Special relativity

Kinematics

$$m_N \frac{d^2 \bar{x}_N}{dt^2} = G \sum_M \frac{m_N m_M (\bar{x}_M - \bar{x}_N)}{|x_M - x_N|^3}$$

invariant under transformations from the Galileo group (GG)

$$\bar{x} \to \bar{x}' = R\bar{x} + \bar{v}t + \bar{d}$$
$$t \to t' = t + \tau$$

The Principle of Galilean Relativity : the laws of motion must be invariant under transformations from the Galileo group

The Maxwell's equations :

$$\bar{\nabla} \cdot \bar{E} = \varepsilon$$
$$\bar{\nabla} \times \bar{B} = \frac{\partial \bar{E}}{\partial t} + \bar{J}$$
$$\bar{\nabla} \cdot \bar{B} = 0$$
$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

– are NOT INVARIANT under transformations from the GG

→ but the M's equations are correct

⇓

Einstein's special relativity

The Principle of Special Relativity : the laws of motion must be invariant under Lorentz transformations

$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$$

where

$$\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta} \qquad \eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$$

$\alpha, \beta, \ldots = 0, 1, 2, 3$, $i, j, k, \ldots = 1, 2, 3$

System of units : $\hbar = c = 1$

$$[c] = \frac{L}{T} \qquad [\hbar] = \underset{\substack{\uparrow \\ \text{energy } (E = \hbar\omega)}}{E} \cdot T = M\left(\frac{L}{T}\right)^2 T = M\frac{L^2}{T}$$

$$\Downarrow \qquad\qquad [L] = [M^{-1}]$$

$$[x^0] = [x^i] = L$$

$$x'^\alpha = \underbrace{\Lambda^\alpha{}_\beta x^\beta}_{\text{summation under } \beta} + e^\alpha = \Lambda^\alpha{}_0 x^0 + \Lambda^\alpha{}_1 x^1 + \Lambda^\alpha{}_2 x^2 + \Lambda^\alpha{}_3 x^3 + e^\alpha$$

$d\tau^2 = dt^2 - d\bar{x}^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$ – the "proper time"

in another inertial frame $\qquad dx'^\alpha = \Lambda^\alpha{}_\beta dx^\beta$

$$d\tau'^2 = \eta_{\alpha\beta}\, dx'^\alpha dx'^\beta =$$

$$= \eta_{\alpha\beta} \Lambda^\alpha{}_\gamma dx^\gamma \Lambda^\beta{}_\delta dx^\delta = \underbrace{\eta_{\alpha\beta} \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta}_{\eta_{\gamma\delta}} dx^\gamma dx^\delta = d\tau^2$$

$\Downarrow$

$d\tau$ is invariant

Michelson – Morley exp.

$d\tau = 0$ for light $\frac{d\bar{x}}{dt} = 1$

$\Downarrow$

the same in all inertial frames

$x^\alpha \to x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$ — the inhomogenous Lorentz group (the Poincaré group)

$a^\alpha = 0$ — the homogenous Lorentz group

$\Lambda^0{}_0 \geq 1 \wedge \mathrm{Det}\,\Lambda = +1$ — proper $\left\{ \begin{array}{l} \text{homogenous} \\ \text{inhomogenous} \end{array} \right.$

$$\left.\begin{array}{ll} \Lambda^0{}_0 \leq -1 \wedge \mathrm{Det}\,\Lambda = +1 & (\text{time inversion}) \\ \Lambda^0{}_0 \geq 1 \wedge \mathrm{Det}\,\Lambda = -1 & (\text{space inversion}) \end{array}\right\} \left.\text{improper}\ \right] \text{violated by Nature}$$

$$\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta} \quad \Big|\Rightarrow \quad (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1$$

$\gamma = \delta = 0$

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1,2,3} (\Lambda^i{}_0)^2 \geq 1$$

$$\Lambda^T \eta \Lambda = \eta \quad \Rightarrow \quad (\mathrm{Det}\,\Lambda)^2 = 1$$

Rotations : $\Lambda^i{}_j = R_{ij} \quad \Lambda^i{}_0 = \Lambda^0{}_i = 0 \quad \Lambda^0{}_0 = 1$

with $\mathrm{Det}\,R = 1 \quad R^T R = \mathbb{1}$

Boosts :

$O : x^\mu = (\overset{t}{\overset{\shortparallel}{x^0}}, 0, 0, 0)$ at rest

$O' : x'^\mu = (x'^0, x'^1, x'^2, x'^3)$ moving

$$dx'^\alpha = \Lambda^\alpha{}_\beta dx^\beta \longrightarrow \begin{array}{l} dt' = \Lambda^0{}_0\, dt \\ dx'^i = \Lambda^i{}_0\, dt \end{array}$$

$\Downarrow$

$$\frac{\Lambda^i{}_0}{\Lambda^0{}_0} = v_i \;\left(= \frac{dx'^i}{dt'}\right) \qquad \Lambda^i{}_0 = v_i \Lambda^0{}_0$$

$$\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta} \quad \Big|\Rightarrow \quad 1 = (\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2$$

$\gamma = \delta = 0$

$$\Lambda^0{}_0 = \gamma \quad \Lambda^i{}_0 = \gamma v_i$$

$\gamma = (1 - v^2)^{-1/2}$

$\Lambda^i{}_j$ are not uniquely determined because of possible rotation, a convenient choice is

$$\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2}$$

$$\Lambda^j{}_0 = \Lambda^0{}_j = \gamma v_j \qquad \Lambda^0{}_0 = \gamma$$

Dynamics

Let us "define" the relativistic force f^α acting on a particle with coordinates $x^\alpha(\tau)$

$$f^\alpha = m \frac{d^2 x^\alpha}{d\tau^2}$$

where f^α is related to the Newtonian force by noting that:

A) if the particle is momentarily at rest then $d\tau = dt$ and $f^\alpha = F^\alpha$ with $F^0 = 0$ and F^i being Cartesian components of the nonrelativistic force $\bar{F}$

B) Under the general Lorentz transformation $dx'^\alpha = \Lambda^\alpha{}_\beta \, dx^\beta$ while $d\tau' = d\tau$ is invariant

$$\Downarrow$$

$$f'^\alpha = \Lambda^\alpha{}_\beta f^\beta \qquad (\text{a four-vector})$$

Suppose that our particle has velocity $\bar{v}$ at some moment t_0 and introduce a new coordinate system x'^α

$$x^\alpha = \Lambda^\alpha{}_\beta(\bar{v}) x'^\beta$$

where $\Lambda(\bar{v})$ is the boost that carries a particle from rest to velocity $\bar{v}$ $\Rightarrow$ our particle is at rest in x'^α

$\Downarrow$

the force four-vector in x'^α at t_0 is equal to the nonrelativistic force F^α, so

$$f^\alpha = \Lambda^\alpha{}_\beta(\bar{v})\, F^\beta$$

since $F^0 = 0$

$$f^i = \Lambda^i{}_j F^j = \left(\delta_{ij} + v_i v_j \frac{\gamma-1}{v^2}\right) F^j = F^i + (\gamma-1)\, v_i \frac{\bar{v}\cdot\bar{F}}{v^2}$$

$$f^0 = \Lambda^0{}_j F^j = \gamma\, \bar{v}\cdot\bar{F}$$

$$f^\alpha = m \frac{d^2 x^\alpha}{d\tau^2} \implies x^\alpha = x^\alpha(\tau)$$

initial conditions : $\quad -1 = \eta_{\alpha\beta} \dfrac{dx^\alpha}{d\tau} \dfrac{dx^\beta}{d\tau}$

Energy and momentum

$$f^\alpha = m \frac{d^2 x^\alpha}{d\tau^2} \implies p^\alpha \equiv m \frac{dx^\alpha}{d\tau}$$

$\Downarrow$

$$f^\alpha = \frac{dp^\alpha}{d\tau} \qquad d\tau = \left(dt^2 - d\bar{x}^2\right)^{1/2} = \left(1 - \bar{v}^2\right)^{1/2} dt$$

$$\begin{cases} p^i = m \dfrac{dx^i}{d\tau} = m\left(1-\bar{v}^2\right)^{-1/2} \dfrac{dx^i}{dt} = m\gamma v^i \\[2ex] p^0 = m \dfrac{dx^0}{d\tau} = m \dfrac{dt}{d\tau} = m\gamma \end{cases} \qquad \bar{v} \equiv \frac{d\bar{x}}{dt}$$

p^α is a four-vector : $\quad p'^\alpha = \Lambda^\alpha{}_\beta\, p^\beta$

$$p^i = m v^i + O(v^3)$$
$$p^0 = m + \tfrac{1}{2} m\bar{v}^2 + O(v^4)$$

$\implies p^0 \equiv E = \left(\bar{p}^2 + m^2\right)^{1/2}$

eliminating $\bar{v}$

p^α are conserved (see Noether's theorem)

$\Downarrow$

p'^α is also conserved

$$\left(\Delta \sum_n p'^\alpha_n = \Lambda^\alpha{}_\beta\, \Delta \sum_n p^\beta_n \right)$$

Vectors and Tensors

contravariant four-vector : $V^\alpha \to V'^\alpha = \Lambda^\alpha{}_\beta V^\beta$ e.g. $dx^\alpha,\ f^\alpha,\ p^\alpha$

for $x^\alpha \to x'^\alpha = \Lambda^\alpha{}_\beta x^\beta$

covariant four-vector : $U_\alpha \to U'_\alpha = \Lambda_\alpha{}^\beta U_\beta$

where $\Lambda_\alpha{}^\beta \equiv \eta_{\alpha\gamma}\eta^{\beta\delta}\Lambda^\gamma{}_\delta$ and $\eta^{\beta\delta} = \eta_{\beta\delta}$

$$\eta^{\beta\delta}\eta_{\alpha\delta} = \delta^\beta{}_\alpha = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Note that $\Lambda_\alpha{}^\beta$ is the inverse of $\Lambda^\alpha{}_\beta$

$$\Lambda_\alpha{}^\gamma \Lambda^\alpha{}_\beta = \eta_{\alpha\delta}\,\eta^{\gamma\sigma}\Lambda^\delta{}_\sigma \Lambda^\alpha{}_\beta = \eta^{\gamma\sigma}\eta_{\sigma\beta} = \delta^\gamma{}_\beta$$

$$\eta_{\sigma\beta}$$

$U_\alpha V^\alpha$ is invariant :

$$U'_\alpha V'^\alpha = \Lambda_\alpha{}^\beta U_\beta \Lambda^\alpha{}_\delta V^\delta = U_\beta V^\beta$$

$$\delta^\beta{}_\delta$$

$V_\alpha \equiv \eta_{\alpha\beta} V^\beta$ is indeed covariant :

$$V_\sigma \to V'_\sigma = \eta_{\sigma\beta} V'^\beta = \eta_{\sigma\beta}\Lambda^\beta{}_\delta V^\delta = \eta_{\sigma\beta}\Lambda^\beta{}_\delta\,\eta^{\delta\alpha} V_\alpha = \Lambda_\sigma{}^\alpha V_\alpha$$

$$\eta^{\delta\alpha} V_\alpha = \underbrace{\eta^{\delta\alpha}\eta_{\alpha\beta}}_{\delta^\delta{}_\beta} V^\beta = V^\delta$$

$$\frac{\partial}{\partial x^\alpha} \to \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta} = \Lambda_\alpha{}^\beta \frac{\partial}{\partial x^\beta} \Rightarrow \frac{\partial}{\partial x^\alpha}$$ – covariant four-vector

$$\Lambda_\alpha{}^\gamma \mid x'^\alpha = \Lambda^\alpha{}_\beta x^\beta \Rightarrow \Lambda_\alpha{}^\gamma x'^\alpha = \underbrace{\Lambda_\alpha{}^\gamma \Lambda^\alpha{}_\beta}_{\delta^\gamma{}_\beta} x^\beta = x^\gamma$$

$$x^\gamma = \Lambda_\alpha{}^\gamma x'^\alpha$$

$$x^\beta = \Lambda_\alpha{}^\beta x'^\alpha \Rightarrow \frac{\partial x^\beta}{\partial x'^\alpha} = \Lambda_\alpha{}^\beta$$

$\frac{\partial}{\partial x^\alpha}$ - covariant , $\frac{\partial}{\partial x_\alpha}$ - contravariant

$$\Box^2 \equiv \eta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} = \frac{\partial^2}{\partial x^0} - \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} = \frac{\partial^2}{\partial t^2} - \nabla^2$$

Lorentz invariant

Tensors

$$T^\gamma{}_{\alpha\beta} \to T'^\gamma{}_{\alpha\beta} = \Lambda^\gamma{}_\delta \Lambda_\beta{}^\xi \Lambda_\alpha{}^\varepsilon T^\delta{}_{\varepsilon\xi}$$

$$T^\alpha{}_\beta \equiv a R^\alpha{}_\beta + b S^\alpha{}_\beta$$ - tensor if $R^\alpha{}_\beta$ and $S^\alpha{}_\beta$ tensors

- direct product : $T^\alpha{}_\beta{}^\gamma = A^\alpha{}_\beta B^\gamma$ - tensor if ...
- contraction : $T^{\alpha\gamma} = T^\alpha{}_\beta{}^{\gamma\beta}$

$$T^{\alpha\gamma} \to T'^{\alpha\gamma} = T'^\alpha{}_\beta{}^{\gamma\beta} = \Lambda^\alpha{}_\delta \underbrace{\Lambda_\beta{}^\varepsilon \Lambda^\gamma{}_\xi \Lambda^\beta{}_\kappa}_{\delta^\varepsilon_\kappa} T^\delta{}_\varepsilon{}^{\xi\kappa} = \Lambda^\alpha{}_\delta \Lambda^\gamma{}_\xi T^{\delta\xi\varepsilon}{}_\varepsilon$$

$\Downarrow$

$T^{\alpha\gamma}$ rank 2 tensor

- differentiation : $T_\alpha{}^{\beta\gamma} = \frac{\partial}{\partial x^\alpha} T^{\beta\gamma}$ - tensor as a product of tensors

- the Minkowski tensor:

definition of the Lorentz transf. : $\eta_{\alpha\beta} = \Lambda^\gamma{}_\alpha \Lambda^\delta{}_\beta \eta_{\gamma\delta} \to \eta_{\alpha\beta}$ tensor covariant

$$\eta^{\varepsilon\xi} = \Lambda^\varepsilon{}_\alpha \Lambda^\xi{}_\beta \eta^{\alpha\beta}$$ contravariant tensor

$\eta_{\alpha\beta} \to \eta'_{\alpha\beta} = \eta_{\alpha\beta} =$

$= \Lambda_\alpha{}^\gamma \Lambda_\beta{}^\delta \eta_{\gamma\delta}$

↑ see the next page

$\cancel{\eta^{\alpha\varepsilon}\eta^{\beta\xi} \mid \eta_{\alpha\beta} = \Lambda^\gamma{}_\alpha \Lambda^\delta{}_\beta \eta_{\gamma\delta}}$

$\cancel{\eta^{\alpha\varepsilon}\eta_{\alpha\beta}\eta^{\beta\xi} = \eta^{\alpha\varepsilon}\Lambda^\gamma{}_\alpha \eta^{\beta\xi}\Lambda^\delta{}_\beta \eta_{\gamma\delta}}$

$\cancel{\eta^{\varepsilon\xi} =}$

$\cancel{\delta^\alpha \quad \eta^{\alpha\gamma}}$

Let us show that $\eta_{\alpha\beta}$ is a covariant ~~four vector~~ second rank tensor 7a which is invariant under the Lorentz transformation

$$\eta_{\alpha\beta} \to \eta'_{\alpha\beta} = \Lambda_\alpha{}^\gamma \Lambda_\beta{}^\delta \eta_{\gamma\delta}$$

from the definition of the LT we have

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta = \eta_{\gamma\delta} \qquad \gamma\uparrow$$

$$\eta_{\alpha\beta} \Lambda^{\alpha\gamma} \Lambda^\beta{}_\delta = \delta^\gamma_\delta \qquad \Big| \, (\Lambda^{-1})^\delta{}_\tau$$

$$\eta_{\alpha\beta} \Lambda^{\alpha\gamma} \underbrace{\Lambda^\beta{}_\delta (\Lambda^{-1})^\delta{}_\tau}_{= \delta^\beta_\tau} = (\Lambda^{-1})^\gamma{}_\tau$$

$$\Lambda_\tau{}^\gamma = \eta_{\alpha\tau} \Lambda^{\alpha\gamma} = (\Lambda^{-1})^\gamma{}_\tau \qquad \Big| \, \Lambda^\alpha{}_\gamma$$

$$\Lambda^\alpha{}_\gamma \Lambda_\tau{}^\gamma = \Lambda^\alpha{}_\gamma (\Lambda^{-1})^\gamma{}_\tau = \delta^\alpha_\tau \qquad \alpha\downarrow$$

$$\Lambda_{\alpha\gamma} \Lambda_\tau{}^\gamma = \eta_{\alpha\tau}$$

$$\eta_{\gamma\delta} \Lambda_\alpha{}^\delta \Lambda_\tau{}^\gamma = \eta_{\alpha\tau}$$

therefore

$$\eta_{\alpha\beta} \to \eta'_{\alpha\beta} = \Lambda_\alpha{}^\gamma \Lambda_\beta{}^\delta \eta_{\gamma\delta} = \eta_{\alpha\beta}$$

- the Levi-Civita tensor

$$\varepsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ even permutation of } 0123 \\ -1 & \text{odd} \\ 0 & \text{otherwise} \end{cases}$$

note that

$$\Lambda^{\alpha}{}_{\varepsilon} \Lambda^{\beta}{}_{\zeta} \Lambda^{\gamma}{}_{\kappa} \Lambda^{\delta}{}_{\lambda} \varepsilon^{\varepsilon\zeta\kappa\lambda} \propto \varepsilon^{\alpha\beta\gamma\delta}$$

as LHS must be odd under any permutation of indices $\alpha\beta\gamma\delta$

Currents and Densities

particles with position $x_n(t)$ and charge e_n

electric current : $\bar{J}(\bar{x},t) = \sum_n e_n \delta^3(x - x_n(t)) \frac{d\bar{x}_n(t)}{dt}$

charge density : $\varepsilon(\bar{x},t) = \sum_n e_n \delta^3(x - x_n(t))$

for $\int d^3x \, \delta^3(\bar{x}-\bar{y}) f(\bar{x}) = f(\bar{y})$

Let us make a "four-vector" prototype, such that

$$J^0 \equiv \varepsilon$$

$$J^\alpha(x) = \sum_n e_n \delta^3(\bar{x} - \bar{x}(t)) \frac{dx_n^\alpha(t)}{dt} \qquad \text{for } x_n^0(t) = t$$

Let us show that $J^\alpha(x)$ is indeed a four-vector

$$J^\alpha(x) = \int dt' \sum_n e_n \delta^4(x - x_n(t')) \frac{dx_n^\alpha(t')}{dt'} = (t' \to \tau) =$$

$$d\tau^2 = dt^2 - d\bar{x}^2$$

$$= \int d\tau \sum_n e_n \underbrace{\delta^4(x - x_n(\tau))} \underbrace{\frac{dx_n^\alpha(\tau)}{d\tau}}_{\text{four-vector}}$$

invariant — invariant since Det $\Lambda = 1$

$$\Downarrow$$

$J^\alpha(x)$ is a four-vector

$\delta^{(4)}(x - x_0)$ is Lorentz invariant

$\beta \,|\quad x^\beta - x_0^\beta = 0 \qquad\qquad x'^\alpha = \Lambda^\alpha{}_\beta\, x^\beta$

$$x'^\alpha - x_0'^\alpha = 0$$

$$\delta[f(x)] = \sum_i \frac{f(x - x_i)}{\left|\frac{df}{dx}\right|_{x = x_i}} \Rightarrow \delta^{(4)}(x^\beta - x_0^\beta) = \delta^{(4)}(x'^\alpha - x_0'^\alpha)$$

$$\int d^4x\, f(x)\, \delta^4(x - x_0) = \int d^4x' \; \underbrace{|\mathrm{Det}\,\Lambda^{-1}|}_{=1} \; f[x(x')]\, \delta^{(4)}(x' - x_0') =$$

$\uparrow$ change of variables

$$= f[x(x_0')] = f(x_0)$$

$$\int d^4x\; \delta^4(x - x_0)\, f(x) = f(x_0)$$

For invariance we require

$$\int d^4x \;\; \delta^4(x' - x_0')\, f(x) = f(x_0)$$

where $\quad x'^\alpha = \Lambda^\alpha{}_\beta\, x^\beta \qquad x_0'^\alpha = \Lambda^\alpha{}_\beta\, x_0^\beta$

$$\int d^4x' \; \det(\Lambda^{-1})\; \delta^4(x' - x_0')\, f(\Lambda^{-1}x') = \det(\Lambda^{-1})\, f(\underbrace{\Lambda^{-1}x_0'}_{x_0}) =$$

change of integration variables $\quad x' = \Lambda x$

$$= \underbrace{\det \Lambda^{-1}}_{=1}\; f(x_0) = f(x_0) \qquad \square$$

Homework $\rightarrow$ $\frac{\partial}{\partial x^\alpha} J^\alpha(x) = 0 \Rightarrow \frac{\partial}{\partial x^0} J^0 + \bar\nabla \bar J = 0$

Now we can show that $Q \equiv \int d^3x\, J^0(x)$ is time independent: $\frac{dQ}{dt} = 0$

$$\frac{dQ}{dt} = \int d^3x \frac{\partial}{\partial x^0} J^0(x) = -\int d^3x \underbrace{\bar\nabla \cdot \bar J(x)}_{\frac{\partial}{\partial x^i} J^i(x)} = 0$$

$\uparrow$ provided all particles are localized at some finite distance $x_n(t)$

Electrodynamics

the Maxwell's equations: (ME)

$$\left.\begin{array}{l} \bar\nabla \cdot \bar E = \varepsilon \\ \bar\nabla \times \bar B = \frac{\partial \bar E}{\partial t} + J \end{array}\right\} (*)$$

$$\left.\begin{array}{l} \bar\nabla \cdot \bar B = 0 \\ \bar\nabla \times \bar E = -\frac{\partial \bar B}{\partial t} \end{array}\right\} (**)$$

$\bar E = (E^1_{\cancel{x}}, E^2_{\cancel{y}}, E^3_{\cancel{z}})$

$\bar B = (B^1_{\cancel{x}}, B^2_{\cancel{y}}, B^3_{\cancel{z}})$

to write the ME in a covariant fashion let us define $F^{\alpha\beta}$ such that

$F^{12} = -B^3_{\cancel{3}}$ $\quad F^{23} = -B^1_{\cancel{1}}$ $\quad F^{31} = -B^2_{\cancel{2}}$ $\qquad F^{\alpha\beta} = -F^{\beta\alpha}$

$F^{01} = -E^1_{\cancel{1}}$ $\quad F^{02} = -E^2_{\cancel{2}}$ $\quad F^{03} = -E^3_{\cancel{3}}$

then $(*)$ reads

$$\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = +J^\beta$$

$\beta = 0$: $\frac{\partial}{\partial x^\alpha} F^{\alpha 0} = +J^0 = +\varepsilon$ $\qquad$ o.k.

$= \frac{\partial}{\partial x^i} F^{i0} = -\frac{\partial}{\partial x^i} F^{0i} = +\frac{\partial}{\partial x^i} E^i_{\cancel{i}}$

$\beta = i$: $\frac{\partial}{\partial x^\alpha} F^{\alpha i} = +J^i$ $\qquad$ o.k.

$$= \frac{\partial}{\partial x^0} F^{0i} + \frac{\partial}{\partial x^j} F^{ji} = -\frac{\partial}{\partial t} E_i + \overbrace{\epsilon_{jik} \frac{\partial}{\partial x^j} B_k}^{-(\bar\nabla \times \bar B)_i}$$

$\epsilon_{abi}\epsilon_{ijk} = \delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}$ $\qquad$ $-B^i_{\cancel{i}} \equiv \frac{1}{2}\epsilon_{ijk} F^{jk} \Rightarrow F^{ji} = -\epsilon_{jik} B^k_{\cancel{k}}$

1. Show that $\frac{\partial}{\partial x^\alpha} J^\alpha(x) = 0$

$$\bar\nabla \cdot \bar J(\bar x, t) = \sum_n e_n \frac{\partial}{\partial x^i} \delta^3(\bar x - \bar x_n(t)) \frac{d x_n^i(t)}{dt} =$$

$$= -\sum_n e_n \underbrace{\frac{\partial}{\partial x_n^i} \delta^3(\bar x - \bar x_n(t)) \frac{d x_n^i(t)}{dt}}_{\frac{\partial}{\partial t} \delta^3(\bar x - \bar x_n(t))} =$$

$$= -\frac{\partial}{\partial t} \underbrace{\sum_n e_n \delta^3(\bar x - \bar x_n(t))}_{\varepsilon(\bar x, t)} = -\frac{\partial \varepsilon(\bar x, t)}{\partial t}$$

$\square$

2. Show that $\eta_{\alpha\beta}$ is a tensor, i.e.

$$\eta_{\alpha\beta} \to \eta_{\alpha\beta}' = \Lambda_\alpha{}^\gamma \Lambda_\beta{}^\delta \eta_{\gamma\delta}$$

and prove that $\eta_{\alpha\beta}' = \eta_{\alpha\beta}$

(**) could be written as

$$\varepsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0 \qquad \left(F_{\gamma\delta} = \eta_{\gamma\alpha} \eta_{\delta\beta} F^{\alpha\beta} \right)$$

check : $\alpha = 0$

$$\varepsilon^{ijk} \frac{\partial}{\partial x^i} F_{jk} = -\varepsilon^{ijk} \frac{\partial}{\partial x^i} \epsilon_{jkl} B^{l} =$$

$$= + \underbrace{\varepsilon^{ijk} \varepsilon_{jkl}}_{2\delta^i{}_l} \frac{\partial}{\partial x^i} B^l = +2 \frac{\partial}{\partial x^i} B^i = 0$$

$\bar{\nabla} \cdot \bar{B} = 0$ o.k.

$\alpha = i$

$$\varepsilon^{i\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = \varepsilon^{i0kl} \frac{\partial}{\partial x^0} F_{kl} + \varepsilon^{ij\gamma\delta} \frac{\partial}{\partial x^j} F_{\gamma\delta} =$$

$$= \underbrace{+\varepsilon^{ikl} \frac{\partial}{\partial x^0} \epsilon_{klm} B^m}_{+2 \frac{\partial B_i}{\partial t}} + \underbrace{\varepsilon^{ij0k} \frac{\partial}{\partial x^j} F_{0k} + \varepsilon^{ijk0} \frac{\partial}{\partial x^j} F_{k0}}_{+\varepsilon^{ijk} \frac{\partial}{\partial x^j} \left(E^k + E^k \right) = +2 \left(\bar{\nabla} \times \bar{E} \right)_i}$$

o.k.

Since $\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = +J^\beta$ and

$\frac{\partial}{\partial x^\alpha}$ and J^β are vectors $\Rightarrow$ $F^{\alpha\beta}$ is a tensor

(not mentioning: if $F^{\alpha\beta}$ has a component $G^{\alpha\beta}$ such that $\partial_\alpha G^{\alpha\beta} = 0$ then we can't tell it transformation) (since $F^{\alpha\beta}$ could be solved in terms of J^β)

The electromagnetic force on a charged particle

(homework)

$$\left. \begin{aligned} f^\alpha &= e\, \eta_{\beta\gamma} F^{\alpha\beta} \frac{dx^\gamma}{d\tau} \\ \frac{dp^\alpha}{d\tau} &= f^\alpha \end{aligned} \right\} \Rightarrow \frac{d\bar{p}}{dt} = e \left(\bar{E} + \bar{v} \times \bar{B} \right)$$

$$\epsilon^{\mu\nu\varsigma\sigma} \epsilon_{\mu\nu}{}^{\varsigma'\sigma'} = -2 \left(\eta^{\varsigma\varsigma'} \eta^{\sigma\sigma'} + - \eta^{\varsigma\sigma'} \eta^{\varsigma'\sigma} \right)$$

An alternative for $\varepsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0$

$$\frac{\partial}{\partial x^\alpha} F_{\beta\gamma} + \frac{\partial}{\partial x^\beta} F_{\gamma\alpha} + \frac{\partial}{\partial x^\gamma} F_{\alpha\beta} = 0$$

see class

$\varepsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0 \Rightarrow$ there exists a four-vector potential

The Energy - Momentum tensor

In analogy to the four-vector current construction

$$\begin{cases} T^{\alpha 0}(t,\bar{x}) \equiv \sum\limits_n \underbrace{p_n^\alpha}_{\text{four-momentum}}(t)\, \delta^3(\bar{x}-\bar{x}_n(t)) & (\text{density of } p^\alpha) \\ T^{\alpha i}(t,\bar{x}) \equiv \sum\limits_n p_n^\alpha(t) \dfrac{dx_n^i(t)}{dt} \delta^3(\bar{x}-\bar{x}_n(t)) & (\text{current of } p^\alpha) \end{cases}$$

$$T^{\alpha\beta}(x) = \sum_n p_n^\alpha(t) \frac{dx_n^\beta(t)}{dt} \delta^3(\bar{x}-\bar{x}_n(t)) \quad \text{where } x_n^0(t) \equiv t$$

from the definition of the four-momenta we have

$$p_n^\beta = E_n \frac{dx_n^\beta}{dt}$$

so

$$T^{\alpha\beta}(x) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\bar{x}-\bar{x}_n(t)) \quad \Rightarrow \quad T^{\alpha\beta} = T^{\beta\alpha}$$

to see that $T^{\alpha\beta}$ is indeed a tensor let's write it as

$$T^{\alpha\beta}(x) = \sum_n \int d\tau \, p_n^\alpha \frac{dx_n^\beta}{d\tau} \delta^4(x - x_n(\tau))$$

Four - momentum (total : including the E-M field)

time independent :

$$p^\alpha_{total} = \int d^3x \, T^{\alpha 0}(t,\bar{x})$$

as a consequence of the tensor conservation

(homework $\partial_\alpha T^{\alpha\beta}_{total} = 0$)

The energy-momentum conservation

$$T^{\alpha\beta}(\bar{x},t) = \sum_n p_n^\alpha(t) \frac{dx_n^\beta(t)}{dt} \delta^3(\bar{x}-\bar{x}_n(t))$$

$$\frac{\partial}{\partial x^i} T^{\alpha i}(x,t) = -\sum_n p_n^\alpha(t) \frac{dx_n^i(t)}{dt} \frac{\partial}{\partial x_n^i} \delta^3(\bar{x}-\bar{x}_n(t)) =$$

$$= -\sum_n p_n^\alpha(t) \frac{\partial}{\partial t} \delta^3(\bar{x}-\bar{x}_n(t)) =$$

$$= -\frac{\partial}{\partial t} T^{\alpha 0}(x,t) + \sum_n \frac{dp_n^\alpha(t)}{dt} \delta^3(\bar{x}-\bar{x}_n(t))$$

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = G^\alpha$$

$\llcorner$ density of force

$$\frac{dp^\alpha}{d\tau} = f^\alpha \quad \Rightarrow \quad \frac{dp^\alpha}{dt}\frac{dt}{d\tau} = f^\alpha$$

$$G^\alpha = \sum_n \frac{d\tau}{dt} f_n^\alpha(t) \delta^3(\bar{x}-\bar{x}_n(t))$$

For free particles $f_n^\alpha = 0$, then

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = 0$$

Consider a pas of charged particles with charges e_n then

$$f_n^\alpha = e_n F^\alpha{}_\gamma(x) \frac{dx_n^\gamma}{d\tau}$$

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = \sum_n \frac{d\tau}{dt} e_n F^\alpha{}_\gamma(x) \frac{dx_n^\gamma}{d\tau} \delta^3(\bar{x}-\bar{x}_n(t)) =$$

$$= \sum_n e_n F^\alpha{}_\gamma(x) \frac{dx_n^\gamma}{dt} \delta^3(\bar{x}-\bar{x}_n(t)) = F^\alpha{}_\gamma(x) J^\gamma(x)$$

Consider $\quad T^{\alpha\beta}_{em} := -F^{\alpha}{}_{\gamma} F^{\beta\gamma} + \frac{1}{4}\eta^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}$

$$T^{00}_{em} = \frac{1}{2}\left(\bar{E}^2 + \bar{B}^2\right) \qquad T^{i0}_{em} = (\bar{E} \times \bar{B})_i$$

$$F^{12} = -B^{3}_{\cancel{3}} \qquad F^{23} = -B^{1}_{\cancel{1}} \qquad F^{31} = -B^{2}_{\cancel{2}}$$

$$F^{01} = -E^{1}_{\cancel{1}} \qquad F^{02} = -E^{2}_{\cancel{2}} \qquad F^{03} = -E^{3}_{\cancel{3}}$$

$$-\frac{\partial}{\partial x^\beta} T^{\alpha\beta}_{em} = F^{\alpha}{}_{\gamma}\,\partial_\beta F^{\beta\gamma} + F^{\beta\gamma}\,\partial_\beta F^{\alpha}{}_{\gamma} - \frac{1}{2}\underbrace{\eta^{\alpha\beta} F_{\gamma\delta}\,\partial_\beta F^{\gamma\delta}}_{F_{\gamma\delta}\,\partial^\alpha F^{\gamma\delta}} =$$

$$= F^{\alpha}{}_{\gamma}\,\underbrace{\partial_\beta F^{\beta\gamma}}_{=+J^\gamma} - \frac{F_{\beta\gamma}}{2}\underbrace{\left[\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta}\right]}_{=0} =$$

Maxwell equations (the Bianchi identity)

$$= +F^{\alpha}{}_{\gamma} J^\gamma$$

$$\partial_\beta T^{\alpha\beta}_{em} = -F^{\alpha}{}_{\gamma} J^\gamma$$

$$\partial_\beta T^{\alpha\beta} = F^{\alpha}{}_{\gamma} J^\gamma$$

$$\Downarrow$$

$$\partial_\beta T'^{\alpha\beta} = 0 \quad \text{for} \quad T'^{\alpha\beta} := T^{\alpha\beta} + T^{\alpha\beta}_{em}$$

$T'^{\alpha\beta}$ – symmetric

Relativistic Hydrodynamics

A perfect fluid is defined as having at each point a velocity $\bar{v}$, such that an observer moving with this velocity sees the fluid around him as isotropic (no viscosity!)

- suppose we are in a frame $(\sim)$ of reference in which fluid is at rest at some particular position and time

$\Downarrow$ from the definition of the perfect fluid

$$\tilde{T}^{ij} = p\,\delta_{ij} \qquad \tilde{T}^{i0} = 0 \qquad \tilde{T}^{00} = \rho$$

p: pressure

ρ: proper energy density

- go to a frame at rest in the Lab and suppose that the fluid in this frame is moving (at the given space-time point) with velocity $\bar{v}$

$$x^{\alpha} = \Lambda^{\alpha}{}_{\beta}(v)\,\tilde{x}^{\beta}$$

$$\Lambda^{0}{}_{0} = \gamma \qquad \Lambda^{i}{}_{0} = \gamma v_i \qquad \gamma = (1 - v^2)^{-1/2}$$

$$\Lambda^{i}{}_{j} = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2} \qquad \Lambda^{0}{}_{j} = \gamma v_j$$

- calculate $\quad T^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma}(v)\,\Lambda^{\beta}{}_{\delta}(v)\,\tilde{T}^{\gamma\delta}$

$$T^{ij} = p\delta_{ij} + (p+\rho)\frac{v_i v_j}{1-v^2}$$

$$T^{i0} = (p+\rho)\frac{v_i}{1-v^2} \qquad T^{00} = \frac{\rho + p v^2}{1-v^2}$$

that could be written as

$$T^{\alpha\beta} = -p\eta^{\alpha\beta} + (p+\rho)\underbrace{U^\alpha U^\beta}_{\text{four velocity vector}}$$

$$\bar{U} = \frac{d\bar{x}}{d\tau} = \gamma\bar{v}$$

$$U^0 = \frac{dt}{d\tau} = \gamma \qquad \text{with} \qquad U_\alpha U^\alpha = +1$$

$$\Downarrow$$

$T^{\alpha\beta}$ is indeed a tensor

$n \equiv \underbrace{\text{particle number}}_{\text{for a conserved quantity}}\ \overbrace{\text{density}}^{\text{at a given point}}$, in the comoving frame:

$\tilde{N}^0 := n \qquad \tilde{N}^i := 0$ particle current

in other frames

$$N^0 = \Lambda^0{}_\beta(\bar{v})\,\tilde{N}^\beta = \gamma n$$

$$N^i = \Lambda^i{}_\beta(\bar{v})\,\tilde{N}^\beta = \gamma v_i n \,, \qquad \text{for}$$

$$\Lambda^0{}_0 = \gamma \qquad \Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma-1}{v^2}$$

$$\Lambda^i{}_0 = \gamma v_i \qquad \Lambda^0{}_j = \gamma v_j \qquad \gamma = (1-v^2)^{-1/2}$$

$$N^\alpha \overset{(1)}{=} n\,\underbrace{U^\alpha}_{\text{the velocity four-vector}}$$

the energy momentum conservation:

$$0 = \partial_\beta T^{\alpha\beta} = \partial_\beta \left[-p \eta^{\alpha\beta} + (p+\rho) U^\alpha U^\beta \right] \qquad N = \int n^0 d^3x$$

conservation of the particle current, the continuity equation

$$0 = \partial_\alpha N^\alpha = \partial_\alpha (n U^\alpha) = \partial_0 (n\gamma) + \bar{\nabla} \cdot (n\gamma \bar{v})$$

$$\bar{\nabla} = \partial_1, \partial_2, \partial_3$$

<u>$\alpha = i$</u>

$$0 = +\partial_i p + \underbrace{\partial_t \left[(p+\rho) U^i U^0 \right]}_{\partial_t T^{i0}} + \bar{\nabla} \left[(p+\rho) U^i \bar{U} \right] =$$

$$U^i = v^i U^0$$

$$= + \partial_i p + \partial_t \left[(p+\rho) v^i U^{0 2} \right] + \partial_j \left[(p+\rho) U^0 v^i U^0 v^j \right] =$$

<u>$\alpha = 0$</u>

$$0 = -\partial_t p + \partial_t \left[(p+\rho) U^{0 2} \right] + \partial_j \left[(p+\rho) U^0 U^0 v^j \right]$$

$$+ \partial_i p + \underbrace{\boxed{v^i \partial_t \left[(p+\rho) U^{02} \right] + v^i \partial_j \left[(p+\rho) U^{02} v^j \right]}}_{+ v^i \partial_t p} + (p+\rho) U^{02} \partial_t v^i + (p+\rho) \underbrace{U^{02}}_{= \gamma^2} v^j \partial_j v^i =$$

$$= + \partial_i p + v^i \partial_t p + (p+\rho) \gamma^2 \left[\partial_t v^i + (\bar{v} \cdot \bar{\nabla}) \cdot v^i \right]$$

$$\partial_t \bar{v} + (\bar{v} \cdot \bar{\nabla}) \cdot \bar{v} = - \frac{1 - \bar{v}^2}{p + \rho} \left(\bar{\nabla} p + \bar{v} \partial_t p \right)$$

the Euler equation

$\rho \to p + \rho$, usually $p \ll \rho$

non-relativistic form: $\partial_t \bar{v} + (\bar{v} \cdot \bar{\nabla}) \bar{v} = - \frac{\bar{\nabla} p}{\rho}$

The scalar equation reads:

$$\partial_\beta\left[-p\eta^{\alpha\beta} + (p+\rho)U^\alpha U^\beta\right] = 0 \qquad \Big|\cdot U_\alpha$$

$$-(\partial^\alpha p)U_\alpha + \partial_\beta\left[(p+\rho)U^\alpha U^\beta\right]U_\alpha =$$

$$0 = \partial_\beta(\underbrace{U_\alpha U^\alpha}_{=+1}) = 2U_\alpha\partial_\beta U^\alpha$$

$$= -(\partial^\beta p)U_\beta + \partial_\beta\left[(p+\rho)U^\beta\right]\underbrace{U^\alpha U_\alpha}_{=+1} + (p+\rho)U^\beta\underbrace{(\partial_\beta U^\alpha)U_\alpha}_{=0} =$$

$$= -U^\beta(\partial_\beta p) + \partial_\beta\left[(p+\rho)U^\beta\right] =$$

from the conservation of the particle current $\partial_\beta(nU^\beta)=0$

$$= U^\beta\left[\cancel{(-\partial_\beta p)} + \partial_\beta(\cancel{p}+\rho)\right] + (p+\rho)\underbrace{\partial_\beta U^\beta}_{=-n^{-1}U^\beta\partial_\beta n} =$$

$$\partial_\beta(nU^\beta) = U^\beta\partial_\beta n + n\partial_\beta U^\beta$$

$$= U^\beta\left[\partial_\beta\rho - (p+\rho)\underbrace{n^{-1}\partial_\beta n}_{-n\,\partial_\beta n^{-1}}\right] = nU^\beta\left[n^{-1}\partial_\beta\rho + (p+\rho)\partial_\beta n^{-1}\right] =$$

$$= nU^\beta\left[p\,\partial_\beta\left(\frac{1}{n}\right) + \partial_\beta\left(\frac{\rho}{n}\right)\right] = 0$$

The second law of thermodynamics tells us

$$p\,d\left(\frac{1}{n}\right) + d\left(\frac{\rho}{n}\right) = T\,dS$$

S – the entropy per particle

$n = \frac{N}{V}$, $\frac{1}{n} = \frac{V}{N}$ – volume per particle (fluid

$$\Rightarrow 0 = U^\beta\partial_\beta S \propto \left[\frac{\partial S}{\partial t} + (\vec{v}\cdot\vec{\nabla})S\right] = 0$$

Consider a fluid composed of structureless point particles, then

$$T^{\alpha\beta} = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\bar{x}-\bar{x}_n)$$

In the comoving frame

$$T^{\alpha\beta} = \mathrm{diag}\,(\rho, p, p, p)$$

so
$$p = \frac{1}{3}\sum_{i=1}^{3} T^{ii} = \frac{1}{3}\sum_n \frac{\bar{p}_n^2}{E_n}\delta^3(\bar{x}-\bar{x}_n)$$

$$\rho = T^{00} = \sum_n E_n \delta^3(\bar{x}-\bar{x}_n)$$

For the particle number density we have

$$n = \sum_n \delta^3(\bar{x}-\bar{x}_n)$$

As $\bar{p}_n^2 = E_n^2 - m_n^2$, we have

$$p = \frac{1}{3}\sum_n \frac{1}{E_n}(E_n^2 - m_n^2)\delta^3(\,) = \frac{1}{3}\rho - \frac{1}{3}\sum_n \frac{m_n^2}{E_n}\delta^3(\bar{x}-\bar{x}_n)$$

$$\Downarrow$$

$$0 \leq p \leq \frac{\rho}{3}$$

- For a non-relativistic gas $E_n \simeq m_n + \frac{\bar{p}_n^2}{2m_n} + \cdots$

then for $m_n = m$ we get

$$\rho \simeq m\cdot n + \sum_n \frac{\bar{p}_n^2}{2m}\delta^3(\bar{x}-\bar{x}_n) = m\cdot n +$$

$$+ \sum_n \frac{\bar{p}_n^2}{2} \frac{1}{E_n - \frac{\bar{p}_n^2}{2m_n} + \cdots}\delta^3(\,) = m\cdot n + \frac{3}{2}p + \cdots$$

$$\rho = m\cdot n + \frac{3}{2}p$$

For a high-relativistic gas $E_n \simeq |\bar{p}_n| \gg m$, then

$$\rho \cong \sum_n |\bar{p}_n| \, \delta^3(\bar{x}-\bar{x}_n) \cong \sum \frac{|\bar{p}_n|^2}{E_n} \delta^3(\bar{x}-\bar{x}_n) = 3p$$

$$\frac{|\bar{p}_n|^2}{|\bar{p}_n|} = \frac{|\bar{p}_n|^2}{E_n + \cdots} \qquad \rho = 3p$$

The variational principle

classical mechaniques :
$$S[x(t)] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \frac{d\bar{x}}{dt}\frac{d\bar{x}}{dt} - V(x)\right] =$$
$$= \int_{t_1}^{t_2} dt\, L(\bar{x}, \dot{\bar{x}})$$

classical trajectories $\Leftrightarrow$ extremum of $S[x(t)]$ $\Updownarrow$ $\delta S[x] = 0$

$$\delta S[x^i] \equiv S[x^i + \delta x^i] - S[x^i] =$$
$$= -\int_{t_1}^{t_2} dt\, \delta x^i \left[m \frac{d^2x^i}{dt^2} + \frac{\partial V(x)}{\partial x^i}\right] + m\left(\delta x^i \frac{dx^i}{dt}\right)\Big|_{t_1}^{t_2} + O(\delta x^2)$$
$= 0$ for any $\delta x^i(t)$ such that $\delta x^i(t_1) = \delta x^i(t_2) = 0$

$$\Downarrow$$
$$m \frac{d^2 x^i}{dt^2} + \frac{\partial V}{\partial x^i} = 0$$

classical field theory :
$$S[q] = \int_0^T dt \left(\sum_a \frac{1}{2} m_a \dot{q}_a^2 - V(q_1, \dots q_N)\right)$$

let us consider $V(\dots) = \sum_{a,b,c} \frac{1}{2} k_{ab} q_a q_b + \frac{1}{2} g_{abc} q_a q_b q_c + \dots$

$\boxed{V_0 + \sum_a e_a q_a}$

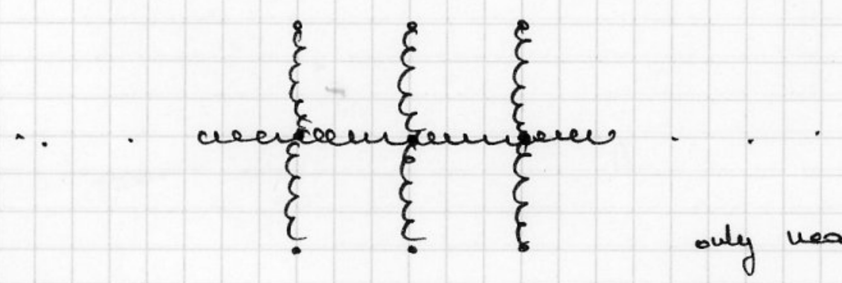
expansion around certain equilibrium

$$\Downarrow$$
$$m_a \ddot{q}_a = -e_a - k_{ab} q_b + \dots$$

harmonic oscylator

constant force

– let us keep only k_{ab} (oscylator) and consider a mattress in the limit $l \to 0$



only nearest neighbors interact

- we can replace the label "a" by a position vector $\bar{x}$

so $q(t,\bar{x})$ instead of $q_a(t)$

tradition $\hookrightarrow \varphi(t,\bar{x}) \leftarrow$ this is called a field

$$\sum_a \frac{1}{2} m_a \dot{q}_a^2 \to \int d^3x \, \frac{1}{2} \sigma \left(\frac{\partial \varphi}{\partial t}\right)^2 \qquad \left(\sum_a \to \frac{1}{L^3} \int d^3x \qquad \frac{m_a}{L^3} \to \sigma \right)$$

$m_a = m$

$$V = \sum_{ab} \frac{1}{2} k_{ab} q_a q_b + \cdots$$

keep only nearest neighbors

$$2 q_a q_b = (q_a - q_b)^2 - q_a^2 - q_b^2$$

$\Downarrow$

$$(q_a - q_b)^2 \simeq L^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 + \cdots$$

$$\sum_{ab} \frac{1}{2} k_{ab} q_a q_b$$

$\uparrow$ in each direction

$\downarrow$

single sum

$$\frac{k_{ab}}{L^3} \to \rho \quad (a,b\text{-inde-pendent})$$

$\downarrow$

$$\frac{1}{L^3} d^3x$$

$$S[q] \to S[\varphi] \equiv \int_0^T dt \int d^3x \, \mathcal{L}(\varphi) = \int_0^T dt \int d^3x \, \frac{1}{2} \left\{ \sigma \left(\frac{\partial \varphi}{\partial t}\right)^2 - \rho (\bar{\nabla}\varphi)^2 + - \tau \varphi^2 - \xi \varphi^3 + \cdots \right\}$$

the Lorentz invariance requires that $\sigma = \rho$

In general we will consider scalar field theories described by:

$$S[\varphi] = \int d^4x \left[\frac{1}{2} \underbrace{(\partial\varphi)^2}_{} - \frac{1}{2} m^2 \varphi^2 - \frac{g}{3!} \varphi^3 - \frac{\lambda}{4!} \varphi^4 + \cdots \right]$$

$$\partial_\mu \varphi \, \partial^\mu \varphi$$

$$\hookrightarrow \left(\frac{\partial}{\partial x^\mu} \varphi\right) \left(\frac{\partial}{\partial x_\mu} \varphi\right)$$

Symmetry : Lorentz invariance + at most two powers of $\frac{\partial}{\partial t}$

$$\Downarrow$$

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \varphi \partial^\mu \varphi - \underbrace{V(\varphi)}_{\substack{\text{polynomial if} \\ \text{expansion is allowed}}}$$

$\delta S[\varphi] = 0 \quad \Rightarrow \quad$ Euler-Lagrange equations (equations of motion)

$$\delta S[\varphi] = \int_M d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi}\, \delta\varphi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\, \delta\left(\partial_\mu \varphi(x)\right)\right] =$$

$$= \int_M d^4x\, \delta\varphi(x) \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right)\right] + \underbrace{\int_{\partial M} d^3x\, n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\, \delta\varphi(x)}$$

$\delta\left[\partial_\mu \varphi(x)\right] = \partial_\mu \delta\varphi(x)$

we will assume

either $\delta\varphi(x) = 0$ for $x \in \partial M$

or $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \to 0$ if $\partial M \to \infty$

(matter localized at finite distance)

$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0}$$

For $\mathcal{L} = \frac{1}{2}\, \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4$ $\quad$ (symmetry $\varphi \to -\varphi$ imposed)

E-L equation is $\partial_\mu \partial^\mu \varphi + m^2 \varphi + \frac{\lambda}{6}\varphi^3 = 0$

$$\underbrace{(\Box + m^2)\varphi}_{\substack{\shortparallel \\ 0}} + \frac{\lambda}{6}\varphi^3 = 0$$

$\frac{\lambda}{6}\varphi^3$ ← interaction terms

if $= 0$ then the Klein-Gordon equation

Nature : pions , kaons , the Higgs boson

Complex scalar field

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi \quad - V(\varphi^* \varphi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = 0 \quad \Rightarrow \quad \begin{array}{ll} \partial^\mu \partial_\mu \varphi + m^2 \varphi = 0 & \leftarrow \delta\varphi^* \\ \partial^\mu \partial_\mu \varphi^* + m^2 \varphi^* = 0 & \leftarrow \delta\varphi \end{array}$$

$\mathcal{L}$ is invariant under a global transformation

α is a constant

$$\varphi(x) \to \varphi'(x) = e^{i\alpha} \varphi(x)$$
$$\varphi^*(x) \to \varphi'^*(x) = e^{-i\alpha} \varphi^*(x)$$

Electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Downarrow$$

E-L equation : $\partial_\mu F^{\mu\nu} = 0$

$$\varepsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

the Bianchi identity

gauge invariance :

$F_{\mu\nu}$ is invariant under $\quad A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$

local !

$$\Downarrow$$

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} \qquad \alpha = \alpha(x)$$

Scalar electrodynamics

$$\mathcal{L} = \underbrace{\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi}_{} \quad \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{}$$

vary under local transformation

$\varphi(x) \to \varphi'(x) = e^{i\alpha(x)} \varphi(x)$

invariant under local transformation

$$\hookrightarrow \quad \partial^\mu \varphi \Rightarrow (\partial^\mu - ieA^\mu)\varphi \equiv D^\mu \varphi$$

then $\quad \varphi(x) \to \varphi'(x) = e^{i\alpha(x)} \varphi$

$$D^\mu \varphi(x) \to (D^\mu \varphi(x))' = e^{i\alpha(x)} D^\mu \varphi(x)$$

Yang – Mills theories (non-Abelian generalization of electrodynamics)

– global non-Abelian symmetries

$$\varphi_i(x) \to \varphi_i'(x) = \omega_{ij}\,\varphi_j(x) \qquad \omega \in SU(N)$$

$$\det \omega = 1 \qquad \omega^+\omega = 1\!\!1$$

$$\varphi_i^*\varphi_i \to \varphi_j'^*\underbrace{\omega_{ij}^*\,\omega_{ik}}_{(\omega^+)_{ji}\,\omega_{ik} = \delta_{jk}}\varphi_k' = \varphi_j'^*\varphi_j'$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}$$

$$\mathcal{L} = \partial_\mu\varphi^+\partial^\mu\varphi - m^2\varphi^+\varphi - \lambda\,(\varphi^+\varphi)^2$$

SU(N) symmetry ⟹ equal masses

– let us generalize from global to local non-Abelian symmetries :

$$\varphi_i(x) \to \varphi_i'(x) = \omega_{ij}(x)\,\varphi_j(x)$$

example $\omega(x) \in SU(2)$

but
$$\partial^\mu\varphi'(x) = \underbrace{(\partial^\mu\omega(x))\,\varphi(x)}_{\text{the problem}} + \omega(x)\,\partial^\mu\varphi(x) \qquad (*)$$

following E-D let us introduce a covariant derivative such that

$$(D^\mu\varphi)' = \omega\,D^\mu\varphi$$

from $(*)$ it is clear that it can be achieved if

$$D^\mu\varphi = (\partial^\mu + \overbrace{A^\mu}^{\text{matrix acting on }\varphi})\,\varphi$$

$$(D^\mu\varphi)' = \partial^\mu\varphi' + A^{\mu\prime}\varphi' = (\partial^\mu\omega)\varphi + \cancel{\omega\partial^\mu\varphi} + A^{\mu\prime}\omega\varphi = \cancel{\omega\partial^\mu\varphi} + \omega A^\mu\varphi$$

$$A^{\mu\prime} = -(\partial^\mu\omega)\,\omega^+ + \omega A^\mu\omega^+ \quad \text{for any } \varphi$$

$$A^\mu(x) \to A'^\mu(x) = \omega A^\mu\omega^+ + \omega\,\partial^\mu\omega^+ \quad \Leftarrow \text{non-Abelian}$$

for $\omega(x) = \exp(i\alpha(x))$ then $-ieA'^\mu = -ieA^\mu - i\partial^\mu\alpha$

$$A_\mu \to A_\mu' = \omega A_\mu \omega^{-1} + \omega \partial_\mu \omega^{-1}$$

Let us consider an infinitesimal transformation

$$\omega(x) = 1 + \varepsilon(x) \qquad \omega^{-1}(x) = 1 - \varepsilon(x)$$

$\in$ Lie algebra of $SU(2)$

anti-Hermitian 2×2 traceless matrix

$A_\mu \in ASU(2)$

i) $\omega \partial_\mu \omega^{-1} = (1+\varepsilon)(-\partial_\mu \varepsilon) = -\partial_\mu \varepsilon + o(\varepsilon^2)$

so $\omega \partial_\mu \omega^{-1} \in ASU(2)$

ii) assume that

$A_\mu \in ASU(2)$, then $\omega A_\mu \omega^{-1} \simeq (1+\varepsilon) A_\mu (1-\varepsilon) =$

$$= A_\mu + [\varepsilon, A_\mu] + o(\varepsilon^2) \qquad \in ASU(2)$$

– now we should (may want) to introduce a kinetic term for the vector field A_μ in analogy to $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ in E-D

- for global transformations we have

$$A_\mu \to A_\mu' = \omega A_\mu \omega^{-1}$$

- we expect that $F_{\mu\nu}$ should contain $\partial_\mu A_\nu - \partial_\nu A_\mu$

global transformation

$$\omega(\partial_\mu A_\nu - \partial_\nu A_\mu)\omega^{-1}$$

so now $F_{\mu\nu}$ is not going to be invariant as in ED

- let us try

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \omega(x) F_{\mu\nu}(x) \omega^{-1}(x)$$

local transformation

$$\Downarrow$$

$$\mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) \text{ is invariant}$$

- the right form of $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad \mathcal{L} = +\frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

- for $SU(2)$

$$A_\mu(x) = -ig\frac{\tau^a}{2}A^a_\mu(x)$$

$$\downarrow$$

$$F_{\mu\nu}(x) = -ig\frac{\tau^a}{2}F^a_{\mu\nu}(x) =$$

$$F_{\mu\nu}(x) = -ig\frac{\tau^a}{2}\underbrace{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc}A^b_\mu A^c_\nu\right)}_{F^a_{\mu\nu}(x)}$$

$$\left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right] = i\varepsilon^{abc}\frac{\tau^c}{2}$$

$$\mathcal{L} = +\frac{1}{2g^2}\mathrm{Tr}(F^2) = \frac{1}{2g^2}(-ig)^2 F^a_{\mu\nu}F^{b\,\mu\nu}\underbrace{\mathrm{Tr}\frac{\tau^a}{2}\frac{\tau^b}{2}}_{\frac{1}{2}\delta^{ab}}$$

$$= -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}$$

A gauge invariant scalar - vector theory is described by the following Lagrangian

$$\mathcal{L} = (D_\mu \varphi)^\dagger (D^\mu \varphi) - V(\varphi^\dagger \varphi) + \frac{1}{2g^2} \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad D_\mu \varphi \equiv \left(\partial_\mu - ig \frac{T^a}{2} A^a_\mu\right) \varphi$$

$$F_{\mu\nu} \equiv -ig \frac{T^a}{2} F^a_{\mu\nu}$$

Fermions

Following Dirac, we are looking for a ~~first order~~ differential equation linear in derivatives, such that its solutions satisfy the Klein - Gordon equation $(\Box^2 + m^2)\varphi = 0$

since the goal was to construct a relativistic covariant generalization of the Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$ which is linear in $\frac{\partial}{\partial t}$

$(-p^2 + m^2)\varphi = 0$

$$(i\gamma^\mu \partial_\mu + m) \Big| \quad (i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \leftarrow \text{postulate}$$

$\uparrow$ undefined coefficients

$$(i\gamma^\mu \partial_\mu + m)(i\gamma^\nu \partial_\nu - m)\psi = \left[\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu - m^2\right]\psi = 0$$

in order to have the K-G equation reproduced we require

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$\leftarrow$ the commutation relation for the Clifford algebra

1. γ^μ – matrices
2. ψ must have several components $\Leftarrow$

One can prove that for a space-time of dimension D the minimal dimension of matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ equals $2^{[\frac{D}{2}]}$, where $[x]$ denotes the integer part of x

$$\left([\tfrac{4}{2}] = 2 \quad , \quad [\tfrac{5}{2}] = 2 \right)$$

$$\Downarrow$$

γ^μ are 4×4 matrices in $D=4$ dim

Hermiticity of the Hamiltonian requires $\alpha_i^+ = \alpha_i$, $\beta^+ = \beta$ defined such that $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\gamma^{0+} = \gamma^0 \qquad \gamma^{i+} = -\gamma^i$$

$$i\hbar\partial_t\Psi = \hat{H}\Psi$$

$$\gamma^0 \; \Big| \; i\gamma^0\partial_t\Psi = (-i\gamma^i\partial_i + m)\Psi$$

$$\gamma^{0\,2} = 1\!\!1 \Leftarrow \{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$$

$$\Downarrow$$

$$i\partial_t\Psi = (-i\underbrace{\gamma^0\gamma^i}_{\alpha_i}\partial_i + \underbrace{\gamma^0}_{\beta} m)\Psi$$

$$\Leftarrow \quad \alpha_i^+ = \alpha_i \quad \beta^+ = \beta$$

"standard representation" of Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (1\!\!1, \sigma^i) \leftarrow \text{Pauli matrices}$$

$$\tilde{\sigma}^\mu = (1\!\!1, -\sigma^i)$$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_4 \end{pmatrix}$$

$$\delta S = 0$$

$$S = \int d^4x \; \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \quad \Rightarrow \quad (\gamma^\mu\partial_\mu - m)\Psi = 0$$

where $\bar{\Psi} \equiv \Psi^+\gamma_0$

— invariance under $\Psi(x) \to \omega\Psi(x)$ global such that $\omega^+\omega = 0$

— to generalize for local transformation one needs to replace ∂_μ by $\partial_\mu + A_\mu$, then

$(i\gamma^\mu D_\mu - m)\Psi(x) = 0$ in the Dirac eq.

for SU(2)

$$\left\{ i(\gamma^\mu)_{\alpha\beta} \left[\delta_{ij}\partial_\mu - i\frac{g}{2}(\tau^a)_{ij} A^a_\mu \right] - m\,\delta_{\alpha\beta}\delta_{ij} \right\} \Psi_{\beta j}(x) = 0$$

Lorentz $\quad$ ← SU(2) index

The Noether's theorem

$$\delta S[\varphi] = 0 \quad \Rightarrow \quad \text{E-L equations} \quad \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = 0$$

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

Let us consider global transformations of fields only (no transformation of space-time coordinates)

$$\phi^i(x) \to \phi'^i(x) = (\delta^{ij} + \varepsilon^a \overbrace{t_a^{ij}}^{\text{numerical constants}})\, \phi^j \qquad (*)$$

$\uparrow$ infinitesimal parameter (independent of x^μ)

Assume that the Lagrangian density is invariant under (*):

$$\delta \mathcal{L} = \mathcal{L}(\phi + \delta\phi, \partial_\mu \phi + \delta \partial_\mu \phi) - \mathcal{L}(\phi, \partial_\mu \phi) = 0$$

$$\delta \phi = \varepsilon^a t_a^{ij} \phi^j \qquad \delta \partial_\mu \phi = \varepsilon^a t_a^{ij} \partial_\mu \phi^j \quad (= \partial_\mu \delta\phi)$$

$$\rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi^i} \varepsilon^a t_a^{ij} \phi^j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \varepsilon^a t_a^{ij} \partial_\mu \phi^j = 0$$

All ε^a are independent, so

$$\frac{\partial \mathcal{L}}{\partial \phi^i} t_a^{ij} \phi^j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} t_a^{ij} \partial_\mu \phi^j = 0$$

$$\left\{ \frac{\partial \mathcal{L}}{\partial \phi^i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right.$$

Using E-L equation we get

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} t_a^{ij} \phi_j \right] = 0 \quad \Rightarrow \quad \partial_\mu j^{a\mu} = 0$$

$$\text{for } j^{a\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} t_a^{ij} \phi_j.$$

(only for fields that satisfy the E-L equation)

$$Q \equiv \int d^3x \, j^0 \quad \text{is time independent} \quad \frac{dQ}{dt} = 0$$

$$\frac{dQ}{dt} = \int d^3x \, \partial_0 j^0 = \int d^3x \, \partial_i j^i = \int d\Sigma_i \, j^i \to 0$$

Example : $\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi^* \varphi)$

let us consider φ and φ^* as independent fields :

$$\Phi^i = (\varphi, \varphi^*)^T \qquad i = 1,2$$

$\mathcal{L}$ is invariant under $\varphi \to \varphi' = e^{i\alpha}\varphi$

$$\varphi^* \to \varphi^{*\prime} = e^{-i\alpha}\varphi^*$$

for infinitesimal transformation $\varphi' = (1+i\alpha)\varphi$

$$\varphi^{*\prime} = (1-i\alpha)\varphi^*$$

$$\Phi = \begin{pmatrix}\varphi \\ \varphi^*\end{pmatrix} \to \Big[1\!\!1 + \underbrace{i\begin{pmatrix}\alpha & 0 \\ 0 & -\alpha\end{pmatrix}}_{\alpha\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}} \Big]\begin{pmatrix}\varphi \\ \varphi^*\end{pmatrix}$$

$$\underset{a=1}{\varepsilon^1 = \alpha} \qquad t_1 = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}$$

$$\Downarrow$$

$$j^\mu = \partial^\mu \varphi^* i \varphi - \partial^\mu \varphi \, i \, \varphi^* = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} t_a^{ij} \Phi_j$$

$$= i\left[(\partial^\mu \varphi^*)\varphi - (\partial^\mu \varphi)\varphi^*\right]$$

it is easy to see that

$\partial_\mu j^\mu = 0$ for fields satisfying the Klein-Gordon eq.

$$(\Box + m^2)\varphi = 0$$

Now let us consider space-time translations

$$\Phi^i(x^\mu) \to \Phi'^i(x^\mu) = \Phi^i(x^\mu + \varepsilon^\mu) = \Phi^i(x^\mu) + \partial_\mu \Phi^i(x^\mu)\varepsilon^\mu + \dots$$

let us assume that the Lagrangian doesn't depend explicitly on x, then

$$\mathcal{L}(\Phi', \partial_\mu \Phi') = \mathcal{L}(x^\mu + \varepsilon^\mu)$$

$$\mathcal{L}(\Phi, \partial_\mu \Phi) + \underbrace{\frac{\partial \mathcal{L}}{\partial \Phi^i}}_{\downarrow} \partial_\mu \Phi^i \varepsilon^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi^i)} \partial_\nu \partial_\mu \Phi^i \varepsilon^\mu = \mathcal{L}(x) + \underbrace{\partial_\mu \mathcal{L}\, \varepsilon^\mu}_{\partial_\nu \delta^\nu{}_\mu \mathcal{L}\, \varepsilon^\mu}$$

$$\partial_\nu \left[\frac{\partial \mathcal{L}}{\quad} \right]$$

$$\partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^i)} \partial_\mu \phi^i - \delta^\nu{}_\mu \mathcal{L} \right] \varepsilon^\mu = 0$$

ε^μ independent parameters, so

$$\partial_\nu T^\nu{}_\mu = 0 \qquad \text{for} \qquad T^\nu{}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^i)} \partial_\mu \phi^i - \delta^\nu{}_\mu \mathcal{L}$$

$$E = \int d^3x \; T^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot\phi^i} \dot\phi^i - \mathcal{L} \right)$$

$$p^i = \int d^3x \; T^{0j} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot\phi^i} \partial^j \phi^i \right)$$

Comment: (class)

– $\bar{j}^a_\mu \equiv j^a_\mu + \partial^\nu f^a_{\mu\nu}$ $\qquad f^a_{\mu\nu} = - f^a_{\nu\mu}$ $\quad (*)$

$$\Downarrow$$

$$\partial^\mu \bar{j}^a_\mu = 0$$

– $\bar{T}^\nu{}_\mu = T^\nu{}_\mu + \partial_\lambda \Omega^{\nu\lambda}{}_\mu$ $\qquad (**)$

$$\Omega^{\nu\lambda}{}_\mu = - \Omega^{\lambda\nu}{}_\mu \Rightarrow \partial_\nu \bar{T}^\nu{}_\mu = 0$$

– $T^{\mu\nu} \neq T^{\nu\mu}$ but one can construct symmetric tensor $\bar{T}^{\mu\nu}$ which is also conserved

Show (claim) that the time-independent charge 22a

$\partial_\mu j^\mu = 0$ remain the same after (*) and (**).

$$\bar{j}^\mu = j^\mu + \partial_\sigma f^{\sigma\mu} \qquad f^{\sigma\mu} = -f^{\mu\sigma}$$

$$\bar{Q} := \int_V d^3x\, \bar{j}^0 \qquad \frac{d\bar{Q}}{dt} = \int_V d^3x\, \partial_t \bar{j}^0 =$$

$$= \int d^3x \left(\partial_t j^0 + \partial_t \partial_\sigma f^{\sigma 0}\right) =$$

$$= \frac{dQ}{dt} + \int d^3x \left(\partial_t \partial_t \underbrace{f^{00}}_{=0} + \partial_i \partial_t f^{i0}\right) =$$

$$= \frac{dQ}{dt} + \underbrace{\int_V \nabla \cdot \bar{F}\, d^3x}_{=0} \qquad \text{for} \quad F^i \equiv \partial_t f^{i0}$$

The same works for the energy-momentum tensor

Show that the canonical energy-momentum tensor can always be made symmetric by

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\sigma \Omega^{\sigma\mu\nu}.$$

Construct $\Omega^{\sigma\mu\nu}$

$$\bar{T}^{\mu\nu} = T^{\mu\nu} + \partial_\sigma \Omega^{\sigma\mu\nu} \qquad \Omega^{\sigma\mu\nu} = -\Omega^{\mu\sigma\nu}$$

$$\partial_\mu \bar{T}^{\mu\nu} = 0$$

$$\bar{T}^{\mu\nu} = \bar{T}^{\nu\mu}$$

$$T^{\mu\nu} + \partial_\sigma \Omega^{\sigma\mu\nu} = T^{\nu\mu} + \partial_\sigma \Omega^{\sigma\nu\mu}$$

$$T^{\mu\nu} - T^{\nu\mu} = \partial_\sigma \Omega^{\sigma\nu\mu} - \partial_\sigma \Omega^{\sigma\mu\nu} = \partial_\sigma\left(\Omega^{\sigma\nu\mu} - \Omega^{\sigma\mu\nu}\right) =$$

$$= \partial_\sigma \Omega^{\sigma[\nu\mu]} \cdot 2$$

Take $\left(\Omega^{\sigma\nu\mu} - \Omega^{\sigma\mu\nu}\right) = x^\mu T^{\sigma\nu} - x^\nu T^{\sigma\mu}$,

then $\partial_\sigma\left(\Omega^{\sigma\nu\mu} - \Omega^{\sigma\mu\nu}\right) = \partial_\sigma\left(x^\mu T^{\sigma\nu} - x^\nu T^{\sigma\mu}\right) =$

$$= \delta_\sigma{}^\mu T^{\sigma\nu} + \underbrace{x^\mu \partial_\sigma T^{\sigma\nu}}_{=0} - \delta_\sigma{}^\nu T^{\sigma\mu} - \underbrace{x^\nu \partial_\sigma T^{\sigma\mu}}_{=0} =$$

$$= T^{\mu\nu} - T^{\nu\mu}$$

$$\therefore \quad \Omega^{\sigma\nu\mu} = \underbrace{\frac{1}{2}\left(x^\mu T^{\sigma\nu} - x^\nu T^{\sigma\mu}\right)}_{\equiv\, \Theta^{\sigma\nu\mu}} + \Psi^{\sigma\nu\mu}$$

for $\Psi^{\sigma\nu\mu} = \Psi^{\sigma\mu\nu}$

But recall that we also used

$$\Omega^{\sigma\mu\nu} = -\Omega^{\mu\sigma\nu}$$

that will allow to determine the symmetric piece

$$\frac{1}{2}\left(x^\nu T^{\sigma\mu} - x^\mu T^{\sigma\nu}\right) + \Psi^{\sigma\mu\nu} = -\frac{1}{2}\left(x^\nu T^{\mu\sigma} - x^\sigma T^{\mu\nu}\right) - \Psi^{\mu\sigma\nu}$$

$$\Theta^{\sigma\mu\nu} + \Psi^{\sigma\mu\nu} = -\Theta^{\mu\sigma\nu} - \Psi^{\mu\sigma\nu}$$

$$\Psi^{\sigma\mu\nu} + \Psi^{\mu\sigma\nu} = -\Theta^{\sigma\mu\nu} - \Theta^{\mu\sigma\nu}$$

So we must satisfy two conditions:

$$\Psi^{\sigma\mu\nu} + \Psi^{\mu\sigma\nu} = -\Theta^{\sigma\mu\nu} - \Theta^{\mu\sigma\nu} \qquad (*)$$

$$\Psi^{\sigma\nu\mu} = \Psi^{\sigma\mu\nu} \qquad (**)$$

Let's guess Ψ as $\Psi^{\sigma\nu\mu} = -\Theta^{\nu\sigma\mu} - \Theta^{\mu\sigma\nu}$

and check $(*)$

$$\text{l.h.s} = \underbrace{-\Theta^{\mu\sigma\nu} - \Theta^{\nu\sigma\mu}}_{\Psi^{\sigma\mu\nu}} \underbrace{-\Theta^{\sigma\mu\nu} - \Theta^{\nu\mu\sigma}}_{\Psi^{\mu\sigma\nu}} =$$

$$= -\Theta^{\sigma\mu\nu} - \Theta^{\mu\sigma\nu} - \underbrace{\left(\Theta^{\nu\sigma\mu} + \Theta^{\nu\mu\sigma}\right)}_{\overset{\shortparallel}{0}} = \text{r.h.s.}$$

since $\Theta^{\sigma\nu\mu} = -\Theta^{\sigma\mu\nu}$

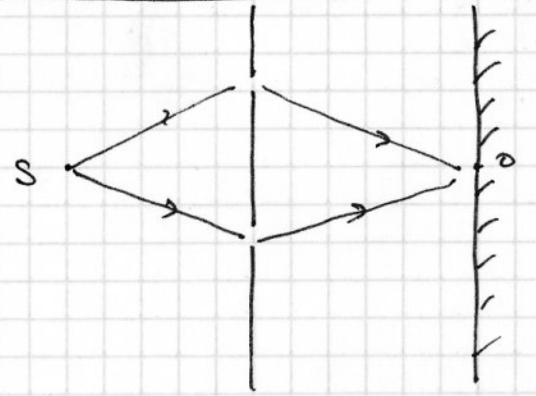
Obviously $(**)$ is satisfied.

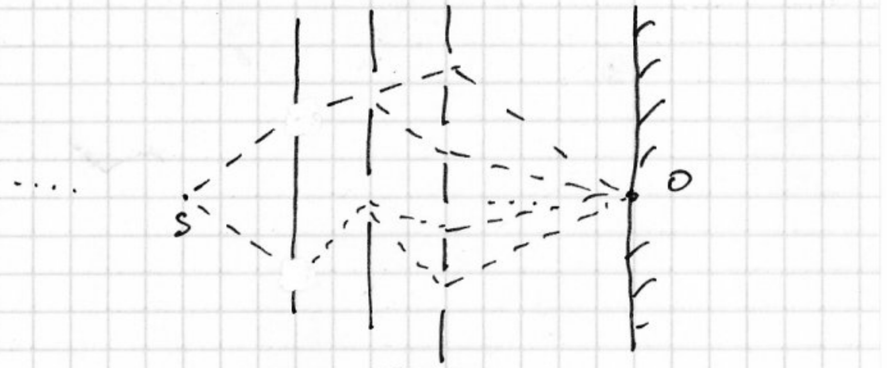
r $\Omega^{\sigma\nu\mu} = \Theta^{\sigma\nu\mu} - \Theta^{\nu\sigma\mu} - \Theta^{\mu\sigma\nu}$, $\quad \Theta^{\sigma\nu\mu} := \frac{1}{2}\left(x^\mu T^{\sigma\nu} - x^\nu T^{\sigma\mu}\right)$

$\bar{T}^{\mu\nu} = T^{\mu\nu} + \partial_\sigma \Omega^{\sigma\mu\nu}$ is symmetric

$$\bar{T}^{\mu\nu} = \bar{T}^{\nu\mu}$$

Path Integral Formulation of Quantum Physics





many screens

$$\mathcal{A}\left(\underset{S\to O}{\text{detection at } O}\right) = \sum_{\substack{i \\ \text{all possible holes}}} \mathcal{A}(S \to A_i \to O)$$ the superposition principle

– at the very end $\underbrace{\text{we remove all the screens}}_{\text{after drilling infinitely many holes}}$ as they are not really there

$$\Downarrow$$

$$\mathcal{A}(S \to O) = \sum_{\text{all paths}} \mathcal{A}(S \xrightarrow{\text{path}} O)$$

<u>Quantum mechanics</u>

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_S = \hat{H}(\hat{p}, \hat{q}) \,|\psi(t)\rangle_S \Rightarrow |\psi(t)\rangle_S = e^{-\frac{i}{\hbar}\hat{H}t} |\psi(0)\rangle_S$$

Schrödinger picture (states evolve)

in the Heisenberg picture the operators are time dependent, so for the position operator

$$-i\hbar \dot{\hat{q}}_H = [\hat{H}, \hat{q}_H]$$

and $\hat{q}_H(t) = e^{i\hat{H}t/\hbar} \hat{q}_S \, e^{-i\hat{H}t/\hbar}$

The time-dependent operator $\hat{q}_H(t)$ has a complete set of eigenstates :

$$\hat{q}_H(t) \,|q, t\rangle = q \,|q, t\rangle$$

$$|q,t\rangle_H = e^{i\hat{H}t/\hbar}|q\rangle\,;\quad \hat{q}_S|q\rangle = q|q\rangle$$

– we will be interested in the following transition amplitude

$$ {}_H\langle q_F, t_F | q_I, t_I\rangle_H = \langle q_F| e^{-iH(t_F-t_I)/\hbar}|q_I\rangle =$$

"Feynman kernel" $$= \langle q_F|e^{-iHT/\hbar}|q_I\rangle$$

– following Dirac we divide the time T into N segments $\delta t = \frac{T}{N}$, then (for $\hbar = 1$)

$$\langle q_f|e^{-iHT}|q_i\rangle = \langle q_f|e^{-iH\delta t}\cdots e^{-iH\delta t}|q_i\rangle$$

$$\int dq\; |q\rangle\langle q| = 1$$

$$= \prod_{j=1}^{N-1}\int dq_j\, \langle q_f|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle\cdots\langle q_1|e^{-iH\delta t}|q_i\rangle$$

let us consider factor $\langle q_{j+1}|e^{-iH\delta t}|q_j\rangle$

and assume free-particle Hamiltonian $H = \frac{\hat{p}^2}{2m}$

$$\hat{p}|p\rangle = p|p\rangle$$

– insert a complete set of states $\mathbb{1} = \int\frac{dp}{2\pi}|p\rangle\langle p|$

$$\langle q|p\rangle = e^{ipq}$$

$$\langle q_{j+1}|e^{-iH\delta t}|q_j\rangle = \langle q_{j+1}|e^{-i\frac{\hat{p}^2}{2m}\delta t}|q_j\rangle = \int\frac{dp}{2\pi}\langle q_{j+1}|e^{-i\frac{\hat{p}^2}{2m}\delta t}|p\rangle\langle p|q_j\rangle$$

$$= \int\frac{dp}{2\pi}\, e^{-i\frac{p^2}{2m}\delta t}\langle q_{j+1}|p\rangle\langle p|q_j\rangle = \underline{\int\frac{dp}{2\pi}\, e^{-i\frac{p^2}{2m}\delta t}\, e^{ip(q_{j+1}-q_j)}}$$

Gaussian integral

$$\int_{-\infty}^{+\infty}dx\; e^{-\frac{1}{2}ax^2+Jx} = \left(\frac{2\pi}{a}\right)^{1/2} e^{J^2/2a}$$

then
$$\langle q_{j+1}| e^{-iH\delta t} |q_j\rangle = \frac{1}{2\pi}\left(\frac{2\pi m}{i\delta t}\right)^{1/2} e^{\frac{m}{2i\delta t}(-)(q_{j+1}-q_j)^2} =$$

$$a = \frac{i\delta t}{m} \qquad J = i(q_{j+1}-q_j)$$

$$= \left(\frac{-im}{2\pi\delta t}\right)^{1/2} e^{i\delta t \frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2}$$

Putting all those factors together one gets

$$\langle q_f| e^{-iHT} |q_i\rangle = \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}} \prod_{j=0}^{N-1}\int dq_j \, e^{i\delta t \frac{m}{2}\sum_{j=0}^{N-1}(q_{j+1}-q_j)^2/\delta t^2}$$

with $q_0 \equiv q_i$, $q_N \equiv q_f$

Go to the limit $\delta t \to 0$: $\left(\frac{q_{j+1}-q_j}{\delta t}\right)^2 \to \dot{q}^2$

$$\delta t \sum_{j=0}^{N-1} \to \int_0^T dt$$

Let's define the path integral

$$\int \mathcal{D}q(t) = \lim_{N\to\infty}\left(\frac{-im}{2\pi\delta t}\right)^{N/2}\prod_{j=0}^{N-1} dq_j$$

Then

$$\langle q_f| e^{-iHT} |q_i\rangle = \int \mathcal{D}q(t) \underbrace{e^{i\int_0^T dt\, \frac{1}{2}m\dot{q}^2}}_{\substack{\text{the amplitude for each}\\ \text{path recall the}\\ \text{superposition principle}}}$$

For $H = \frac{\hat{P}^2}{2m} + V(q)$

$$\langle q_f| e^{-iHT} |q_i\rangle = \int \mathcal{D}q(t)\, e^{\underbrace{i\int_0^T \underbrace{\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]}_{=L} dt}_{S[q(t)]}}$$

For $\mathcal{D}q(t)$ to emerge it is crucial that $[p,q]\neq 0$.

In the field theory we will often be interested in transition amplitudes between vacuum states

$$Z \equiv \langle 0| e^{-iHT} |0\rangle = \int \mathcal{D}q(t)\, e^{iS[q(t)]}$$

vacuum $\equiv$ ground state $\equiv$ state of the lowest energy

$\langle 0 | \cdots | 0 \rangle$ transitions are not the most interesting, we will consider most vacuum – vacuum transitions in the presence of some external disturbance (like an external force in mechanics : $J_a(t)\, q_a(t)$ extra in the Lagrangian) :

$$Z = \int D\varphi(x)\, e^{i\int d^4x \left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi) + J(x)\varphi(x)\right]}$$

classical mechanics		field theory
$q_a(t)$	$\rightarrow$	$\varphi(\bar{x}) = \varphi(x)$
a	$\rightarrow$	$\bar{x}$
$\sum_a$	$\rightarrow$	$\int d^D x$

Let us consider the free scalar field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

so

$$Z = \int D\varphi\, e^{i\int d^4x \left[-\frac{1}{2}\varphi\left(\partial^2+m^2\right)\varphi + J\varphi\right]}$$

$\varphi \to 0$, $x \to \infty$, therefore the boundary terms where neglected

Let us imagine latticizing (discretizing) space-time :

$$\varphi(x) \rightarrow \varphi_i = \varphi(i\overset{\text{lattice spacing}}{a})$$

$$\partial\varphi \rightarrow \frac{1}{a}\left(\varphi_{i+1} - \varphi_i\right) \equiv \sum_j M_{ij}\,\varphi_j$$

$$\int d^4x\, J(x)\varphi(x) \rightarrow a^4 \sum_j J_j\,\varphi_j$$

The integrals to perform:

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dq_1 \ldots dq_N \; e^{\frac{i}{2} q \cdot A \cdot q + i J q} = \left[\frac{(2\pi i)^N}{\det A} \right]^{\frac{1}{2}} e^{-\frac{i}{2} J \cdot A^{-1} \cdot J}$$

$$A A^{-1} = \mathbb{1}$$

$$A \leftrightarrow -(\partial^2 + m^2)$$

$$-(\partial^2 + m^2)\, D(x-y) = \delta^{(4)}(x-y)$$

$$Z[J] = C\, e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\, D(x-y)\, J(y)} \equiv C\, e^{i W[J]}$$

$$Z[J] = Z[0]\, e^{i W[J]}$$

$$W[J] = -\frac{1}{2} \int d^4x\, d^4y\, J(x)\, D(x-y)\, J(y)$$

In order to have convergent integrals : $m^2 \to m^2 - i\varepsilon$

then the factor $e^{-\frac{\varepsilon}{2}\int d^4x\, \varphi^2}$ $\quad \left(\frac{\varepsilon}{2} \equiv \varepsilon \right)$

suppresses the integral if φ is large

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}$$

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4}\, D(k)\, e^{ik(x-y)}$$

$$-(\partial^2 + m^2)\, D(x-y) = \delta^{(4)}(x-y)$$

$$-\int \frac{d^4k}{(2\pi)^4}\, D(k) \left(-k^2 + \underset{-i\varepsilon}{m^2} \right) e^{ik(x-y)} \overset{!}{=} \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}$$

$$D(k) = \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$W[J] = -\frac{1}{2}\int d^4x\, d^4y\; J(x)\, D(x-y)\, J(y)$$

$$\int d^4x\, e^{-ik'x} \;\Big|\qquad J(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, J(k)$$

$$\int d^4x\, e^{-ik'x} J(x) = \int \frac{d^4k}{(2\pi)^4}\int d^4x\, e^{i(k-k')x}\, J(k) =$$

$$= J(k')$$

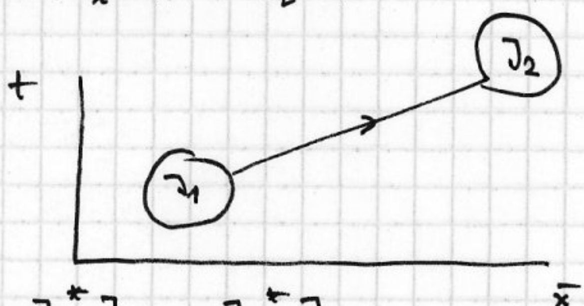
$$W[J] = -\frac{1}{2}\int d^4x\, d^4y \int \frac{d^4k}{(2\pi)^4}\, e^{ikx} J(k) \int \frac{d^4l}{(2\pi)^4}\, e^{il(x-y)} D(l) \int \frac{d^4p}{(2\pi)^4}\, e^{ipy} J(p) =$$

$$= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, d^4l\, d^4p\; \delta^{(4)}(k+l)\, \delta^{(4)}(p-l)\, J(k)\, D(l)\, J(p) =$$

$$= -\frac{1}{2}\int \frac{d^4l}{(2\pi)^4}\, J(-l)\, D(+l)\, J(l) = -\frac{1}{2}\int \frac{d^4l}{(2\pi)^4}\, J^*(l)\, D(l)\, J(l)$$

since $J(x)$ is real we have $J(-l) = J^*(l)$

Consider $\quad J(x) = J_1(x) + J_2(x)$



$$J_1^* J_1, \quad \underbrace{J_1^* J_2, \quad J_2^* J_1}_{\text{let us consider these}}, \quad J_2^* J_2$$

$$W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \underset{1}{J_2^*(k)}\, \frac{1}{k^2 - m^2 + i\varepsilon}\, \underset{2}{J_1(k)}$$

the main contribution to the integral comes from $k^2 \cong m^2$ so from an exchange of a real (on-shell) particle, which was created at e.g. x_1 the region where $J_1(x)$ is concentrated and annihilated at x_2 where $J_2(x)$ is located

We choose $J_a(x) = \delta^{(3)}(\bar{x} - \bar{x}_a) \qquad a = 1, 2$

since we are interested in interactions between the two sources we neglect $J_a^* J_a$ terms

$$W[J] = -\int dx^0 dy^0 \int \frac{dk^0}{2\pi} e^{ik^0(x^0-y^0)} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}(\bar{x}_1-\bar{x}_2)}}{k^2 - m^2 + i\varepsilon}$$

$J_2^* J_1 + J_1^* J_2$

$$J_2^*(k)D(k)J_1(k) = \int d^4x \, e^{ikx} \delta^{(3)}(\bar{x}-\bar{x}_2) \, d^4y \, e^{-iky} \delta^{(3)}(\bar{y}-\bar{x}_1) \, D(k) =$$

$$\int dx^0 dy^0 \, e^{ik^0(x^0-y^0)} \, e^{-i\bar{k}(\bar{x}_2-\bar{x}_1)}$$

$$\int dy^0 \, e^{-ik^0y^0} = 2\pi \, \delta(k^0) \qquad \text{so}$$

$$W[J] = +\left(\int dx^0\right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}(\bar{x}_1-\bar{x}_2)}}{\bar{k}^2 + m^2}$$

$i\varepsilon$ not needed since $\bar{k}^2 + m^2 \neq 0$

Remember that

$$Z[J] = C \, e^{iW[J]} = \langle 0 | e^{-iHT} | 0 \rangle =$$

see Weinberg vol. II sec. 16.3

$$= e^{-iET}$$ where E is the vacuum energy due to the presence of the two sources acting on each other

$$-iE\cancel{T} = iW[J] = i\left(\cancel{\int dx^0}\right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}(\bar{x}_1-\bar{x}_2)}}{\bar{k}^2 + m^2}$$

$$E = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k}(\bar{x}_1-\bar{x}_2)}}{\bar{k}^2 + m^2}$$

E is the potential energy between two static sources

$$E = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2}$$

- oscillations cut off the integral for large $|\vec{x}_1-\vec{x}_2|$

- since $\frac{1}{m}$ is the only scale in the problem we expect E to be strongly suppressed for $|\vec{x}_1-\vec{x}_2| \gtrsim \frac{1}{m}$

$\Leftarrow$ m determines the range of the attractive force

Direct calculation shows that

$$E = -\frac{1}{4\pi r} e^{-mr} \qquad \text{for } r = |\vec{x}_1-\vec{x}_2|$$

$$\frac{dE}{dr} > 0 \implies \text{attraction !}$$

<u>Conclusion</u>

An exchange of scalar quanta leads to an attractive force (if sources are identical!)

Electrodynamics : why like charges repel ?

Between like objects Coulomb's force is repulsive.

Newtonian gravity : Newton's gravitational force is attractive

Can we understand that within a quantum field theory ?

Let us try.

In order to simplify arguments we will assume that photon has a tiny mass, then for QED

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu$$

external source: conserved current: $\partial_\mu J^\mu = 0$

in analogy with $\frac{1}{2} m^2 \varphi^2$ in the scalar theory

$$Z \equiv e^{iW[J]} = \int DA_\mu \, e^{iS[A_\mu]} \quad \text{for}$$

$$S[A_\mu] = \int d^4x \, \mathcal{L} = \int d^4x \left\{ \frac{1}{2} A_\mu \left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\nu + A_\mu J^\mu \right\}$$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) =$$

$$= -\frac{1}{4} \left(2 \partial_\mu A_\nu \partial^\mu A^\nu - 2 \partial_\mu A_\nu \partial^\nu A^\mu \right) \rightarrow \text{by parts}$$

$$\rightarrow \frac{1}{2} A_\nu \underset{\partial^2}{\underset{\parallel}{\Box}} A^\nu - \frac{1}{2} A_\nu \partial_\mu \partial^\nu A^\mu = \frac{1}{2} A_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu$$

find the potential energy we need the photon propagator:

$$\left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] D_{\nu\lambda}(x) = \delta^\mu{}_\lambda \, \delta^{(4)}(x)$$

in the momentum space we have

$$D_{\nu\lambda}(x) = \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k) \, e^{ikx} \quad \text{and therefore}$$

$$\left[(-k^2 + m^2) g^{\mu\nu} + k^\mu k^\nu \right] D_{\nu\lambda}(k) = \delta^\mu{}_\lambda \quad \text{then} \quad (*)$$

$$D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + \frac{k_\nu k_\lambda}{m^2}}{k^2 - m^2}$$

inserting into we can see that this is indeed the solution

So, for $W[J]$ we get

$$W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, J^\mu(k)^* \, \frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}}{\underbrace{k^2 - m^2 + i\varepsilon}} \, J^\nu(k)$$

The current J_μ is conserved, so $\partial^\mu J_\mu(x) = 0$

$$\Downarrow$$

$\frac{k_\mu k_\nu}{m^2}$ could be dropped $\Leftarrow$ $k^\mu J_\mu(k) = 0$

$$\Downarrow$$

$$W[J] = +\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, J^\mu(k)^* \, \frac{1}{k^2 - m^2 + i\varepsilon} \, J_\mu(k)$$

the spin structure (Lorentz transformation properties) produce the extra "−" sign!

$$\Downarrow$$

$J^\mu = (J^0, \vec{0})$

$\frac{dE}{dr} < 0$ The electromagnetic force between like charges is repulsive!

$E = \varepsilon^2 \frac{1}{4\pi r} e^{-mr}$

$\xrightarrow{m \to 0}$ $\varepsilon^2 \frac{1}{4\pi r}$

$J^0(x) = \varepsilon\, \delta^{(3)}(x - x_0)$

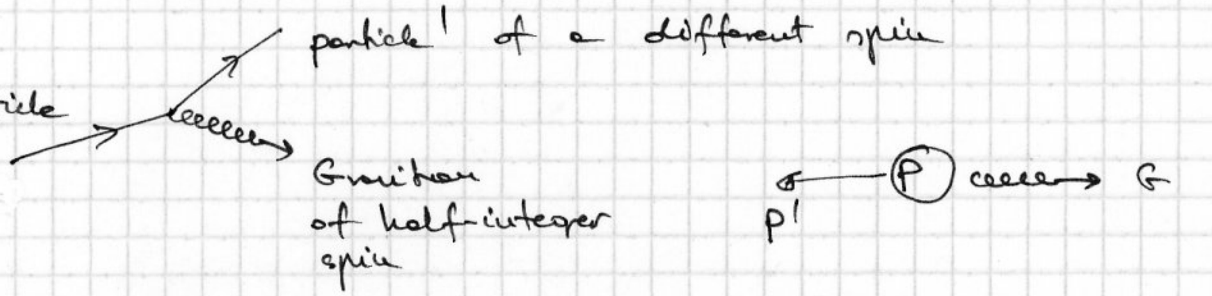
$\uparrow$ charge density

To accomodate positive and negative charges we shall use $J^\mu = J_+^\mu - J_-^\mu$, then we will see that

for $J_+^0 J_-^0$ the energy is negative $E < 0$, so opposite charges attract each other

Let us try gravity now: the goal is to explain $F = -\frac{G m_1 m_2}{r}$

Feynman : " In order to produce a static force and not just scattering, the emission or absorption of a single graviton by either particle must leave both particles in the same internal state "



$\Downarrow$

gravity can't be mediated by an exchange of half-integer graviton

- $s = 0 \Rightarrow$ attractive, so could φ mediate gravity?

assume that gravity couples to the energy-momentum tensor $T_{\mu\nu}$, then we would get

$\varphi T_\mu{}^\mu$ as the only Lorentz invariant possibility

however for QED we have

$$T_\mu{}^\mu = 0 \quad \Leftarrow \quad T_{\mu\nu} = - F_{\mu\lambda} F_\nu{}^\lambda + \frac{1}{4}\eta_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}$$

$$\eta^{\mu\nu} T_{\mu\nu} = - F_{\mu\lambda}{}^{\mu\lambda} + F_{\lambda\rho} F^{\lambda\rho} = 0$$

$\Downarrow$

for $\varphi T_\mu{}^\mu$ there would be no gravity-light interactions, but bending of light is observed! in ~~presence~~ the vicinity of Sun

Conclusion As electromagnetic interactions need $s=1$, the next option for gravity is $s=2$

- to calculate the potential energy we need the graviton propagator!
- let us rederive the propagator for a massive vector field

$$D_{\nu\lambda}(k) = \frac{-G_{\nu\lambda}}{k^2-m^2} \qquad G_{\nu\lambda} \equiv g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2}$$

Let us understand the physics behind $G_{\nu\lambda}$.

A massive spin 1 particle has 3 polarization vectors:

$$k^\mu = (m,0,0,0) \qquad k^\mu \varepsilon^a_\mu = 0 \qquad a=1,2,3$$

(in the rest frame, the spin vector can point in three different directions)

$$\varepsilon^1_\mu = (0,1,0,0)$$
$$\varepsilon^2_\mu = (0,0,1,0)$$
$$\varepsilon^3_\mu = (0,0,0,1)$$

a possible choice

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{a=1}^{3} \varepsilon^a_\mu(k)\left(a^a(k)e^{-ikx} + a^{a+}(k)e^{ikx}\right)$$

$\partial^\mu A_\mu = 0$

consequence of the E-L equations for $m \neq 0$

$$\underbrace{[(\partial^2+m^2)g^{\mu\nu} - \partial^\mu\partial^\nu]A_\nu = 0}$$

- the amplitude for a particle to be created at the source is proportional to ε^a_λ
- the same for the absorption $\propto \varepsilon^a_\nu$

$$\sum_a \varepsilon^a_\nu \varepsilon^a_\lambda$$ – sum (coherent) over a

$\Downarrow$

we expect that $\sum_a \varepsilon^a_\nu \varepsilon^a_\lambda \propto G_{\nu\lambda}$

From the Lorentz covariance:

$$k^\nu \;\Big|\; \sum_a \varepsilon^a_\nu \varepsilon^a_\lambda = A\, g_{\nu\lambda} + B\, k_\nu k_\lambda$$

$$\hookrightarrow \sum_a \varepsilon^a_\nu \varepsilon^a_\lambda = -Bm^2\left(g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2}\right)$$

choose $\lambda = \nu = 1$

$$\sum_a \varepsilon_1^a \varepsilon_1^a = 1 = -Bm^2(-1-0) = Bm^2$$

$$\sum_a \varepsilon_\nu^a \varepsilon_\lambda^a = -G_{\nu\lambda} = -\left(g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2}\right)$$

Having the experience from the massive photon propagator let's consider the graviton propagator.

$s = 2 \Rightarrow 2s+1 = 5$ degrees of freedom

$\Downarrow$

5 polarization tensors $\varepsilon^a_{\mu\nu}$ $a = 1, \dots 5$

- $\varepsilon^a_{\mu\nu} = \varepsilon^a_{\nu\mu}$ (symmetric tensor describes $s = 2$)
- $k^\mu \varepsilon^a_{\mu\nu} = 0$ (transverse)
- $g^{\mu\nu} \cdot \varepsilon^a_{\mu\nu} = 0$ (tracelessness)

$\Leftarrow$ equations of motion for spin 2, $h_{\mu\nu}$ - graviton field with $m \neq 0$

degrees of freedom: $\underbrace{10}_{\text{symmetric tensor}} - \underbrace{4}_{\partial^\mu h_{\mu\nu}=0} - \underbrace{1}_{g^{\mu\nu} h_{\mu\nu}=0} = 5$!

The numerator of the propagator (in analogy with $s=1$)

$$\sum_a \varepsilon^a_{\mu\nu}(k)\, \varepsilon^a_{\lambda\sigma}(k) = G_{\mu\lambda} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\lambda} - \frac{2}{3} G_{\mu\nu} G_{\lambda\sigma}$$

$\Downarrow$ $\leftarrow$ homework

$$D_{\mu\nu,\lambda\sigma}(k) = \frac{G_{\mu\lambda} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\lambda} - \frac{2}{3} G_{\mu\nu} G_{\lambda\sigma}}{k^2 - m^2}$$

Assume that the gravity (gravitons) couples to $T^{\mu\nu}$

$$h^{\mu\nu} T_{\mu\nu}$$

$\leftarrow$ energy momentum tensor

Let us find the potential energy :

$$W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{\mu\nu}(k)^* \frac{G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda} - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2 - m^2 + i\varepsilon} T^{\lambda\sigma}(k)$$

$$\partial_\mu T^{\mu\nu} = 0 \implies G_{\mu\nu} \to g_{\mu\nu}$$

$$\hookrightarrow k_\mu T^{\mu\nu}(k) = 0$$

$$G_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$$

For pure energy sources only $T^{00} \neq 0$ then

$$W[T] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{00}(k)^* \frac{1 + 1 - \frac{2}{3}}{k^2 - m^2} T^{00}(k)$$

$$\Downarrow$$

since $1 + 1 - \frac{2}{3} > 0$ therefore two masses attract!

- For a particle at $\bar{x} = \bar{x}_0$ $\quad T^{\alpha\beta} = \frac{p^\alpha p^\beta}{E}\delta^3(\bar{x} - \bar{x}_0)$
- if the particle is at rest $T^{00} = p^0 \delta^3(\bar{x} - \bar{x}_0) =$
$= m\,\delta^3(\bar{x} - \bar{x}_0)$

The Lagrangian for the gravitational field

QED: $S_{QED} = \int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right) d^4x$, spin 1 $\Rightarrow$ A_μ

Gravity: • spin 2 $\Rightarrow$ described by symmetric tensor $h_{\mu\nu}$

• our guess (in analogy to $J^\mu A_\mu$) for the interaction term: $-\lambda h_{\mu\nu} T^{\mu\nu}$

• kinetic term is however difficult to guess because of too many indices:

$$\mathcal{L} = a\, \partial_\sigma h^{\mu\nu} \partial^\sigma h_{\mu\nu} + b\, \partial^\nu h^{\mu\sigma} \partial_\sigma h_{\mu\nu} + c\, \partial^\nu h^\mu{}_\mu \partial_\sigma h^\sigma{}_\nu + \dots$$

For QED we had

$$\delta S_{QED} = 0 \Rightarrow \left. \begin{array}{l} \partial^\nu F_{\nu\mu} = -J_\mu \\ \partial^\nu \partial_\nu A_\mu - \partial^\nu \partial_\mu A_\nu = -J_\mu \end{array} \right\} \overset{\text{notation}}{\Longrightarrow} A_{\mu,\nu}{}^{|\nu} - A_{\nu,\mu}{}^{|\nu} = -J_\mu$$

Note that $A_{\mu,\nu}{}^{|\nu\mu} - A_{\nu,\mu}{}^{|\nu\mu} = 0$ is an identity!

$\Downarrow$

current conservation: $J_\mu{}^{|\mu} = 0$

For gravity a similar argument could be adopted to constrain the coefficients $a, b, c, ..$ relative to each other: • let's write gravitational e.o.m. as

$$G_{\mu\nu} \propto T_{\mu\nu}$$

from the kinetic term, contain $a, b, c, \dots$ — from $\lambda h_{\mu\nu} T^{\mu\nu}$ in $\mathcal{L}_g$

• since $T^{\mu\nu}{}_{,\nu} = 0$ (energy-momentum conservation)

therefore we expect that $G_{\mu\nu}{}^{|\nu}$ vanishes identically

The equations for the gravitational field

We try to construct the kinetic terms for the symmetric tensor $h_{\mu\nu}$:

- if the two tensor indices are different from the derivative index we have:

  1. $h_{\mu\nu,\sigma}\, h^{\mu\nu,\sigma}$
  2. $h_{\mu\nu,\sigma}\, h^{\mu\sigma,\nu}$

- if there are two indices which are equal:

  3. $h^{\mu\nu}{}_{,\nu}\, h^{\sigma}{}_{\mu,\sigma}$
  4. $h^{\mu\nu}{}_{,\nu}\, h^{\sigma}{}_{\sigma,\mu}$
  5. $h^{\nu}{}_{\nu,\mu}\, h_{\sigma}{}^{\sigma,\mu}$

- but 2 can be converted to 3 by integration by parts:

$$h_{\mu\nu,\sigma}\, h^{\mu\sigma,\nu} \to -h_{\mu\nu,\sigma}{}^{,\nu}\, h^{\mu\sigma} \to -- \underbrace{h_{\mu\nu}{}^{,\nu}}_{h_{\nu\mu}{}^{,\nu} = h^{\nu}{}_{\mu,\nu}}\, h^{\mu\sigma}{}_{,\sigma}$$

$\square$

$$S_g = \int d^4x \Big[\underset{1}{a\, h^{\mu\nu,\sigma}\, h_{\mu\nu,\sigma}} + \underset{3}{b\, h^{\mu\nu}{}_{,\nu}\, h^{\sigma}{}_{\mu,\sigma}} +$$

$$+ \underset{4}{c\, h^{\mu\nu}{}_{,\nu}\, h^{\sigma}{}_{\sigma,\mu}} + \underset{5}{d\, h^{\nu}{}_{\nu,\mu}\, h_{\sigma}{}^{\sigma,\mu}} - \lambda T^{\mu\nu} h_{\mu\nu} \Big]$$

$\delta S_g = 0 \Rightarrow$ e.o.m. for $h_{\mu\nu}$

$$\delta S_g = \int d^4x \left(\frac{\partial \mathcal{L}_g}{\partial h_{\mu\nu}} \delta h_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial h_{\mu\nu,\sigma}} \delta\, h_{\mu\nu,\sigma} + \frac{\partial \mathcal{L}}{\partial h^{\sigma}{}_{\mu,\sigma}} \delta\, h^{\sigma}{}_{\mu,\sigma} + \dots \right) = 0$$

$$a\,2 h_{\alpha\beta,\sigma}{}^{,\sigma} + b\left(h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma}\right) + c\left(h^{\sigma}{}_{\sigma,\alpha\beta} + \eta_{\alpha\beta} h^{\mu\nu}{}_{,\nu\mu}\right) + \qquad (*)$$

$$+ 2d\, \eta_{\alpha\beta} h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda T_{\alpha\beta}$$ E-L equation for $h_{\mu\nu}$

a-term : $$\frac{\partial \mathcal{L}}{\partial h_{\mu\nu,\sigma}} \delta h_{\mu\nu,\sigma} = 2a\, h^{\mu\nu,\sigma} \delta h_{\mu\nu,\sigma}$$

$\downarrow$ by parts

$$-2a\, h^{\mu\nu,\sigma}{}_{,\sigma}\, \delta h_{\mu\nu} = -2a\, h_{\alpha\beta,\sigma}{}^{,\sigma}\, \delta h^{\alpha\beta}$$

b-term : $$\frac{\partial}{\partial h^{\sigma}{}_{\mu,\sigma}} \Big(\underbrace{b\, h^{\mu\nu}{}_{,\nu}\, h^{\sigma}{}_{\mu,\sigma}}_{\eta^{\mu\rho} h^{\nu}{}_{\rho,\nu} h^{\sigma}{}_{\mu,\sigma}} \Big) \cdot \delta h^{\sigma}{}_{\mu,\sigma} =$$ by parts

$$= -2b\, h^{\rho\mu}{}_{,\rho}{}^{,\nu}\, \delta h_{\mu\nu} = -\cancel{2} b \frac{1}{\cancel{2}} \left(h^{\rho\mu}{}_{,\rho}{}^{,\nu} + h^{\rho\nu}{}_{,\rho}{}^{,\mu} \right) \delta h_{\mu\nu} =$$

$(\mu,\nu) \to (\alpha,\beta)$

$\rho \to \sigma$

$$= -b\left(h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma} \right) \delta h^{\alpha\beta}$$

only symmetric part contributes

:

etc.

Now let's calculate of $(*)$ with respect to the index β and require that it vanishes identically:

$$2a\, h^{\alpha\beta,\sigma}{}_{,\sigma\beta} + b\, h^{\alpha\sigma,\beta}{}_{,\sigma\beta} + b\, h^{\beta\sigma,\alpha}{}_{,\sigma\beta} + c\, h^{\sigma}{}_{\sigma}{}^{,\alpha\beta}{}_{,\beta} +$$

$$+ c\, h^{\mu\nu}{}_{,\mu\nu}{}^{,\alpha} + 2d\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu\alpha} = 0$$ since $T^{\alpha\beta}{}_{,\beta} = 0$

$$(2a+b)\, h^{\alpha\beta,\sigma}{}_{,\sigma\beta} + (c+2d)\, h^{\sigma}{}_{\sigma}{}^{,\alpha\beta}{}_{,\beta} + (b+c)\, h^{\beta\sigma,\alpha}{}_{,\sigma\beta} = 0$$

independent tensors

$2a + b = 0$

$c + 2d = 0$ $\qquad\qquad$ $d = -\frac{1}{2}$

$$\mathcal{L}_g = \frac{1}{2} h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\sigma,\mu} - \frac{1}{2} h^{\nu}{}_{\nu,\mu} h^{\sigma}{}_{\sigma}{}^{,\mu} +$$
$$- \lambda T^{\mu\nu} h_{\mu\nu}$$

Let us define for arbitrary second rank tensor

$$\bar{X}_{\mu\nu} := \frac{1}{2}\left(X_{\mu\nu} + X_{\nu\mu}\right) - \frac{1}{2}\eta_{\mu\nu} X^{\sigma}{}_{\sigma}$$

For a symmetric tensor such as $h_{\mu\nu}$ we have

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\sigma}{}_{\sigma} \quad \text{and} \quad \bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$$

$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{\sigma}{}_{\sigma} = h_{\mu\nu}$$

Define $h = h^{\sigma}{}_{\sigma}$ (trace of $h^{\mu}{}_{\nu}$) $\underbrace{\; = -\bar{h}^{\sigma}{}_{\sigma}\;}$

and note that $\bar{h}^{\sigma}{}_{\sigma} = -h$.

Let us rewrite e.o.m. in terms of the "bar" $T_{\alpha\beta}$

$$h_{\alpha\beta,\sigma}{}^{,\sigma} - \left(h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma}\right) + \left(h^{\sigma}{}_{\sigma,\alpha\beta} + \eta_{\alpha\beta} h^{\mu\nu}{}_{,\nu\mu}\right) + \qquad (*)$$
$$- \eta_{\alpha\beta} h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda T_{\alpha\beta}$$

, first contract $(*)$ with $\eta^{\alpha\beta}$

$$h^{\alpha}{}_{\alpha,\sigma}{}^{,\sigma} - 2 h_{\alpha\sigma}{}^{,\alpha\sigma} + h^{\sigma}{}_{\sigma,\alpha}{}^{,\alpha} + 4 h^{\mu\nu}{}_{,\mu\nu} - 4 h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda T_{\alpha}{}^{\alpha} \quad \Big| \times \left(-\frac{1}{2}\eta_{\alpha\beta}\right)$$

and add to $(*)$

$$\boxed{h_{\alpha\beta,\sigma}{}^{,\sigma} - h_{\alpha\sigma,\beta}{}^{,\sigma} - h_{\beta\sigma,\alpha}{}^{,\sigma} + h^{\sigma}{}_{\sigma,\alpha\beta} = -\lambda \bar{T}_{\alpha\beta}}$$

The graviton equation of motion

40 a

$$h_{\alpha\beta,\sigma}{}^{,\sigma} - \left(h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma}\right) + h^{\sigma}{}_{\sigma,\alpha\beta} + \eta_{\alpha\beta}\, h^{\mu\nu}{}_{,\mu\nu} - \eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda\, T_{\alpha\beta} \qquad |\eta$$

$$h^{\alpha}{}_{\alpha,\sigma}{}^{,\sigma} - 2 h_{\alpha\sigma}{}^{,\alpha\sigma} + h^{\sigma}{}_{\sigma,\alpha}{}^{,\alpha} + 4 h^{\mu\nu}{}_{,\mu\nu} - 4 h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda T_{\alpha}{}^{\alpha} \qquad |\times\left(-\tfrac{1}{2}\eta_{\alpha\beta}\right.$$

$$-\tfrac{1}{2}\eta_{\alpha\beta}\, h^{\mu}{}_{\mu,\sigma}{}^{,\sigma} + \eta_{\alpha\beta}\, h_{\mu\sigma}{}^{,\mu\sigma} - \tfrac{1}{2}\eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} - 2\eta_{\alpha\beta}\, h^{\mu\nu}{}_{,\mu\nu} + 2\eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = +\tfrac{1}{2}\lambda\,\eta_{\alpha\beta}\, T_{\alpha}{}^{\alpha}$$

$1 + 3 \Downarrow$

$$h_{\alpha\beta,\sigma}{}^{,\sigma} - \tfrac{1}{2}\eta_{\alpha\beta}\, h^{\mu}{}_{\mu,\sigma}{}^{,\sigma} - 2\bar{h}_{\alpha\sigma,\beta}{}^{,\sigma} + h^{\sigma}{}_{\sigma,\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} - \eta_{\alpha\beta}\, h^{\mu\nu}{}_{,\mu\nu} + \eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda\, \bar{T}_{\alpha\beta}$$

✓ ✓

$$-2\bar{h}_{\alpha\sigma,\beta}{}^{,\sigma} - \eta_{\alpha\beta}\, h^{\mu\nu}{}_{,\mu\nu} = -2\left[\tfrac{1}{2}\left(h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma}\right) - \tfrac{1}{2}\eta_{\alpha\beta}\, h_{\alpha\sigma}{}^{,\alpha\sigma}\right] - \eta_{\alpha\beta}\, h^{\mu\nu}{}_{,}$$

$$= -h_{\alpha\sigma,\beta}{}^{,\sigma} - h_{\beta\sigma,\alpha}{}^{,\sigma}$$

$$h_{\alpha\beta,\sigma}{}^{,\sigma} - h_{\alpha\sigma,\beta}{}^{,\sigma} - h_{\beta\sigma,\alpha}{}^{,\sigma} + h^{\sigma}{}_{\sigma,\alpha\beta} = -\lambda\, \bar{T}_{\alpha\beta}$$

gravity : $h_{\alpha\beta,\sigma}{}^{,\sigma} - h_{\alpha\sigma,\beta}{}^{,\sigma} - h_{\beta\sigma,\alpha}{}^{,\sigma} + h^{\sigma}{}_{\sigma,\alpha\beta} = -\lambda \bar{T}_{\alpha\beta}$

QED : $A_{\alpha,\beta}{}^{,\beta} - A^{\beta}{}_{,\alpha\beta} = -J_\alpha$ invariant under $A_\alpha \to A_\alpha' = A_\alpha + X_{,\alpha}$

For gravity we guess in analogy to

$$h_{\alpha\beta} \to h_{\alpha\beta}' = h_{\alpha\beta} + \underbrace{X_{\alpha,\beta} + X_{\beta,\alpha}}_{\text{symmetric}} \qquad \text{gauge transformation}$$

Is the e.o.f. invariant under $h_{\alpha\beta} \to h'_{\alpha\beta}$?

$$h_{\alpha\beta,\sigma}{}^{,\sigma} \to h_{\alpha\beta,\sigma}{}^{,\sigma} + \underbrace{(X_{\alpha,\beta} + X_{\beta,\alpha})_{,\sigma}{}^{,\sigma}}_{= X_{\alpha,\beta\sigma}{}^{,\sigma} + X_{\beta,\alpha\sigma}{}^{,\sigma}}$$

$$-h_{\alpha\sigma,\beta}{}^{,\sigma} \to -h_{\alpha\sigma,\beta}{}^{,\sigma} \underbrace{- (X_{\alpha,\sigma} + X_{\sigma,\alpha})_{,\beta}{}^{,\sigma}}_{= -X_{\alpha,\sigma\beta}{}^{,\sigma} - X_{\sigma,\alpha\beta}{}^{,\sigma}}$$

$\Rightarrow$ e.o.m. is invariant!

$$-h_{\beta\sigma,\alpha}{}^{,\sigma} \to -h_{\beta\sigma,\alpha}{}^{,\sigma} \underbrace{- (X_{\beta,\sigma} + X_{\sigma,\beta})_{,\alpha}{}^{,\sigma}}_{= -X_{\beta,\sigma\alpha}{}^{,\sigma} - X_{\sigma,\beta\alpha}{}^{,\sigma}}$$

$$h^{\sigma}{}_{\sigma,\alpha\beta} \to h^{\sigma}{}_{\sigma,\alpha\beta} + \underbrace{(X^{\sigma}{}_{,\sigma} + X^{\sigma}{}_{,\sigma})_{,\alpha\beta}}_{=2X_{\sigma,\alpha\beta}{}^{,\sigma}}$$

QED : the Lorentz gauge : $A_\alpha{}^{,\alpha} = 0 \Rightarrow \Box A_\beta = -J_\beta$

gravity : the Lorentz (harmonic) gauge : $\bar{h}_{\beta\alpha}{}^{,\alpha} = 0$
(de Donder)

$$\Updownarrow \quad h_{\beta\alpha}{}^{,\alpha} - \frac{1}{2} h^{\alpha}{}_{\alpha,\beta} = 0$$

$$\Box h_{\alpha\beta} = -\lambda \bar{T}_{\alpha\beta}$$

$$\left(-h_{\alpha\sigma,\beta}{}^{,\sigma} - h_{\beta\sigma,\alpha}{}^{,\sigma} + h^{\sigma}{}_{\sigma,\alpha\beta} = 0 \right)$$

$$h'_{\alpha\beta} = h_{\alpha\beta} + x_{\alpha|\beta} + x_{\beta|\alpha}$$

$$\bar{h}_{\beta\alpha} = h_{\beta\alpha} - \frac{1}{2}\eta_{\beta\alpha} h^{\sigma}{}_{\sigma}$$

$$\overline{h'}_{\beta\alpha} = h'_{\beta\alpha} - \frac{1}{2}\eta_{\beta\alpha}\left(h^{\sigma}{}_{\sigma} + 2x^{\sigma}{}_{|\sigma}\right)$$

$$\overline{h'}_{\beta\alpha}{}^{|\alpha} = h'_{\beta\alpha}{}^{|\alpha} - \frac{1}{2}\left(h^{\sigma}{}_{\sigma|\beta} + 2x^{\sigma}{}_{|\sigma\beta}\right) = 0$$

$\leftarrow$ the gauge condition

$$x^{\sigma}{}_{|\sigma\beta} = h'_{\beta\alpha}{}^{|\alpha} - \frac{1}{2}h^{\sigma}{}_{\sigma|\beta} =$$

$$= h_{\beta\alpha}{}^{|\alpha} + \underbrace{x_{\alpha|\beta}{}^{|\alpha}}_{= x^{\sigma}{}_{|\beta\sigma}} + \underbrace{x_{\beta|\alpha}{}^{|\alpha}}_{= \Box x_{\beta}} - \frac{1}{2}h^{\sigma}{}_{\sigma|\beta}$$

$$\Box x_{\beta} = -\left(h_{\beta\alpha}{}^{|\alpha} - \frac{1}{2}h^{\sigma}{}_{\sigma|\beta}\right) = -\bar{h}_{\beta\alpha}{}^{|\alpha}$$

$$\Box x_{\beta} = -\bar{h}_{\beta\alpha}{}^{|\alpha}$$

$$h_{\alpha\beta,\sigma}{}^{,\sigma} - \left(h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma}\right) + h^{\sigma}{}_{\sigma,\alpha\beta} + \eta_{\alpha\beta}\, h^{\mu\nu}{}_{,\mu\nu} - \eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda T_{\alpha\beta}$$

Let's "bar" the above equation :

$$\bar{X}_{\mu\nu} := \frac{1}{2}\left(X_{\mu\nu} + X_{\nu\mu}\right) - \frac{1}{2}\eta_{\mu\nu}\, X_{\alpha}{}^{\alpha}$$

$$\bar{\eta}_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\,\underbrace{\eta_{\sigma}{}^{\sigma}}_{=4} = -\eta_{\alpha\beta}$$

$$\overline{h_{\alpha\beta,\sigma}{}^{,\sigma}} = h_{\alpha\beta,\sigma}{}^{,\sigma} - \frac{1}{2}\eta_{\alpha\beta}\, h^{\mu}{}_{\mu,\sigma}{}^{,\sigma}$$

$$\overline{h_{\alpha\sigma,\beta}{}^{,\sigma}} = \overline{h_{\beta\sigma,\alpha}{}^{,\sigma}} = \frac{1}{2}\left(h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma}\right) - \frac{1}{2}\eta_{\alpha\beta}\, h_{\mu\sigma}{}^{,\mu\sigma}$$

$$\overline{h^{\sigma}{}_{\sigma,\alpha\beta}} = h^{\sigma}{}_{\sigma,\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu}$$

$$\Downarrow$$

$$h_{\alpha\beta,\sigma}{}^{,\sigma} \cancel{- \frac{1}{2}\eta_{\alpha\beta}\, h^{\mu}{}_{\mu,\sigma}{}^{,\sigma}} - 2\,\overline{h_{\alpha\sigma,\beta}{}^{,\sigma}} + h^{\sigma}{}_{\sigma,\alpha\beta} \cancel{- \frac{1}{2}\eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu}} +$$

$$- \eta_{\alpha\beta}\, h^{\mu\nu}{}_{,\mu\nu} \cancel{+ \eta_{\alpha\beta}\, h^{\sigma}{}_{\sigma,\mu}{}^{,\mu}} = -\lambda \bar{T}_{\alpha\beta}$$

$$h_{\alpha\beta,\sigma}{}^{,\sigma} - 2\,\overline{h_{\alpha\sigma,\beta}{}^{,\sigma}} + h^{\sigma}{}_{\sigma,\alpha\beta} - \eta_{\alpha\beta}\, h^{\mu\nu}{}_{,\mu\nu} = -\lambda \bar{T}_{\alpha\beta}$$

Note that $(\bar{h}_{\alpha\sigma})_{,\beta}{}^{,\sigma} = \left(h_{\alpha\sigma} - \frac{1}{2}\eta_{\alpha\sigma} h^{\mu}{}_{\mu}\right)_{,\beta}{}^{,\sigma} = h_{\alpha\sigma,\beta}{}^{,\sigma} - \frac{1}{2}\eta_{\alpha\sigma} h^{\mu}{}_{\mu,\beta}{}^{,\sigma} =$

$$= h_{\alpha\sigma,\beta}{}^{,\sigma} - \frac{1}{2} h^{\mu}{}_{\mu,\beta\alpha}$$

$$(\bar{h}_{\beta\sigma})_{,\alpha}{}^{,\sigma} = h_{\beta\sigma,\alpha}{}^{,\sigma} - \frac{1}{2} h^{\mu}{}_{\mu,\alpha\beta}$$

therefore

$$(\bar{h}_{\alpha\sigma})_{,\beta}{}^{,\sigma} + (\bar{h}_{\beta,\sigma})_{,\alpha}{}^{,\sigma} = \underbrace{h_{\alpha\sigma,\beta}{}^{,\sigma} + h_{\beta\sigma,\alpha}{}^{,\sigma}}_{2\,\overline{h_{\alpha\sigma,\beta}{}^{,\sigma}} + \eta_{\alpha\beta}\, h_{\mu\sigma}{}^{,\mu\sigma}} - h^{\mu}{}_{\mu,\alpha\beta}$$

$$\Downarrow$$

$$\boxed{h_{\alpha\beta,\sigma}{}^{,\sigma} - \bar{h}_{\alpha\sigma,\beta}{}^{,\sigma} - \bar{h}_{\beta\sigma,\alpha}{}^{,\sigma} = -\lambda \bar{T}_{\alpha\beta}}$$

$$S_{QED} = \int d^4x\, \mathcal{L}_{QED} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right] =$$

$$= \int d^4x \left[\frac{1}{2} A_\mu \left(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu + A_\mu J^\mu \right]$$

$\mathcal{L}_{QED}$ is invariant under $A_\mu \to A_\mu - \partial_\mu \Lambda$

(zero eigen-value)

$Q^{\mu\nu} \partial_\nu \Lambda = 0$

$\Updownarrow$

there is no inverse

$\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu = Q^{\mu\nu}$

$\Downarrow$

$$I = \int DA_\mu\, e^{i S_{QED}}$$ sums over equivalent (connected by a gauge transformation) field configurations $\Leftarrow$

Faddeev - Popov $\Longrightarrow$

$$S_{QED} \to \int d^4x\, \mathcal{L}_{QED} - \frac{1}{2\xi} \int d^4x\, (\partial_\mu A^\mu)^2$$

the R_ξ gauge

$$Q^{\mu\nu}_{eff} = \partial^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu$$ has an inverse

$$Q^{\mu\nu}_{eff}(k) \left[-\eta_{\nu\lambda} + (1-\xi)^{-1} \frac{k_\nu k_\lambda}{k^2} \right] \cancel{\frac{1}{k^2}} k^2 = \delta^\mu_\lambda$$

$$\overbrace{\frac{-k^2 \eta^{\mu\nu} + \left(1-\xi \cancel{\tfrac{1}{\xi}}\right) \frac{k^\mu k^\nu}{k^2}}{k^2}}^{\parallel}$$

the photon propagator : $\frac{-1}{k^2}\left[\eta_{\mu\nu} - \left(1-\xi \cancel{\tfrac{1}{\xi}}\right) \frac{k_\mu k_\nu}{k^2} \right]$

$\xi = 1$ Feynman gauge

$$S_g = \int d^4x\, \mathcal{L}_g = \int d^4x \Big[\frac{1}{2} h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\sigma,\mu} +$$

$$- \frac{1}{2} h^{\nu}{}_{\nu,\mu} h^{\sigma}{}_{\sigma}{}^{,\mu} - \lambda T^{\mu\nu} h_{\mu\nu} \Big]$$

$\mathcal{L}_g$ invariant under $h_{\alpha\beta} \to h_{\alpha\beta} + X_{\alpha,\beta} + X_{\beta,\alpha}$

to fix the harmonic gauge : $\bar{h}_{\beta\alpha}{}^{,\alpha} = 0 \iff h_{\beta\alpha}{}^{,\alpha} - \frac{1}{2} h^{\alpha}{}_{\alpha,\beta} = 0$

we will add the gauge fixing term

$$\mathcal{L}_g \to \mathcal{L}_g + \underbrace{\left(h_{\mu\nu}{}^{,\mu} - \frac{1}{2} h^{\mu}{}_{\mu,\nu}\right)\left(h_{\mu}{}^{\nu,\mu} - \frac{1}{2} h^{\mu}{}_{\mu}{}^{,\nu}\right)}_{\left(h_{\mu\nu}{}^{,\mu} - \frac{1}{2} h^{\mu}{}_{\mu,\nu}\right)^2 = \left(\bar{h}_{\mu\nu}{}^{,\mu}\right)^2} = \mathcal{L}_{eff}$$

$$\mathcal{L}_{eff} = \frac{1}{2} h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - \cancel{h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\mu,\sigma}} + h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\sigma,\mu} - \frac{1}{2} h^{\nu}{}_{\nu,\mu} h^{\sigma}{}_{\sigma}{}^{,\mu} +$$

$$+ \cancel{h_{\mu\nu}{}^{,\mu} h_{\sigma}{}^{\nu,\sigma}} + \frac{1}{4} h^{\mu}{}_{\mu,\nu} h^{\sigma}{}_{\sigma}{}^{,\nu} \underbrace{- \frac{1}{2} h^{\mu}{}_{\mu,\nu} h_{\sigma}{}^{\nu,\sigma} - \frac{1}{2} h_{\mu\nu}{}^{,\mu} h^{\sigma}{}_{\sigma}{}^{,\nu}}_{= - h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\sigma,\mu}} + \dots$$

$$= \underbrace{\frac{1}{2} \left[h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - \frac{1}{2} h^{\nu}{}_{\nu,\mu} h^{\sigma}{}_{\sigma}{}^{,\mu} \right]}_{\equiv\, h^{\mu\nu} K_{\mu\nu;\lambda\sigma} h^{\lambda\sigma}} - \lambda h_{\mu\nu} T^{\mu\nu}$$

Let's determine the quadratic operator $K_{\mu\nu;\lambda\sigma}$, its inverse will give the graviton propagator

$$\int d^4x \left[h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - \frac{1}{2} h^{\nu}{}_{\nu,\mu} h^{\sigma}{}_{\sigma}{}^{,\mu} \right] = \text{by parts}$$

$$= \int d^4x \left[- h^{\mu\nu} \eta_{\mu\lambda}\eta_{\nu\sigma} \Box h^{\lambda\sigma} + \frac{1}{2} \underbrace{h^{\nu}{}_{\nu} \Box h^{\sigma}{}_{\sigma}}_{h^{\mu\nu} \eta_{\mu\nu}\eta_{\lambda\sigma} \Box h^{\lambda\sigma}} \right] =$$

$$= \int d^4x \; \underbrace{h^{\mu\nu} \frac{1}{2}\left[\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma} \right]}_{= K_{\mu\nu;\lambda\sigma}} (-\Box) h^{\lambda\sigma}$$

Note that

(home work) $\quad K_{\mu\nu;\lambda\sigma} K^{\lambda\sigma}{}_{\rho\omega} = \frac{1}{2} \left(\eta_{\mu\rho}\eta_{\nu\omega} + \eta_{\mu\omega}\eta_{\nu\rho} \right) = I_{\mu\nu;\rho\omega}$

identity acting in a linear space spanned by <u>symmetric</u> second rank tensors

$$\Downarrow$$

the graviton propagator in the harmonic gauge

$$D_{\mu\nu;\lambda\sigma}(k) = \frac{1}{2} \frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i\varepsilon}$$

Massive gravitons

- Symmetric tensor $h_{\mu\nu}$ has 10 independent components
- Spin 2 field should have $2s+1 = 5$ components
- Vector and scalar components of $h_{\mu\nu}$ should be removed by the following Lorentz invariant conditions:

$$\partial^\mu h_{\mu\nu} = 0 \quad \text{and} \quad h_\nu{}^\nu = 0$$

- Lorentz invariant mass term:

$$+ \frac{1}{2} m^2 \left(e \left(h_\sigma{}^\sigma\right)^2 + f\, h_{\mu\nu} h^{\mu\nu} \right)$$

QED with massive photons:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$

$m^2 \neq 0 \Rightarrow$ breaking of the gauge invariance

$$A_\mu \to A_\mu' = A_\mu + X_{,\mu}$$

$$A_\mu A^\mu \to A_\mu' A'^\mu = (A_\mu + X_{,\mu})(A^\mu + X^{,\mu}) = A_\mu A^\mu + A_\mu X^{,\mu} +$$

$$+ X_{,\mu} A^\mu + X_{,\mu} X^{,\mu} \neq A_\mu A^\mu$$

Gravity with massive gravitons:

$$\mathcal{L} = \frac{1}{2} h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - h^{\mu\nu}{}_{,\nu} h^\sigma{}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu} h^\sigma{}_{\sigma,\mu} - \frac{1}{2} h^\nu{}_{\nu,\mu} h^\sigma{}_\sigma{}^{,\mu} +$$

$$+ \frac{1}{2} m^2 \left(e\, h^2 + f\, h_{\mu\nu} h^{\mu\nu} \right) - \lambda T^{\mu\nu} h_{\mu\nu}$$

$$h \equiv h_\sigma{}^\sigma$$

gauge transformation

$h_{\mu\nu} \to h_{\mu\nu}' = h_{\mu\nu} + X_{\mu,\nu} + X_{\nu,\mu}$ changes the mass term $m^2 (e\, h^2 + f\, h_{\mu\nu} h^{\mu\nu})$

For instance

$$h^2 = (h_\sigma{}^\sigma)^2 \to (h'_\sigma{}^\sigma)^2 = (h_\sigma{}^\sigma + X_\sigma{}^{,\sigma} \cdot 2)^2 =$$

$$= h^2 + 4 h X_\sigma{}^{,\sigma} + 4 (X_\sigma{}^{,\sigma})^2$$

We require that E-L equations derived from $\mathcal{L}$ (with $m^2 \neq 0$) will lead to the constraints that are necessary to remove vector and scalar compo-nents of symmetric $h_{\mu\nu}$, i.e.

$$\partial^\nu h_{\mu\nu} = 0 \quad \text{and} \quad h_\nu{}^\nu = 0$$

$\delta S = 0$

$$\Downarrow$$

$$h_{\mu\nu,\sigma}{}^{,\sigma} - (h_{\mu\sigma,\nu}{}^{,\sigma} + h_{\nu\sigma,\mu}{}^{,\sigma}) + (h^\sigma{}_{\sigma,\mu\nu} + \eta_{\mu\nu} h^{\sigma\lambda}{}_{,\sigma\lambda}) +$$

$$- \eta_{\mu\nu} h^\sigma{}_{\sigma,\lambda}{}^{,\lambda} - m^2 (\eta_{\mu\nu} e h + f h_{\mu\nu}) = - \lambda T_{\mu\nu}$$

Derivative (∂^ν) and contraction with $\eta^{\mu\nu}$ should lead to $\partial^\nu h_{\mu\nu} = 0$ and $h_\nu{}^\nu = 0$.

- Note that for $m^2 = 0$ the Lagrangian is identical as the one we discussed earlier where we demonstrated that ∂^ν (left hand side of equation of motion)$_{\mu\nu} = 0$ as a consequence of conservation of the energy momentum tensor i.e. $\partial^\nu T_{\mu\nu} = 0$.

$$\Downarrow \text{ from } \partial^\nu$$

$$-m^2 (e\, h_\sigma{}^\sigma{}_{,\mu} + f\, h_{\mu\nu}{}^{,\nu}) = 0$$

Now let's take the trace

$$h^{\mu}{}_{\mu,\sigma}{}^{,\sigma} - \left(h_{\mu\sigma}{}^{,\mu\sigma} + h^{\mu}{}_{\sigma,\mu}{}^{,\sigma} \right) + \left(h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} + 4\, h^{\sigma\lambda}{}_{,\sigma\lambda} \right) +$$

$$- 4\, h^{\sigma}{}_{\sigma,\lambda}{}^{,\lambda} - m^2 \left(4 e h_{\sigma}{}^{\sigma} + f h_{\sigma}{}^{\sigma} \right) = - \lambda T_{\mu}{}^{\mu}$$

Since we want to get $h_{\mu\nu}{}^{,\nu} = 0$ and $h_{\sigma}{}^{\sigma} = 0$ we have to assume $T_{\mu}{}^{\mu} = 0$ for consistency, then

$$- 2\, h^{\mu}{}_{\mu,\sigma}{}^{,\sigma} + 2\, h_{\mu\sigma}{}^{,\mu\sigma} - m^2 (4e + f)\, h_{\sigma}{}^{\sigma} = 0$$

$$\begin{cases} - 2 \left(h_{\sigma}{}^{\sigma}{}_{,\mu} - h_{\mu\nu}{}^{,\nu} \right)^{,\mu} - m^2 (4e + f)\, h_{\sigma}{}^{\sigma} = 0 \\ e\, h_{\sigma}{}^{\sigma}{}_{,\mu} + f\, h_{\mu\nu}{}^{,\nu} = 0 \end{cases} \qquad (\text{from } \partial^{\nu})$$

$$- 2 \left(h_{\sigma}{}^{\sigma}{}_{,\mu} + \frac{e}{f}\, h_{\sigma}{}^{\sigma}{}_{,\mu} \right)^{,\mu} - m^2 (4e + f)\, h_{\sigma}{}^{\sigma} = 0 \quad \Big| \; \frac{1}{4e + f}$$

$$- 2\, \frac{1 + \frac{e}{f}}{4e + f}\, \Box h - m^2 h = 0$$

this equation leads to $h = 0$ only if $1 + \frac{e}{f} = 0$

$$\Downarrow$$

$$e = -f = 1 \leftarrow \text{choice of } m^2$$

$$\mathcal{L}_{mass} = + \frac{1}{2} m^2 \left(h^2 - h_{\mu\nu} h^{\mu\nu} \right)$$

the Fierz – Pauli model

(see Nieuwenhuizen NPB 60 (1973) 478 for the proof that $e = -f$ is the only choice which leads to a theory free of tachyons)

Equation of motion for a massive graviton:

$$\Big[\left(\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\nu} \eta^{\alpha\beta} \right) \Box + \left(\eta^{\mu\nu} \partial^{\alpha} \partial^{\beta} + \eta^{\alpha\beta} \partial^{\mu} \partial^{\nu} - \eta^{\mu\alpha} \partial^{\nu} \partial^{\beta} - \eta^{\nu\beta} \partial^{\mu} \partial^{\alpha} \right) +$$

$$+ m^2 \left(\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\nu} \eta^{\alpha\beta} \right) \Big]\, h_{\alpha\beta} = - \lambda T^{\mu\nu}$$