the Maxwell equations from the Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$

Solution

For $\mathcal{L} = -1/4 \, F_{\alpha\beta} F^{\alpha\beta}$ we have

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\alpha)} = - F^{\mu\alpha}$$

E-L :
$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \qquad \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = (-) F^{\mu\nu}$$

$$\Downarrow$$

$$- \partial_\mu (-) F^{\mu\nu} = 0$$

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

$\square$

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12:48

Show that

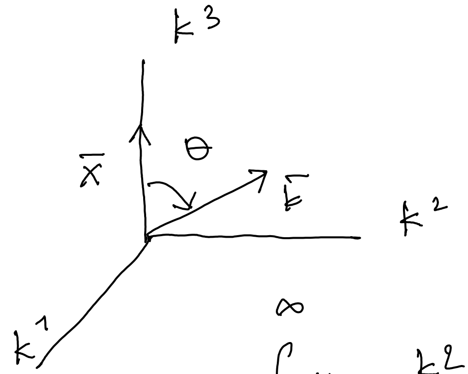
$$V(x) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}\bar{x}}}{\bar{k}^2 + m^2} = -\frac{e^{-mr}}{4\pi r}$$

where $r \equiv (x^i x^i)^{1/2}$

Solution

$k \equiv |\bar{k}|$

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}\bar{x}}}{\bar{k}^2+m^2} = \int \frac{k^2 dk\, d\varphi\, d\theta \cancel{\cos\theta}}{(2\pi)^3} \frac{e^{ikr\cos\theta}}{k^2+m^2} =$$



$u = \cos\theta \qquad \bar{k}\cdot\bar{x} = k^3 r\cos\theta$

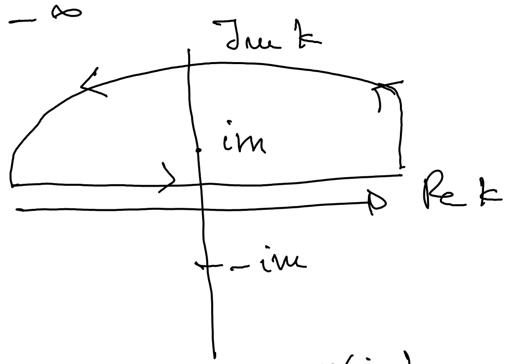
$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dk \frac{k^2}{k^2+m^2} \underbrace{\int_{-1}^{+1} du\, e^{ikru}}_{\frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right) = \frac{2}{kr}\sin kr} =$$

$$= \frac{2}{(2\pi)^2} \frac{1}{r} \underbrace{\int_0^\infty dk \frac{k \sin kr}{k^2+m^2}}$$

$$\hookrightarrow \frac{1}{2}\int_{-\infty}^{+\infty} dk \frac{k \sin kr}{k^2+m^2} = \frac{1}{2i}\int_{-\infty}^{+\infty} dk \frac{k e^{ikr}}{\underbrace{k^2+m^2}_{\shortparallel}} =$$

Im k

$$\hookrightarrow \frac{1}{2}\int_{-\infty} dk \, \frac{k \sin kr}{k^2+m^2} = \frac{1}{2i}\int_{-\infty} dk \, \frac{k e}{\underbrace{k^2+m^2}_{=(k-im)(k+im)}} =$$



$$= \cancel{-}\frac{\cancel{i}}{2}\cancel{2}\pi\cancel{i}\,\frac{\cancel{im}\, e^{ir(im)}}{\cancel{im}+\cancel{im}} = \frac{\pi}{2} e^{-rm}$$

$$V(x) = -\frac{\cancel{2}}{2\cancel{4}\pi^{\cancel{2}}}\,\frac{1}{r}\,\frac{\cancel{\pi}}{2} e^{-rm} = -\frac{e^{-rm}}{4\pi r}$$

Find the Feynman kernel for a free particle.

Solution

$$S[q] = \int_{t_i}^{t_f} \frac{m\dot{q}^2}{2}\,dt \qquad (*)$$

Expand $S[q]$ around certain trajectory $q_0 = q_0(t)$, that satisfies E-L equation for $*$

$$S[q] = S[q_0] + \underbrace{\int_{t_i}^{t_f} \frac{\partial \mathcal{L}}{\partial \dot{q}}\Big|_{q=q_0} \delta\dot{q}(t)\,dt}_{} + \frac{1}{2}\int_{t_i}^{t_f} \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2}\Big|_{q=q_0} \delta\dot{q}(t)^2\,dt$$

$$\underbrace{\frac{\partial \mathcal{L}}{\partial q}}_{\downarrow\, 0} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

$$\hookrightarrow \int \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d}{dt}\delta q(t)\,dt = -\int \underbrace{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{= \frac{\partial \mathcal{L}}{\partial q} \to 0} \delta q(t)\,dt = 0$$

$$S[q] = S[q_0] + \frac{1}{2}\int_{t_i}^{t_f} \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2}\delta\dot{q}(t)^2\,dt$$

The classical path q_0 is a straight line

$$q_0(t) = \frac{q_f(t-t_i) - q_i(t-t_f)}{t_f - t_i} \Rightarrow \dot{q}_0(t) = \frac{q_f - q_i}{t_f - t_i}$$

$$S[q_0] = \frac{1}{2}m\left(\frac{q_f-q_i}{t_f-t_i}\right)^2$$

The kernel $\quad K(q_f, t_f; q_i, t_i) = \int \mathcal{D}q(t)\, e^{iS[q]}$

$$K = e^{iS[q_0]} \cdot \int \mathcal{D}\delta q(t)\, \exp\left[\frac{1}{2}\int_{t_i}^{t_f} \underbrace{\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2}\Big|_{q=q_0}}_{=m} \delta\dot{q}(t)^2\,dt\right] = e^{iS[q_0]}\, K(0, t_f; 0, t_i)$$

Path integrals

In general we have

$$K(q_f, t_f; q_i, t_i) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{\frac{N+1}{2}} \int \prod_{n=1}^{N} dq_n \exp\left[+i\varepsilon\sum_{n=1}^{N}\left(\frac{m\dot{q}_n^2}{2} - V(q_n)\right)\right]$$

for $V(q) = 0$

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}ax^2 + jx} = \left(\frac{2\pi}{a}\right)^{1/2} e^{j^2/2a}$$

$$K = \lim_{N\to\infty}\left(\quad\right)^{\frac{N+1}{2}} I_N \qquad q_{N+1} = q_f \qquad q_0 = q_i$$

$$I_N = \int \prod_{n=1}^{N} dq_n \exp\left[i\varepsilon\sum_{n=1}^{N+1}\frac{m\dot{q}^2}{2}\right] = \int dq_1 \cdots dq_N \exp\left[i\frac{m\varepsilon}{2}\left(\left(\frac{q_f - q_N}{\varepsilon}\right)^2 + \cdots + \left(\frac{q_1 - q_i}{\varepsilon}\right)^2\right)\right] =$$

$$(q_f - q_N)^2 + (q_N - q_{N-1})^2 + \cdots + (q_n - q_{n-1})^2 + \cdots + (q_1 - q_i)^2 =$$

$$= q_f^2 - 2q_f q_N + \underbrace{q_N^2 + q_N^2}_{2q_N^2} - 2q_N q_{N-1} + \cdots + q_{n+1}^2 - 2q_{n+1}q_n + \underbrace{q_n^2 + q_n^2}_{2q_n^2} - 2q_n q_{n-1} + q_n^2 + \cdots q_1^2 - 2q_1 q_i + q_i^2 =$$

$$2(q_1, \ldots, q_N)\underbrace{\begin{pmatrix} 1 & 0 & \cdots & & \\ -1 & 1 & 0 & \cdots & \\ & -1 & 1 & 0 & \cdots \\ & & & \ddots & \\ & & -1 & 1 & 0 \\ & & & -1 & 1 \end{pmatrix}}_{a}\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \\ q_N \end{pmatrix} + 2(-q_i, 0, \ldots, -q_f)\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} + q_f^2 + q_i^2$$

$$q_i a_{ij} q_j = q_i \frac{1}{2}(a_{ij} + a_{ji}) q_j$$

$$\frac{\cancel{1}}{\cancel{2}} q A q = \frac{\cancel{i}m\cancel{\varepsilon}}{\cancel{2}\varepsilon\cancel{2}} q \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} q$$

$$A = \frac{m}{\varepsilon}\begin{pmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \\ 0 & -1 & 2 & -1 \\ \vdots & & -1 & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix} \qquad A^{-1} = \frac{\varepsilon}{m}\frac{1}{N+1}\begin{pmatrix} N & N-1 & N-2 & \cdots & 1 \\ N-1 & & & & 2 \\ N-2 & & & & \vdots \\ \vdots & & & & \\ 1 & & \cdots & N-1 & N \end{pmatrix}$$

$$\det A = \left(\frac{m}{\varepsilon}\right)^N (N+1) \qquad \cancel{i} Jq = \cancel{i}\frac{m(-)\cancel{2}}{\cancel{2}\varepsilon}(q_i,\dots,q_f)\,q$$

$$J = \frac{-m}{\varepsilon}\begin{pmatrix} q_i \\ 0 \\ 0 \\ \vdots \\ \vdots \\ q_f \end{pmatrix} \qquad \int dq_1 \dots dq_N\, e^{\frac{i}{2}qAq + iJq} = \left[\frac{(2\pi i)^N}{\det A}\right]^{1/2} e^{-\frac{i}{2}JA^{-1}J}$$

$$\left[\frac{(2\pi i)^N}{\det A}\right]^{1/2} = \left[\frac{(2\pi i)^N}{m^N}\varepsilon^N\frac{1}{N+1}\right]^{1/2} = \left(\frac{2\pi i\varepsilon}{m}\right)^{N/2}\frac{1}{(N+1)^{1/2}}$$

$$-\frac{i}{2}JA^{-1}J = -\frac{i}{2}\left(\frac{-m}{\varepsilon}\right)^{\cancel{2}} \times \frac{1}{N+1}\frac{\cancel{\varepsilon}}{\cancel{m}} \times (q_i, 0, \dots, q_f)\begin{pmatrix} N & N-1 & \dots & 1 \\ & & & \\ 1 & \dots & & N \end{pmatrix}\begin{pmatrix} q_i \\ 0 \\ \vdots \\ q_f \end{pmatrix} = \frac{-mi}{(N+1)\varepsilon 2}\left[q_i(Nq_i + q_f) + q_f(q_i + Nq_f)\right]$$

$$I_N = \frac{1}{(N+1)^{1/2}}\left(\frac{2\pi i\varepsilon}{m}\right)^{N/2}\exp\underbrace{\left\{i\frac{m\cancel{\varepsilon}}{2}\frac{1}{\varepsilon\cancel{\varepsilon}}(q_f^2 + q_i^2) - i\frac{m}{2(N+1)\varepsilon}\left[Nq_i^2 + 2q_iq_f + Nq_f^2\right]\right\}}_{\frac{im}{2\varepsilon}\frac{1}{N+1}\left[(\cancel{N}+1)q_f^2 + (\cancel{N}+1)q_i^2 - \cancel{Nq_i^2} - 2q_iq_f - \cancel{Nq_f^2}\right] = \frac{im}{2\varepsilon(N+1)}(q_f - q_i)^2}$$

$$I_N = \frac{1}{(N+1)^{1/2}}\left(\frac{2\pi i\varepsilon}{m}\right)^{N/2}\exp\left[\frac{im}{2\varepsilon(N+1)}(q_f - q_i)^2\right]$$

$$K(q_f, t_f; q_i, t_i) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{\frac{N+1}{2}}\boxed{\int\prod_{n=1}^{N}dq_n \exp\left[+i\varepsilon\sum_{n=1}^{N}\left(\frac{m\dot{q}_n^2}{2} - V(q_n)\right)\right]} =$$

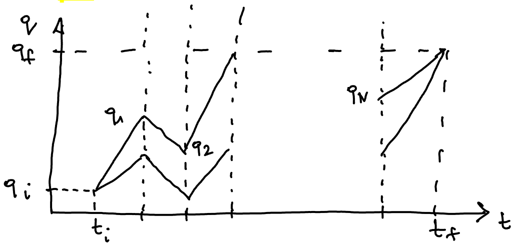
$$= \lim_{\substack{N\to\infty \\ \varepsilon\to 0}}\left[\frac{m}{2\pi i\varepsilon(N+1)}\right]^{1/2}\cancel{\left(\frac{2\pi i\varepsilon}{m}\right)^{N/2}}\cancel{\left(\frac{m}{2\pi i\varepsilon}\right)^{N/2}}\exp\left[\frac{im}{2\varepsilon(N+1)}(q_f - q_i)\right]$$

$$t_f - t_i = (N+2)\varepsilon = (N+1)\varepsilon + \varepsilon \qquad N\varepsilon \to t_f - t_i$$

$$N\to\infty,\ \varepsilon\to 0$$

$$t_i \quad t_1 \quad \dots \quad t_N \quad t_f$$

$$\boxed{K(q_f, t_f; q_i, t_i) = \left(\frac{m}{2\pi i(t_f - t_i)}\right)^{1/2}\exp\left[\frac{im(q_f - q_i)^2}{2(t_f - t_i)}\right]}$$



QM:

$$K = \langle q_f | e^{-i\hat{H}T} | q_i\rangle = \langle q_f | e^{-i\frac{\hat{p}^2}{2m}T}\int\frac{dp}{2\pi}|p\rangle\langle p|q_i\rangle = \int\frac{dp}{2\pi}e^{-i\frac{p^2}{2m}}\underbrace{\langle q_f|p\rangle\langle p|q_i\rangle}_{e^{-ipq_i}} =$$

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

$$= \int\frac{dp}{(2\pi)}e^{-i\frac{p^2}{2m}T + ip(q_f - q_i)} = \longrightarrow \frac{1}{2\pi}\left(\frac{2\pi}{iT}m\right)^{1/2}\exp\left[\frac{-(q_f - q_i)^2}{2iT}m\right]$$

$$\int dx\, e^{-\frac{a}{2}x^2 + jx} = \left(\frac{2\pi}{a}\right)^{1/2}e^{j^2/2a}, \qquad a = i\frac{T}{m}, \quad j = i(q_f - q_i)$$

$$K(q_f, t_f; q_i, t_i) = \left(\frac{m}{2\pi i T}\right)^{1/2} \exp\left[i\,\frac{(q_f - q_i)^2 m}{2T}\right]$$

Verify if the Lorentz and harmonic gauges are achievable.

Solution

Lorentz : $A_\mu{}^{,\mu} = 0$

$$A_\mu \to A'_\mu = A_\mu + \lambda_{,\mu}$$

$$A'_\mu{}^{,\mu} = 0 \Rightarrow A_\mu{}^{,\mu} + \lambda_{,\mu}{}^{,\mu} = 0$$

$$\boxed{\Box\lambda = -A_\mu{}^{,\mu}}$$

harmonic : $\bar{h}_{\beta\alpha}{}^{,\alpha} = 0 \Rightarrow h_{\beta\alpha}{}^{,\alpha} - \frac{1}{2} h^{\alpha}{}_{\alpha,\beta} = 0$

$$h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} + X_{\alpha,\beta} + X_{\beta,\alpha}$$

we require: $h'_{\beta\alpha}{}^{,\alpha} - \frac{1}{2} h'^{\alpha}{}_{\alpha,\beta} = 0$

$$(h_{\alpha\beta} + X_{\alpha,\beta} + X_{\beta,\alpha})^{,\alpha} - \frac{1}{2}\left(h^{\alpha}{}_{\alpha} + 2X_{\alpha}{}^{,\alpha}\right)_{,\beta} = 0$$

$$h_{\alpha\beta}{}^{,\alpha} + (\cancel{X_{\alpha,\beta}} + X_{\beta,\alpha})^{,\alpha} - \frac{1}{2} h^{\alpha}{}_{\alpha,\beta} - \cancel{X_{\alpha}{}^{,\alpha}}{}_{,\beta} = 0$$

$$\boxed{-\bar{h}_{\beta\alpha}{}^{,\alpha} = \Box X_\beta}$$

Show that the quadratic operator for the spin 2 field satisfies the identity:

$$K_{\mu\nu;\lambda\sigma}\, K^{\lambda\sigma}{}_{\rho\sigma} = \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right)$$

Solution

$$K_{\mu\nu\lambda\sigma} = \frac{1}{2}\left(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}\right)$$

$$K_{\mu\nu\lambda\sigma}\, K^{\lambda\sigma}{}_{\rho\sigma} = \frac{1}{4}\left(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}\right) \times$$
$$\times \left(\eta^{\lambda}{}_{\rho}\eta^{\sigma}{}_{\sigma} + \eta^{\lambda}{}_{\sigma}\eta^{\sigma}{}_{\rho} - \eta^{\lambda\sigma}\eta_{\rho\sigma}\right) =$$

$$= \frac{1}{4}\Big(2\eta_{\mu\rho}\eta_{\nu\sigma} + 2\eta_{\mu\sigma}\eta_{\nu\rho} - \underbrace{\eta_{\mu\lambda}\eta^{\lambda}{}_{\nu}}_{\eta_{\mu\nu}}\eta_{\rho\sigma} +$$

$$+ \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \overbrace{\eta_{\mu}{}^{\lambda}\eta_{\nu\lambda}}^{\eta_{\mu\nu}}\eta_{\rho\sigma} +$$

$$- \eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\nu}\eta_{\sigma\rho} + \eta_{\mu\nu}\eta_{\rho\sigma}\cdot 4\Big) =$$

$$= \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right)$$

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Relation between $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$

$g_{\mu\nu}$ defined as $g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$

the connection as

$$\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \Gamma^\lambda_{\mu\nu} \frac{\partial \xi^\alpha}{\partial x^\lambda}$$

The notation we used was over-compact since we suppressed X dependence of ξ^α, we should have written

$$g_{\mu\nu}(X) = \left(\frac{\partial \xi^\alpha_X(x)}{\partial x^\mu} \frac{\partial \xi^\beta_X(x)}{\partial x^\nu} \eta_{\alpha\beta} \right)_{x=X} \qquad (*)$$

$$\left(\frac{\partial^2 \xi^\alpha_X(x)}{\partial x^\mu \partial x^\nu} \right)_{x=X} = \Gamma^\lambda_{\mu\nu}(X) \left(\frac{\partial \xi^\alpha_X(x)}{\partial x^\lambda} \right)_{x=X} \qquad (**)$$

Differentiating with respect to X we get, also terms of the type

$$\left(\frac{\partial^2 \xi^\alpha_X(x)}{\partial X^\lambda \partial x^\mu} \right)_{x=X}$$

These terms have nothing to do with the metric or the connection. Therefore at this point we have to sharpen our definition of the "locally inertial frame by saying that

The locally inertial coordinates ξ^α_X that we construct at a given point X can be chosen so that the first derivatives of the metric tensor vanish at X.

In the coordinate system ξ^α_X the metric tensor at a point X' is given by $(*)$:

$$\overset{X}{g}_{\gamma\delta}(X') = \left(\frac{\partial \xi^\alpha_{X'}(x)}{\partial \xi^\gamma_X(x)} \frac{\partial \xi^\beta_{X'}(x)}{\partial \xi^\delta_X(x)} \eta_{\alpha\beta} \right)_{x=X'}$$

From we know that $\overset{X}{g}_{\gamma\delta}(X')$ is stationary at $X'=X$, so in a laboratory frame x^μ we can write

$$g_{\mu\nu}(X') = \left(\frac{\partial \xi^\alpha_{X'}(x)}{\partial x^\mu} \frac{\partial \xi^\beta_{X'}(x)}{\partial x^\nu} \eta_{\alpha\beta} \right)_{x=X'} = \left(\frac{\partial \xi^\alpha_{X'}(x)}{\partial \xi^\gamma_X} \frac{\partial \xi^\gamma_X(x)}{\partial x^\mu} \frac{\partial \xi^\beta_{X'}(x)}{\partial \xi^\delta_X} \frac{\partial \xi^\delta_X(x)}{\partial x^\nu} \eta_{\alpha\beta} \right) =$$

$$g_{\mu\nu}(X') = \left(\frac{\partial \xi^{\alpha}_{X'}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}_{X'}}{\partial x^{\nu}} \eta_{\alpha\beta}\right)_{x=X'} = \left[\frac{\partial \xi^{\alpha}_{X'}}{\partial \xi^{\gamma}_{X}(x)} \frac{\partial \xi^{\gamma}_{X}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}_{X'}}{\partial \xi^{\delta}_{X}(x)} \frac{\partial \xi^{\delta}_{X}}{\partial x^{\nu}} \eta_{\alpha\beta}\right]_{x=X'} =$$

$$= \overset{X}{g}_{\gamma\delta}(X') \left(\frac{\partial \xi^{\gamma}_{X}(x)}{\partial x^{\mu}} \frac{\partial \xi^{\delta}_{X}(x)}{\partial x^{\nu}}\right)_{x=X'} \qquad \overset{X}{g}_{\gamma\delta}(X')$$

Differentiating with respect to X' and putting $X'=X$ one gets

$$\left(\frac{\partial g_{\mu\nu}(X')}{\partial X'^{\lambda}}\right)_{X'=X} = \frac{\partial g_{\mu\nu}(X)}{\partial X^{\lambda}} = \overset{X}{g}_{\gamma\delta}(X) \left(\frac{\partial}{\partial x^{\lambda}}\left[\frac{\partial \xi^{\gamma}_{X}(x)}{\partial x^{\mu}} \frac{\partial \xi^{\delta}_{X}(x)}{\partial x^{\nu}}\right]\right)_{x=X} =$$

$$\frac{\partial \overset{X}{g}_{\gamma\delta}(X')}{\partial X'^{\lambda}} = 0 \qquad = \eta_{\gamma\delta}$$

$$= \eta_{\gamma\delta}\left[\frac{\partial^2 \xi^{\gamma}_{X}(x)}{\partial x^{\lambda}\partial x^{\mu}} \frac{\partial \xi^{\delta}_{X}(x)}{\partial x^{\nu}} + \frac{\partial \xi^{\gamma}_{X}(x)}{\partial x^{\mu}} \frac{\partial^2 \xi^{\delta}_{X}(x)}{\partial x^{\lambda}\partial x^{\nu}}\right]_{x=X}$$

The above agrees with our naive derivation shown during the lecture.

Derive the following identity: $\mathrm{Tr}\left[M^{-1}(x)\,\partial_\lambda M(x)\right] = \partial_\lambda \ln \mathrm{Det}\, M(x)$

Solution:

Consider variation of $\ln(\mathrm{Det}\, M(x))$ owing to a variation δx^λ:

$$\delta \ln \mathrm{Det}\, M \equiv \ln \mathrm{Det}(M+\delta M) - \ln \mathrm{Det}\, M = \ln\left[\frac{\mathrm{Det}(M+\delta M)}{\mathrm{Det}\, M}\right] = \ln\left[\mathrm{Det}\left(M^{-1}(M+\delta M)\right)\right]$$

$$\delta M = \frac{\partial M}{\partial x^\lambda}\,\delta x^\lambda$$

$$= \ln\left[\mathrm{Det}\left(\mathbb{1} + M^{-1}\delta M\right)\right] = 1 + \mathrm{Tr}\left(M^{-1}\partial_\lambda M\right)\delta x^\lambda + o(M^4)$$

$$\det(\mathbb{1}+A) = \det\begin{pmatrix} 1+A_{11} & A_{12} & \cdots \\ A_{21} & 1+A_{22} & \\ \vdots & & \end{pmatrix} = (1+A_{11})\underbrace{\begin{vmatrix} 1+A_{22} & A_{23} & \cdots \\ A_{32} & & \\ \vdots & & \end{vmatrix}}_{} +$$

$$+ A_{21}\begin{vmatrix} A_{12} & A_{13} & \cdots \\ A_{31} & A_{32} & 1+A_{33} & \cdots \\ \vdots & & \end{vmatrix} + \cdots = (1+A_{11})(1+A_{22})\cdots(1+A_{nn}) + O(A_{ij}^2) =$$

$$= 1 + \mathrm{Tr}\, A + o(A^2)$$

$$\det(\mathbb{1}+A) = 1 + \mathrm{Tr}\, A + o(A^2)$$

On the other hand $\delta \ln \mathrm{Det}\, M = \partial_\lambda(\ln \mathrm{Det}\, M)\,\delta x^\lambda$

$$\partial_\lambda(\ln \mathrm{Det}\, M) = \mathrm{Tr}\left(M^{-1}\partial_\lambda M\right) \quad \square$$

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Show that $\frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\eta}}{\partial x^{\lambda}} \frac{\partial x'^{\xi}}{\partial x^{\varkappa}} \varepsilon^{\mu\nu\lambda\varkappa} = \left| \frac{\partial x'}{\partial x} \right| \varepsilon^{\rho\sigma\eta\xi}$

Solution

$$A^{\rho}{}_{\mu} \equiv \frac{\partial x'^{\rho}}{\partial x^{\mu}} \quad , \quad \det A \equiv A^{0}{}_{\mu} A^{1}{}_{\nu} A^{2}{}_{\lambda} A^{3}{}_{\varkappa} \cdot \underbrace{\mathrm{sgn}(\mu,\nu,\lambda,\varkappa)}$$

sign of the perturbation of the set $(0,1,2,3)$

$$\shortparallel$$

$$\varepsilon^{\mu\nu\lambda\varkappa}$$

in general

$$A^{\rho}{}_{\mu} A^{\sigma}{}_{\nu} A^{\eta}{}_{\lambda} A^{\xi}{}_{\varkappa} \varepsilon^{\mu\nu\lambda\varkappa} = |A| \varepsilon^{\rho\sigma\eta\xi}$$

Show that $\varepsilon_{\rho\sigma\eta\xi} = g\,\varepsilon^{\rho\sigma\eta\xi}$, for $g = -\mathrm{Det}\, g_{\mu\nu}$

Solution

$$\varepsilon^{\mu\nu\lambda\varkappa} = \begin{cases} +1 & \text{even perm. } 0,1,2,3 \\ -1 & \text{odd} \text{ — " — " —} \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_{\rho\sigma\eta\xi} = g_{\rho\mu}\, g_{\sigma\nu}\, g_{\eta\lambda}\, g_{\xi\varkappa}\, \varepsilon^{\mu\nu\lambda\varkappa}$$

in locally inertial frame $\varepsilon_{\rho\sigma\eta\xi} = \eta_{\rho\mu}\, \eta_{\sigma\nu}\, \eta_{\eta\lambda}\, \eta_{\xi\varkappa}\, \varepsilon^{\mu\nu\lambda\varkappa}$

$$\varepsilon_{0123} = \underbrace{\eta_{00}\, \eta_{11}\, \eta_{22}\, \eta_{33}}_{-1}\, \varepsilon^{0123} = -\det \eta_{\mu\nu} \cdot \varepsilon^{0123}$$

one can check that indeed $\varepsilon_{\rho\sigma\eta\xi} = -\det \eta_{\mu\nu}\, \varepsilon^{\rho\sigma\eta\xi}$

$$J_\mu = -ie\left[\phi^* D_\mu \phi - (D_\mu \phi)^* \phi\right] \qquad \text{where} \qquad D_\mu \phi \equiv \partial_\mu \phi - ie A_\mu \phi$$

Show that $\partial^\mu J_\mu = 0$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - V(\phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \qquad\qquad \frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial V}{\partial \phi} + (D_\mu \phi)^*(-ie) A^\mu$$

$$\Downarrow \qquad\qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = (D^\mu \phi)^*$$

$$-\frac{\partial V}{\partial \phi} - ie (D^\mu \phi)^* A_\mu = \partial_\mu (D^\mu \phi)^*$$

$$-\frac{\partial V}{\partial \phi} = (\partial_\mu + ie A_\mu)(D^\mu \phi)^* = (\eta^{\mu\nu} D_\mu D_\nu \phi)^*$$

$$\boxed{\left(\eta^{\mu\nu} D_\mu D_\nu \phi + \frac{\partial V}{\partial \phi^*}\right) = 0}$$

$$\partial_\mu J^\mu = -ie\, \partial_\mu \left[\phi^* D^\mu \phi - (D^\mu \phi)^* \phi\right] = -ie\left[(\partial_\mu \phi)^* D^\mu \phi + \phi^* \partial_\mu D^\mu \phi - (\partial_\mu D^\mu \phi)^* \phi + - (D^\mu \phi)^* \partial_\mu \phi\right] =$$

$$= -ie\left[(\partial_\mu \phi)^* (\cancel{\partial^\mu \phi} + ie A^\mu \phi) + \phi^* \partial_\mu D^\mu \phi - (\partial_\mu D^\mu \phi)^* \phi - (\cancel{\partial^\mu \phi^*} - ie A^\mu \phi^*) \partial_\mu \phi\right] =$$

$$= -ie\left[-(+\underbrace{\partial_\mu}_{D_\mu - ieA_\mu} D^\mu \phi + ie A_\mu \underbrace{\partial^\mu \phi}_{D^\mu - ieA^\mu})^* \phi + \phi^* (\underbrace{\partial_\mu}_{D_\mu - ieA_\mu} D^\mu \phi + ie A_\mu \underbrace{\partial^\mu \phi}_{D^\mu - ieA^\mu})\right] =$$

$$= -ie\left[-(D_\mu D^\mu \phi)^* \phi - (-ie \cancel{A_\mu D^\mu \phi})^* \phi + (ie \cancel{A_\mu D^\mu \phi})^* \phi - e^2 \cancel{A_\mu A^\mu} \phi^* \phi + \right.$$
$$\left. + \phi^* D_\mu D^\mu \phi - ie \phi^* \cancel{A_\mu D^\mu} \phi + ie \phi^* \cancel{A_\mu D^\mu} \phi + e^2 \phi^* \cancel{A_\mu A^\mu} \phi\right] =$$

$$= -ie\left[\phi^* \underbrace{D_\mu D^\mu \phi}_{-\frac{\partial V}{\partial \phi^*}} - \underbrace{(D_\mu D^\mu \phi)^*}_{-\frac{\partial V}{\partial \phi}} \phi\right] = -ie\left[-\phi^* \frac{\partial V}{\partial Z} \phi + \frac{\partial V}{\partial Z} \phi^* \phi\right] = 0 \qquad \square$$

$$V = V(\phi^* \phi)$$

$$\frac{\partial V}{\partial \phi^*} = \frac{\partial V}{\partial Z} \phi \qquad\qquad \frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial Z} \phi^*$$

Show that the covariant derivative along a curve $\frac{DA^\mu}{D\tau} \equiv \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu$ is a vector

solution

$$\frac{DA'^\mu}{D\tau} = \frac{dA'^\mu}{d\tau} + \Gamma'^\mu_{\nu\lambda}\frac{dx'^\lambda}{d\tau}A'^\nu$$

$$A'^\mu(\tau) = \frac{\partial x'^\mu}{\partial x^\nu}\Big|_{x^\nu = x^\nu(\tau)} \cdot A^\nu(\tau)$$

$$\frac{dA'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dA^\nu}{d\tau} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}A^\nu$$

$$\frac{dx'^\lambda}{d\tau} = \frac{\partial x'^\lambda}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}$$

$$\Gamma'^\mu_{\nu\lambda} = \frac{\partial x^\mu}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}$$

$$\frac{DA'^\mu}{D\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dA^\nu}{d\tau} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}A^\nu + \left[\frac{\partial x^\mu}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\rho\partial x^\sigma}\right] \cdot$$

$$\cdot \frac{\partial x'^\lambda}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} \cdot \frac{\partial x'^\nu}{\partial x^\beta}A^\beta =$$

$$= \frac{\partial x'^\mu}{\partial x^\nu} \cdot \frac{dA^\nu}{d\tau} + \frac{\partial x^\mu}{\partial x^\rho}\delta^\tau_\beta \delta^\sigma_\alpha \Gamma^\rho_{\tau\sigma}\frac{dx^\alpha}{d\tau}A^\beta +$$

$$+ \cancel{\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}A^\nu} - \delta^\rho_\alpha \delta^\sigma_\beta \cancel{\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\frac{dx^\alpha}{d\tau}A^\beta} =$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\left(\frac{dA^\nu}{d\tau} + \Gamma^\nu_{\beta\alpha}\frac{dx^\alpha}{d\tau}A^\beta\right) = \frac{\partial x'^\mu}{\partial x^\nu}\frac{DA^\nu}{D\tau} \quad \square$$

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Show that $\frac{\partial}{\partial x^\rho}\left(g^{\mu\nu} D^\alpha{}_\mu D^\beta{}_\nu\right) = 0$ for

$D^\alpha{}_\mu$ satisfying $\underbrace{\frac{\partial D^\alpha{}_\mu}{\partial x^\nu}} = \Gamma^\lambda_{\mu\nu} D^\alpha{}_\lambda$

Solution

$$\frac{\partial}{\partial x^\rho}\left(g^{\mu\nu} D^\alpha{}_\mu D^\beta{}_\nu\right) = \left(\partial_\rho g^{\mu\nu}\right) D^\alpha{}_\mu D^\beta{}_\nu + g^{\mu\nu}\left(\partial_\rho D^\alpha{}_\mu\right) D^\beta{}_\nu + g^{\mu\nu} D^\alpha{}_\mu \left(\partial_\rho D^\beta{}_\nu\right) =$$

$$g^{\mu\nu}{}_{;\rho} = 0 \Rightarrow \underbrace{\partial_\rho g^{\mu\nu}} + \Gamma^\mu_{\rho\sigma} g^{\sigma\nu} + \Gamma^\nu_{\rho\lambda} g^{\mu\lambda} = 0$$

$$= \left(-\Gamma^\mu_{\rho\sigma} g^{\sigma\nu} - \Gamma^\nu_{\rho\lambda} g^{\mu\lambda}\right) D^\alpha{}_\mu D^\beta{}_\nu + g^{\overset{\sigma}{\cancel{\mu}}\nu} \Gamma^{\overset{\mu}{\cancel{\lambda}}}_{\underset{\sigma}{\cancel{\mu}}\rho} D^\alpha{}_{\underset{\mu}{\cancel{\lambda}}} D^\beta{}_\nu + g^{\mu\overset{\lambda}{\cancel{\nu}}} D^\alpha{}_\mu \Gamma^{\overset{\nu}{\cancel{\lambda}}}_{\underset{\lambda}{\cancel{\nu}}\rho} D^\beta{}_{\underset{\nu}{\cancel{\lambda}}} = 0$$

Alternative solution:

$$g^{\mu\nu}{}_{;\rho} = 0 \quad \& \quad \underbrace{\partial_\nu D^\alpha{}_\mu - \Gamma^\lambda_{\mu\nu} D^\alpha{}_\lambda = 0}_{\Downarrow} \quad \Big| \Rightarrow \left(g^{\mu\nu} D^\alpha_\mu D^\beta_\nu\right)_{;\rho} = 0$$

for a given α $D^\alpha{}_{\mu;\nu} = 0$

but $g^{\mu\nu} D^\alpha{}_\mu D^\beta{}_\nu$ is a scalar (α, β just number vectors), so

$$\partial_\rho\left(g^{\mu\nu} D^\alpha{}_\mu D^\beta{}_\nu\right) = 0 \quad \square$$

Show that for two freely falling particles (nearby) the following equation holds:
$$\frac{D^2}{D\tau^2}(\delta x^\lambda) = R^\lambda{}_{\mu\nu\varsigma}\,\delta x^\mu \frac{dx^\nu}{d\tau}\frac{dx^\varsigma}{d\tau}$$
where $\delta x^\lambda(\tau)$ is the (small) separation between their trajectories.

<u>Solution</u>

Equations of motion for $x^\mu(\tau)$ and $x^\mu(\tau)+\delta x^\mu(\tau)$:
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$
$$\frac{d^2}{d\tau^2}\left(x^\mu+\delta x^\mu\right) + \Gamma^\mu_{\nu\lambda}(x+\delta x)\frac{d}{d\tau}(x^\nu+\delta x^\nu)\frac{d}{d\tau}(x^\lambda+\delta x^\lambda) = 0$$
subtracting the first equation from the second one and keeping leading terms in δx one gets
$$\frac{d^2}{d\tau^2}\delta x^\mu + \frac{\partial}{\partial x^\varsigma}\Gamma^\mu_{\nu\lambda}(x)\,\delta x^\varsigma\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau} = 0$$
The derivative along the curve was defined as
$$\frac{DA^\mu}{d\tau} \equiv \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu$$
For the second derivative we would get
$$\frac{D^2A^\mu}{d\tau^2} = \frac{D}{d\tau}\left(\frac{DA^\mu}{d\tau}\right) = \frac{D}{d\tau}\left[\frac{dA^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu\right] =$$
$$= \frac{d^2A^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{dA^\nu}{d\tau} + \frac{d}{d\tau}\left(\Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu\right) + \Gamma^\mu_{\varsigma\alpha}\frac{dx^\alpha}{d\tau}\left(\Gamma^\varsigma_{\sigma\lambda}\frac{dx^\lambda}{d\tau}A^\sigma\right) =$$

e. of m.

$$= \frac{d^2A^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{dA^\nu}{d\tau} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\varsigma}\frac{dx^\varsigma}{d\tau}\frac{dx^\lambda}{d\tau}A^\nu + \Gamma^\mu_{\nu\lambda}\frac{d^2x^\lambda}{d\tau^2}A^\nu + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{dA^\nu}{d\tau} +$$
$$+ \Gamma^\mu_{\alpha\varsigma}\frac{dx^\varsigma}{d\tau}\left(\Gamma^\alpha_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu\right) = \frac{d^2A^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{dA^\nu}{d\tau} + \frac{dx^\varsigma}{d\tau}\frac{dx^\lambda}{d\tau}A^\nu\left[\frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\varsigma} + \Gamma^\mu_{\alpha\varsigma}\Gamma^\alpha_{\nu\lambda}\right] +$$
$$+ \Gamma^\mu_{\nu\alpha}A^\nu\left(-\Gamma^\alpha_{\varsigma\lambda}\frac{dx^\varsigma}{d\tau}\frac{dx^\lambda}{d\tau}\right) =$$
$$= \frac{d^2A^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{dA^\nu}{d\tau} + \frac{dx^\varsigma}{d\tau}\frac{dx^\lambda}{d\tau}A^\nu\left[\frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\varsigma} + \Gamma^\mu_{\alpha\varsigma}\Gamma^\alpha_{\nu\lambda} - \Gamma^\mu_{\nu\alpha}\Gamma^\alpha_{\varsigma\lambda}\right] = \frac{D^2A^\mu}{D\tau^2}$$

$$\underbrace{\frac{d^2}{d\tau^2}\delta x^\mu + \frac{\partial}{\partial x^\nu}\Gamma^\mu_{\varsigma\lambda}(x)\,\delta x^\nu\frac{dx^\varsigma}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau}}_{=\;\frac{D^2\delta x^\mu}{D\tau^2} - \frac{dx^\varsigma}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu\left[\frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\varsigma} + \Gamma^\mu_{\alpha\varsigma}\Gamma^\alpha_{\nu\lambda} - \Gamma^\mu_{\alpha\nu}\Gamma^\alpha_{\varsigma\lambda}\right]} = 0$$

So we get

$$\frac{D^2\delta x^\mu}{D\tau^2} + \frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu \underbrace{\left[\frac{\partial\Gamma^\mu_{\rho\lambda}}{\partial x^\nu} - \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\lambda} - \Gamma^\mu_{\alpha\nu}\Gamma^\alpha_{\rho\lambda}\right]}_{= R^\mu{}_{\lambda\rho\nu}} = 0$$

$$\frac{D^2\delta x^\mu}{D\tau^2} = - R^\mu{}_{\lambda\rho\nu}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu = 0 \qquad (*)$$

$\square$

Note that in the case of non-trivial gravity ($R^\mu{}_{\lambda\rho\nu} \neq 0$), even though a freely falling particle appears to be at rest in the system falling with the particle, a pair of nearby freely falling particle will exhibit a relative motion that can detect the presence of gravitational interaction. Note that rhs of $(*)$ vanishes in the limit $\delta x^\nu \to 0$, as it should.

Show that $\dfrac{\partial g_{\lambda\sigma}}{\partial x^{\kappa}} = \Gamma^{\zeta}_{\kappa\lambda} g_{\zeta\sigma} + \Gamma^{\zeta}_{\kappa\sigma} g_{\zeta\lambda}$

solution

We use $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\dfrac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \dfrac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \dfrac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right) \Big| \; g_{\lambda\sigma}$

$$g_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right)$$

$$+ \quad g_{\nu\sigma}\Gamma^{\sigma}_{\lambda\nu} = \frac{1}{2}\left(\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}}\right)$$

$$\Gamma^{\sigma}_{\mu\nu} g_{\lambda\sigma} + \Gamma^{\sigma}_{\lambda\nu} g_{\nu\sigma} = \frac{\partial g_{\mu\nu}}{\partial x^{\nu}}$$

Find G_{00} in the weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in the leading order in $h_{\mu\nu}$

Solution

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \qquad R = g^{\mu\nu} R_{\mu\nu}$$

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}\left[\frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 g_{\mu\beta}}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 g_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\beta \partial x^\alpha}\right] + $$
$$+ g_{\eta\sigma}\left(\Gamma^\eta_{\beta\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\eta_{\nu\alpha}\Gamma^\sigma_{\mu\beta}\right)$$

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right) = \frac{1}{2}\eta^{\nu\sigma}\left(\frac{\partial h_{\mu\nu}}{\partial x^\lambda} + \frac{\partial h_{\lambda\nu}}{\partial x^\mu} - \frac{\partial h_{\mu\lambda}}{\partial x^\nu}\right) + O(h^2)$$

$$\Gamma^\sigma_{\lambda\mu} \sim h^1 \Rightarrow R_{\alpha\mu\beta\nu} \simeq \frac{1}{2}\left[\frac{\partial^2 h_{\alpha\beta}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_{\mu\beta}}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 h_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\beta \partial x^\alpha}\right] + O(h^2)$$

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = \frac{1}{2}\eta^{\alpha\beta}\left(\frac{\partial^2 h_{\alpha\beta}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_{\mu\beta}}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 h_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\beta \partial x^\alpha}\right) + O(h^2)$$

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\left(\frac{\partial^2 h_{\alpha\beta}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_{\mu\beta}}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 h_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\beta \partial x^\alpha}\right) + O(h^2) =$$

$h \equiv \eta^{\alpha\beta} h_{\alpha\beta}$

$$= \frac{1}{2}\left(\Box h - \frac{\partial^2 h^{\nu\alpha}}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 h^{\beta\mu}}{\partial x^\beta \partial x^\mu} + \Box h\right) + O(h^2) =$$
$$= \Box h - \frac{\partial^2 h^{\alpha\beta}}{\partial x^\alpha \partial x^\beta} + O(h^2)$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_\mu{}^\alpha}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 h^\beta{}_\nu}{\partial x^\beta \partial x^\mu} + \Box h_{\mu\nu}\right) +$$
$$- \frac{1}{2}\eta_{\mu\nu}\left(\Box h - \frac{\partial^2 h^{\alpha\beta}}{\partial x^\alpha \partial x^\beta}\right) + O(h^2)$$

For stationary fields one gets for G_{00}

$g_{00} \simeq (1 + 2\phi) + \cdots$

$h_{00} = 2\phi$

$$G_{00} = \frac{1}{2}\left(0 - 0 - 0 + \Box h_{00}\right) - \frac{1}{2}\left(\Box h - \frac{\partial^2 h^{\alpha\beta}}{\partial x^\alpha \partial x^\beta}\right) + O(h^2) =$$
$$= -\frac{1}{2}\nabla^2 h_{00} - \frac{1}{2}(-)\nabla^2\left(h^0{}_0 + h^i{}_i\right) + \frac{1}{2}\frac{\partial^2 h^{ij}}{\partial x^i \partial x^j}$$

$$\Big\} \; R_{00} = -\frac{1}{2}\nabla^2 h_{00} + O(h^2)$$

$$\Box h^\mu{}_\mu = \frac{\partial^2}{\partial x^{0\,2}} h^\mu{}_\mu - \frac{\partial^2 h^\mu{}_\mu}{\partial x^i \partial x^i} \rightarrow -\nabla^2 h^\mu{}_\mu = -\nabla^2\left(h_{00} - h_{ii}\right)$$

?

$$\Box h_{00} = \left(\frac{\partial^2}{\partial x^0 \partial x^0} - \frac{\partial^2}{\partial x^i \partial x^i}\right) h_{00} \rightarrow -\nabla^2 h_{00}$$

$$R_{ij} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^i \partial x^j} - \frac{\partial^2 h_i{}^\alpha}{\partial x^j \partial x^\alpha} - \frac{\partial^2 h^\alpha{}_j}{\partial x^\alpha \partial x^i} + \Box h_{ij}\right) + O(h^2)$$

$$\frac{1}{2} g_{ij} R = \frac{1}{2}(-)\delta_{ij}\left(\Box h - \frac{\partial^2 h^{\alpha\beta}}{\partial x^\alpha \partial x^\beta}\right) + O(h^2) \qquad h_{ij} =$$

$$R_{ij} \overset{2}{\simeq} \frac{1}{2} g_{ij} R$$

Show that $G_{00} \simeq c_1 \bar{\nabla}^2 g_{00}$ in the weak field approximation if $G_{ij} \simeq 0$

Solution

$$G_{\mu\nu} = C_1 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$$

$$G_{\mu\nu} = -8\pi G\, T_{\mu\nu} \qquad T_{00} \gg T_{ij} \text{ for non-relativistic matter} \Rightarrow G_{00} \gg G_{ij}$$

$\Downarrow$

$$R = R^{\mu}{}_{\mu} = R_{00} - R_{ii} \simeq R_{00} + \frac{3}{2} R \qquad \longleftarrow \qquad R_{ij} \simeq \frac{1}{2} g_{ij} R$$

$$R \simeq -2 R_{00} \qquad\qquad R_{ii} \simeq \frac{1}{2}(-3) R$$

$\Downarrow$

$$G_{00} \simeq c_1 \left(R_{00} + \frac{1}{2} 2 R_{00}\right) \simeq 2 c_1 R_{00}$$

$$R_{00} \simeq -\frac{1}{2} \bar{\nabla}^2 h_{00} + O(h^2) \qquad\qquad h_{00} = 2\phi$$

$\Downarrow$

$$G_{00} \simeq 2 c_1 (-) \frac{1}{2} \bar{\nabla}^2 h_{00} \simeq -2 c_1 \bar{\nabla}^2 \phi$$

$\Downarrow$

$$-2 c_1 \bar{\nabla}^2 \phi = -8\pi G T_{00} = -8\pi G \rho$$

$$c_1 \bar{\nabla}^2 \phi = 4\pi G \rho$$

$\Downarrow$

$$c_1 = 1$$

Alternative derivation of $c_1 = 1$. (no need to assume $G_{ij} \simeq 0$)

$$g^{\mu\nu} \cdot \Big| \; c_1 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = -8\pi G\, T_{\mu\nu}$$

$$c_1 \left(R - \frac{1}{2} 4 R\right) = -8\pi G\, T^{\mu}{}_{\mu}$$

$$c_1 R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, 8\pi G T^{\alpha}{}_{\alpha} = -8\pi G T_{\mu\nu} \quad \Longleftarrow \quad c_1 R = 8\pi G T^{\mu}{}_{\mu}$$

$$c_1 R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha}\right)$$

$$c_1 R_{00} \simeq -c_1 \frac{1}{2} \bar{\nabla}^2 h_{00} \simeq -8\pi G \left(T_{00} - \frac{1}{2} T_{00}\right) = -8\pi G \frac{1}{2} \rho$$

$\nwarrow$ $T_{ij} \ll T_{00}$ for non-relativistic stuff

$$-c_1 \bar{\nabla}^2 h_{00} = -8\pi G \rho$$

$$h_{00} = 2\phi$$

$$c_1 \bar{\nabla}^2 \phi = 4\pi G \rho$$

Show that for $N \leq 3$ dimensional space-time there is no curvature in vacuum.

Solution

First we determine the number of independent components of $R_{\lambda\mu\nu\kappa}$

- treat $R_{\lambda\mu\nu\kappa}$ as a matrix with $(\lambda\mu)$ and $(\nu\kappa)$ indices first
- since $R_{\lambda\mu\nu\kappa} = -R_{\mu\nu\nu\kappa} = -R_{\lambda\mu\kappa\mu}$ $(\lambda\mu)$ and $(\nu\kappa)$ have number of independent components equal to the number of independent components of antisymmetric matrix of dim N : $\frac{1}{2}(N^2 - N) = \frac{N}{2}(N-1)$
- since $R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$ therefore R is symmetric in $(\lambda\mu)$ $(\nu\kappa)$

  so one gets $\frac{N}{2}(N-1) + \left\{\left[\frac{N}{2}(N-1)\right]^2 - \frac{N}{2}(N-1)\right\}\frac{1}{2} = \frac{1}{2}\left\{\frac{N}{2}(N-1)\left[\frac{N}{2}(N-1)+1\right]\right\}$
- $\boxed{\begin{array}{l} R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \\ R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = \\ = R_{\mu\lambda\kappa\nu} \end{array}}$ $R_{\lambda\mu\nu\kappa} + R_{\mu\nu\kappa\lambda} + R_{\nu\kappa\lambda\mu}$ implies that tensor is completely antisymmetric

  $\Downarrow$

  number of further contraints:

  $$\frac{N(N-1)(N-2)(N-3)}{4!}$$
- finally $c_N = \frac{1}{12} N^2 (N^2 - 1)$

N	2	3	4
number of equations $R_{\mu\nu} = 0$	3	6	10
number of $R_{\lambda\mu\nu\kappa}$ components	1	6	20

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta} = 0 \qquad \square$$