Covariant differentiation along a curve

Let's consider tensors that are defined only over a curve $x^\mu(\tau)$, not over all space-time, e.g. the momentum $P^\mu(\tau)$ of a single particle. We are going to define a covariant differentiation with respect to the invariant τ (as the ordinary covariant differentiation is meaningless in that case).

$$\frac{d}{d\tau} \Big| \qquad A'^\mu(\tau) = \frac{\partial x'^\mu}{\partial x^\nu}\Bigg|_{x^\nu = x^\nu(\tau)} A^\nu(\tau) \qquad \text{for a contravariant vector}$$

$$\frac{d A'^\mu(\tau)}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{d A^\nu(\tau)}{d\tau} + \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}}\frac{d x^\lambda}{d\tau} A^\nu(\tau)$$

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} + \underbrace{\frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}}$$

the same as the inhomogeneous term for $\Gamma^\mu_{\lambda\nu}$ transform

$$= -\frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}$$

Let's define a covariant derivative along a curve, as

$$\frac{DA^\mu}{D\tau} \equiv \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} \frac{dx^\lambda}{d\tau} A^\nu$$

Then

$$\frac{DA'^\mu}{D\tau} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{DA^\nu}{D\tau} \ , \text{ so } \frac{DA^\mu}{D\tau} \text{ is a } \overset{\text{contravariant}}{\text{vector}}$$

Similarly for a covariant vector $B_\mu(\tau)$

$$\frac{DB_\mu}{D\tau} \equiv \frac{dB_\mu}{d\tau} - \Gamma^\lambda_{\mu\nu} \frac{dx^\nu}{d\tau} B_\lambda \quad \text{and}$$

$$\frac{DB'_\mu}{D\tau} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{DB_\nu}{D\tau}$$

For other tensors defined over a curve

$$\frac{DT^\mu{}_\nu}{D\tau} \equiv \frac{dT^\mu{}_\nu}{d\tau} + \Gamma^\mu_{\lambda\rho} \frac{dx^\lambda}{d\tau} T^\rho{}_\nu - \Gamma^\sigma_{\lambda\nu} \frac{dx^\lambda}{d\tau} T^\mu{}_\sigma$$

Homework: prove that $\frac{DT^\mu{}_\nu}{D\tau}$ is a tensor

For a tensor field $T^\mu{}_\nu = T^\mu{}_\nu(x)$ the covariant derivative can also be calculated

$$\frac{D T^\mu{}_\nu}{D\tau} = T^\mu{}_{\nu;\lambda} \frac{dx^\lambda}{d\tau}$$

$$\frac{\partial}{\partial x^\lambda} T^\mu{}_\nu + \Gamma^\mu_{\lambda\rho} T^\rho{}_\nu - \Gamma^\sigma_{\lambda\nu} T^\mu{}_\sigma$$

Often a vector $A^\mu(\tau)$ carried along a curve by a particle doesn't change at τ if viewed from a reference frame $\xi_{x(\tau)}$ that is locally inertial at $x(\tau)$, e.g. this is a case for a particle's momentum. In this frame the affine connection also vanishes, so

$$\frac{DA^\mu}{D\tau} = 0$$

← i) a covariant statement
ii) true in the locally inertial system

⇓

$$\frac{dA^\mu}{d\tau} = -\Gamma^\mu_{\nu\lambda} \frac{dx^\lambda}{d\tau} A^\nu$$

⇐ holds in all coordinate systems

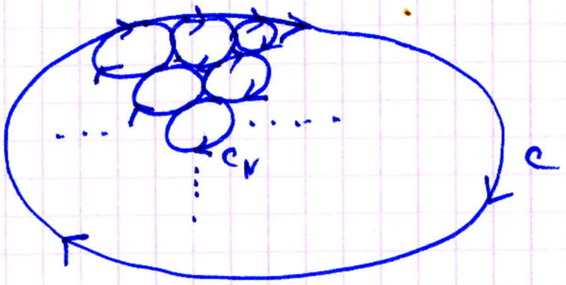
- This defines A^μ for any τ, given $A^\mu = A^\mu(\tau_0)$. $A^\mu(\tau)$ defined that way is said to be defined by parallel transport.
- Similar construction can be made for any tensor, by requiring its covariant derivative along a curve to vanish.
- Tensors defined along curves cannot always be promoted to tensor fields.

Round Trips by Parallel Transport

Let's consider a parallel transport of a vector S_μ

$$(*) \qquad \frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu} \frac{dx^\nu}{d\tau} S_\lambda$$

Question : Does the vector S_μ returns to its original value when carried along a closed curve C ?



The change in S_μ when parallel-transported along C :

$$\Delta S_\mu = \sum_N \underbrace{\Delta_N S_\mu}_{\text{change for a curve } C_N}$$

Let's consider a small cell C_N, if it is small enough we can expand around some point, $\underbrace{X \equiv x(\tau_0)}_{\text{on the curve}}$

$$\Gamma^\lambda_{\mu\nu}(x) = \Gamma^\lambda_{\mu\nu}(X) + (x^\rho - X^\rho)\frac{\partial}{\partial X^\rho}\Gamma^\lambda_{\mu\nu}(X) + \dots$$

The eq. (*) could be solved to first order in $(x^\rho - X^\rho)$:

$$S_\mu(\tau) = S_\mu(\tau_0) + \Gamma^\lambda_{\mu\nu}(X)\underbrace{\left[x^\nu(\tau) - X^\nu\right]}_{\left.\frac{dx^\nu}{d\tau}\right|_{\tau=\tau_0}(\tau-\tau_0)+\dots} S_\lambda(\tau_0) + \dots$$

Then we can get an equation valid to second order:

$$S_\mu(\tau) \simeq S_\mu(\tau_0) + \int_{\tau_0}^{\tau} \underbrace{\left[\Gamma^\lambda_{\mu\nu}(X) + (x^\rho(\tau) - X^\rho)\frac{\partial}{\partial X^\rho}\Gamma^\lambda_{\mu\nu}(X) + \dots\right]}_{=\Gamma^\lambda_{\mu\nu}(x)} \times$$

$$\times \underbrace{\left[S_\lambda(\tau_0) + S_\sigma(\tau_0)\Gamma^\sigma_{\lambda\rho}(X)(x^\rho(\tau) - X^\rho) + \dots\right]}_{S_\lambda(\tau)} \frac{dx^\nu(\tau)}{d\tau}\, d\tau$$

formal integration of (*)

Expanding in powers of $(x - X)$ we get

$$S_\mu(\tau) = S_\mu(\tau_0) + \Gamma^\lambda_{\mu\nu}(X)\, S_\lambda(\tau_0) \int_{\tau_0}^{\tau} \frac{dx^\nu(\tau)}{d\tau}\, d\tau \; +$$

$$+ \left\{ \frac{\partial}{\partial X^\rho} \Gamma^\lambda_{\mu\nu}(X)\, S_\lambda(\tau_0) + \Gamma^\sigma_{\lambda\rho}(X)\, \Gamma^\lambda_{\mu\nu}(X)\, S_\sigma(\tau_0) \right\} \int_{\tau_0}^{\tau} (x^\rho - X^\rho) \frac{dx^\nu}{d\tau}\, d\tau + \dots$$

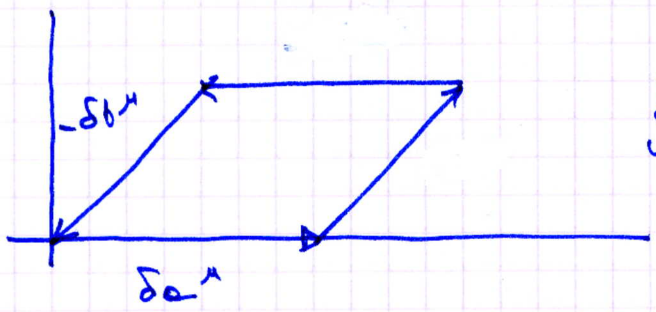
If $x^\mu(\tau)$ returns to its original value $X^\mu = x^\mu(\tau_0)$ at some $\tau = \tau_1$, then, of course

$$\int_{\tau_0}^{\tau_1} \frac{dx^\nu}{d\tau}\, d\tau = 0$$

, then, up to second order

$$\Delta S_{\nu\mu} \equiv S_\mu(\tau_1) - S_\mu(\tau_0) = \left\{ \frac{\partial}{\partial X^\rho} \Gamma^\sigma_{\mu\nu}(X) + \Gamma^\sigma_{\lambda\rho}(X)\, \Gamma^\lambda_{\mu\nu}(X) \right\} S_\sigma(\tau_0) \underbrace{\oint x^\rho dx^\nu}_{= \int_{\tau_0}^{\tau_1} x^\rho(\tau) \frac{dx^\nu}{d\tau} d\tau}$$

In general $\oint x^\rho dx^\nu \neq 0$, e.g.
if our curve was a small parallelogram with edges δa^μ, δb^μ, then



Homework: show that

$$\oint x^\rho dx^\nu = \delta a^\rho \delta b^\nu - \delta b^\rho \delta a^\nu$$

$\oint x^\rho dx^\nu$ is antisymmetric:

$$\oint x^\rho dx^\nu = \underbrace{\int_{\tau_0}^{\tau_1} \frac{d}{d\tau}(x^\rho x^\nu)\, d\tau}_{= 0} - \int_{\tau_0}^{\tau_1} \frac{dx^\rho}{d\tau} x^\nu d\tau = -\oint x^\nu dx^\rho$$

$\Downarrow$ only antisymmetric part of $\left\{ \frac{\partial}{\partial X^\rho} \Gamma^\sigma_{\mu\nu}(X) \dots \right\}$ enters, but that is $\frac{1}{2} R^\sigma_{\mu\nu\rho}$

$$\Delta S_{\nu\mu} = \frac{1}{2} R^\sigma_{\mu\nu\rho}\, S_\sigma \oint x^\rho dx^\nu$$

$x^\rho(\tau) = \cancel{\delta a^\rho} + \delta b^\rho - \cancel{\delta a^\rho} + - \delta b^\rho \tau$

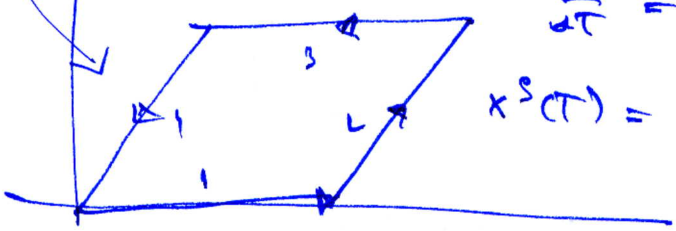
$\frac{dx^\rho}{d\tau} = -\delta b^\rho$

$x^\rho(\tau) = \delta a^\rho + \delta b^\rho + - \delta a^\rho \tau$

$\frac{dx^\nu}{d\tau} = - \delta a^\nu$

$\frac{dx^\nu}{d\tau} = \delta b^\nu$

$x^\rho(\tau) = \delta a^\rho + \delta b^\rho \tau$



δa^μ

$x^\rho(\tau) = \delta a^\rho \cdot \tau \qquad \frac{dx^\nu}{d\tau} = \delta a^\nu$

1. $\int_0^1 x^\rho \frac{dx^\nu}{d\tau} d\tau = \delta a^\rho \delta a^\nu \int_0^1 \tau \, d\tau = \frac{1}{2} \delta a^\rho \delta a^\nu$

2. $\int_0^1 \cdots = \int_0^1 (\delta a^\rho + \delta b^\rho \tau) \delta b^\nu d\tau = \delta a^\rho \delta b^\nu + \frac{1}{2} \delta b^\rho \delta b^\nu$

3. $-(\delta a^\rho + \delta b^\rho) \delta a^\nu + \frac{1}{2} \delta a^\rho \delta a^\nu$

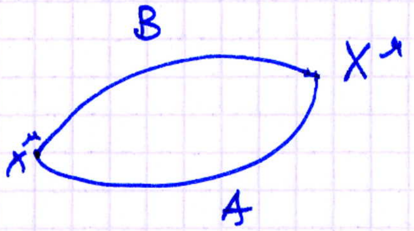
4 $-\delta b^\rho \delta b^\nu + \frac{1}{2} \delta b^\rho \delta b^\nu$

$\oint x^\rho dx^\nu = \delta a^\rho \delta b^\nu - \delta b^\rho \delta a^\nu$

Conclusion: An arbitrary vector S_μ will not change when parallel-transported around an arbitrary small curve at X, if and only if $R^\sigma{}_{\mu\nu\rho}$ vanishes at X.

Since $\Delta S_\mu = \sum_N \Delta_N S_\mu$, therefore if $R^\sigma{}_{\mu\nu\rho}$ vanishes inside the contour C, then S_μ doesn't change when parallel-transported around C.

- Suppose that $R^\sigma{}_{\mu\nu\rho}$ vanishes



, then $\Delta^A_{X\to x} S_\mu + \Delta^B_{x\to X} S_\mu = 0$

$$\Delta^B_{x\to X} S_\mu = -\Delta^B_{X\to x} S_\mu$$

$$\Delta^A_{X\to x} S_\mu = \Delta^B_{X\to x} S_\mu$$

$\Rightarrow$ Given S_μ at X we may construct <u>a field</u> $S_\mu(x)$, defined through-out the space-time region where $R^\sigma{}_{\mu\nu\rho} = 0$, by parallel transport from X to x ($S_\mu(x)$ will depend on x only, no dependence on the path chosen from X to x). For this field

$$\frac{dS_\mu}{d\tau} = \frac{\partial S_\mu}{\partial x^\nu}\frac{dx^\nu(\tau)}{d\tau} \quad \text{(derivative along a curve)}$$

Since S_μ is defined through the parallel transport therefore S_μ satisfies $\frac{DS_\mu}{D\tau} = 0$ so

$$\frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu}\frac{dx^\nu}{d\tau} S_\lambda$$

$\frac{dx^\nu}{d\tau}$ is arbitrary

$$\frac{\partial S_\mu}{\partial x^\nu} = \Gamma^\lambda_{\mu\nu} S_\lambda \iff S_{\mu;\nu} = 0$$

So, if $R^\sigma{}_{\mu\nu\rho}$ vanishes we can always find a solution of $S_{\mu;\nu} = 0$ with any given $S_\mu(X)$.

Conversely, if there exists any covariant vector field such that $S_{\mu;\nu} = 0$, then, of course

$$\frac{dS_\mu}{d\tau} = \frac{\partial S_\mu}{\partial x^\nu}\frac{dx^\nu(\tau)}{d\tau} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu(\tau)}{d\tau}$$

Since parallel transport cannot change a field when carried about any closed loop, therefore

$$\Delta S_\mu = \frac{1}{2} R^\sigma{}_{\mu\nu\rho} S_\sigma \oint x^\rho dx^\nu = 0$$

$$\Downarrow$$

$$R^\sigma{}_{\mu\nu\rho} S_\sigma = 0$$

in the region where $S_{\mu;\nu} = 0$

Gravitational versus Curvilinear coordinates

How do we know that a metric tensor $g_{\mu\nu}(x)$ which is not $\eta_{\mu\nu}$ is not equivalent to $\eta_{\mu\nu}$?
In other words; how can we tell if there exists a gravitational field or if $g_{\mu\nu}$ is just $\eta_{\mu\nu}$ written in curvilinear coordinates?

$$\eta^{\alpha\beta} = g^{\mu\nu}\frac{\partial \xi^\alpha(x)}{\partial x^\mu}\frac{\partial \xi^\beta(x)}{\partial x^\nu}$$

Is there a set of Minkowskian coordinates $\xi^\alpha(x)$ such that holds everywhere?

Our definition of $g^{\mu\nu}$ is:

(*) $\eta^{\alpha\beta} = \frac{\partial \xi^\alpha(x)}{\partial x^\mu}\frac{\partial \xi^\beta(x)}{\partial x^\nu} \cdot g^{\mu\nu}$ where ξ^α are locally inertial coordinates

Note that when we construct a locally inertial system, we do so at a specific point X, therefore $\xi^\alpha(x) = \xi^\alpha_X(x)$ and (*) holds in an infinitesimal neighborhood of X!

For instance if

(*) $g_{rr} = -1 \quad g_{\theta\theta} = -r^2 \quad g_{\varphi\varphi} = -r^2 \sin^2\theta$ and $g_{tt} = +1$

then, as we know, there exists a set of ξ^α s

$\xi^1 = r \sin\theta \cos\varphi \quad \xi^2 = r \sin\theta \sin\varphi \quad \xi^3 = r\cos\theta \quad \xi^0 = t$

such that

$$g^{\mu\nu} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} = \eta^{\alpha\beta} \qquad \text{(homework)}$$

where $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$ for $g_{\nu\lambda}$

This proves that (*) is purely an effect of curvilinear coordinates: there is no gravity involved!

But how could we have told that (*) is indeed equivalent to the Minkowski metric $\eta_{\alpha\beta}$, if we weren't clever enough to guess the coordinates ξ^α?

The answer:

The necessary and sufficient conditions for a metric $g_{\mu\nu}(x)$ to be equivalent to the Minkowski metric $\eta_{\alpha\beta}$ are:

i) $R^\lambda{}_{\mu\nu\chi} = 0$ everywhere

ii) at some point X the matrix $g^{\mu\nu}(X)$ has three negative and one positive eigenvalues

The necessity: suppose we find $\xi^\alpha(x)$ such that

$$\eta^{\alpha\beta} = g^{\mu\nu} \frac{\partial \xi^\alpha(x)}{\partial x^\mu} \frac{\partial \xi^\beta(x)}{\partial x^\nu},$$

— in this coordinate system all components of the affine connection vanish, $\Gamma^\mu_{\nu\lambda} = 0$, and hence the Riemann tensor $R^\alpha{}_{\beta\gamma\delta} = 0$

$$\underbrace{\frac{\partial \xi^\alpha}{\partial x^\mu}}_{O_{\alpha\mu}} g^{\mu\nu} \underbrace{\frac{\partial \xi^\beta}{\partial x^\nu}}_{O_{\beta\nu}} = O_{\alpha\mu} g^{\mu\nu} O_{\beta\nu} = \left(O g^{\mu\nu} O^T\right)_{\alpha\beta}$$

$$O_{\alpha\mu} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \cdots \\ 0 & \vdots & \vdots & \ddots \\ 0 & & & \end{array}\right) \end{array}$$

$$x^1 = \cancel{g^{1\mu} x}\ r$$
$$x^2 = \theta$$
$$x^3 = \varphi$$

$$\xi^1 = r \sin\theta \cos\varphi$$
$$\xi^2 = r \sin\theta \sin\varphi$$
$$\xi^3 = r \cos\theta$$

$$\frac{\partial \xi^1}{\partial x^1} = \sin\theta\cos\varphi \qquad \frac{\partial \xi^1}{\partial x^2} = r\cos\theta\cos\varphi \qquad \frac{\partial \xi^1}{\partial x^3} = -r\sin\theta\sin\varphi$$

$$\frac{\partial \xi^2}{\partial x^1} = \sin\theta\sin\varphi \qquad \frac{\partial \xi^2}{\partial x^2} = r\cos\theta\sin\varphi \qquad \frac{\partial \xi^2}{\partial x^3} = r\sin\theta\cos\varphi$$

$$\frac{\partial \xi^3}{\partial x^1} = \cos\theta \qquad \frac{\partial \xi^3}{\partial x^2} = -r\sin\theta \qquad \frac{\partial \xi^3}{\partial x^3} = 0$$

see spher.m : $\left(O g^{\mu\nu} O^T\right)_{\alpha\beta} = \eta^{\alpha\beta}$

$$\underbrace{\frac{\partial \xi^\alpha}{\partial x^\mu}}_{O^\alpha{}_\mu} g^{\mu\nu} \underbrace{\frac{\partial \xi^\beta}{\partial x^\nu}}_{O^\beta{}_\nu} \overset{?}{=} \eta^{\alpha\beta}$$

$$O^\beta{}_\nu = (O^T)_\nu{}^\beta$$

$$(O\, g\, O^T)^{\alpha\beta} \overset{!}{=} \eta^{\alpha\beta}$$

$g^{\mu\nu}$

$$g^{tt} = 1 \qquad g^{rr} = -1 \qquad g^{\theta\theta} = -\frac{1}{r^2} \qquad g^{\varphi\varphi} = -\frac{1}{r^2 s_\theta^2}$$

$$O^\alpha{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s_\theta c_\varphi & r c_\theta c_\varphi & -r s_\theta s_\varphi \\ 0 & s_\theta s_\varphi & r c_\theta s_\varphi & r s_\theta c_\varphi \\ 0 & c_\theta & -r s_\theta & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -\frac{1}{r^2} & \\ & & & -\frac{1}{r^2 s_\theta^2} \end{pmatrix} O^T{}_\nu{}^\beta =$$

$g^{\mu\nu}(O^T)_\nu{}^\beta$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -s_\theta c_\varphi & -\frac{1}{r} c_\theta c_\varphi & \frac{1}{r}\frac{s_\varphi}{s_\theta} \\ 0 & -s_\theta s_\varphi & -\frac{1}{r} c_\theta s_\varphi & -\frac{1}{r}\frac{c_\varphi}{s_\theta} \\ 0 & -c_\theta & \frac{s_\theta}{r} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s_\theta c_\varphi & s_\theta s_\varphi & c_\theta \\ 0 & r c_\theta c_\varphi & r c_\theta s_\varphi & -r s_\theta \\ 0 & -r s_\theta s_\varphi & r s_\theta c_\varphi & 0 \end{pmatrix}}_{(O^T)_\nu{}^\beta} =$$

$$= \begin{pmatrix} 1 & & & \\ & -s_\theta^2 c_\varphi^2 - c_\theta^2 c_\varphi^2 - s_\varphi^2 & & \\ & & -s_\theta^2 s_\varphi^2 - c_\theta^2 s_\varphi^2 - c_\varphi^2 & \\ & & & -c_\theta^2 - s_\theta^2 \end{pmatrix}$$

- but the vanishing of a tensor ($R^\alpha{}_{\beta\gamma\delta}=0$) is an invariant statement, so $R^\alpha{}_{\beta\gamma\delta}$ must have vanished in the original coordinate system as well.

- Congruence

$$g_{\mu\nu} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

Define

$$D_{\alpha\mu} \equiv \frac{\partial\xi^\alpha}{\partial x^\mu}$$, then we can write

$$g = D^T\eta D \qquad (\text{here } g_{\mu\nu} \text{ is } 4\times4 \text{ matrix})$$

We always assume that ~~the~~ transformations to locally inertial coordinates are nonsingular: i.e. both $\frac{\partial\xi^\alpha}{\partial x^\mu}$ and $\frac{\partial x^\mu}{\partial\xi^\alpha}$ exist, then

$$D^{-1}{}_{\mu\alpha} \equiv \frac{\partial x^\mu}{\partial\xi^\alpha}$$ must exist and

$$(D^{-1}D)_{\mu\nu} = \frac{\partial x^\mu}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial x^\nu} = \delta^\mu_\nu \Rightarrow \text{Det } D \neq 0$$

If $\text{Det } D\neq 0$ then the relation $\underbrace{g = D^T\eta D}$ is called a congruence.

not a similarity transformation as $DD^T \neq \mathbb{1}$

$\Downarrow$

eigenvalues of g and η are not (in general) the same

The Sylvester's law of inertia.

$\Downarrow$

number of eigenvalues that are respectively positive, negative and zero do not change under the congruence

$\Downarrow$

$g_{\mu\nu}$ must have 3 negative and 1 positive (no zero) eigenvalues $\Rightarrow$ the necessity has been proven

The sufficiency: we are going to construct $\xi^\alpha(x)$

such that $\quad \eta^{\alpha\beta} = g^{\mu\nu} \dfrac{\partial \xi^\alpha}{\partial x^\mu} \dfrac{\partial \xi^\beta}{\partial x^\nu}$.

At any point X we can find a matrix $d^\alpha{}_\beta$ such that

$$\eta^{\alpha\beta} = g^{\mu\nu}(X)\, d^\alpha{}_\mu\, d^\beta{}_\nu \quad .$$

Note that for any symmetric matrix, ($g^{\mu\nu}(x)$ is symmetric) there exist an orthogonal matrix $O^\alpha{}_\mu$ such that

$$O^\alpha{}_\mu\, g^{\mu\nu}\, O^\beta{}_\nu = \begin{cases} D^\alpha & \text{for } \alpha=\beta \\ 0 & \text{--''--}\ \alpha\neq\beta \end{cases}$$

($O g O^T$ is diagonal)

We assume that $g^{\mu\nu}(X)$ has three negative and one positive eigenvalues. Let's choose the ordering of D^α such that: $D^0 > 0$ and $D^i < 0$. Then choosing

$$d^0{}_\mu = \frac{O^0{}_\mu}{\sqrt{D^0}} \quad \text{and} \quad d^i{}_\mu = \frac{O^i{}_\mu}{\sqrt{-D^i}} \quad \text{we satisfy}$$

$$g^{\mu\nu}(X)\, d^\alpha{}_\mu\, d^\beta{}_\nu = \eta^{\alpha\beta}$$

Now we define $D^\alpha{}_\mu(x)$ by the equation

$$\frac{\partial D^\alpha{}_\mu}{\partial x^\nu} = \Gamma^\lambda_{\mu\nu} D^\alpha{}_\lambda \quad \text{with the initial condition:}$$
$$D^\alpha{}_\mu = d^\alpha{}_\mu \ \text{at}\ x = X$$

But we have shown that the equation can be always solved, providing that $R^\lambda{}_{\mu\nu\varkappa} = 0$ ($D^\alpha{}_\mu$ is treated as 4 covariant four vectors).

Since $\dfrac{\partial D^\alpha{}_\mu}{\partial x^\nu}$ must be symmetric in $\mu\nu$ we can write

$$D^\alpha{}_\mu = \frac{\partial \xi^\alpha}{\partial x^\mu} \quad \left(\text{then } \frac{\partial D^\alpha{}_\mu}{\partial x^\nu} = \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu} \right.$$
is symmetric in $\mu-\nu$)

To check that these ξ coordinates satisfy

$$\eta^{\alpha\beta} = g^{\mu\nu} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

Let's note that

$$\frac{\partial}{\partial x^\rho}\left(g^{\mu\nu} D^\alpha{}_\mu D^\beta{}_\nu\right) = 0 \qquad \text{Homework : to prove}$$

$$\Downarrow$$

$$g^{\mu\nu}(x) D^\alpha{}_\mu D^\beta{}_\nu = \text{const.}$$

However at $x = X$ where $D^\alpha{}_\mu = d^\alpha{}_\mu$

$$\left. g^{\mu\nu}(X)\, d^\alpha{}_\mu d^\beta{}_\nu = \eta^{\alpha\beta} \right\} \Rightarrow \text{const.} = \eta^{\alpha\beta}$$

So we get $\quad g^{\mu\nu}(x)\, D^\alpha{}_\mu D^\beta{}_\nu = \eta^{\alpha\beta}$ everywhere $\quad \square$

Commutation of covariant derivatives

Consider the second covariant derivative of a covariant vector V_λ :

$$V_{\mu;\nu;\kappa} = \frac{\partial V_{\mu;\nu}}{\partial x^\kappa} - \Gamma^\lambda_{\nu\kappa} V_{\mu;\lambda} - \Gamma^\lambda_{\mu\kappa} V_{\lambda;\nu} =$$

$$= \frac{\partial^2 V_\mu}{\partial x^\nu \partial x^\kappa} - \frac{\partial V_\lambda}{\partial x^\kappa} \Gamma^\lambda_{\mu\nu} - V_\lambda \frac{\partial}{\partial x^\kappa} \Gamma^\lambda_{\mu\nu} +$$

$$- \Gamma^\lambda_{\nu\kappa} \frac{\partial V_\mu}{\partial x^\lambda} + \Gamma^\lambda_{\nu\kappa} \Gamma^\sigma_{\mu\lambda} V_\sigma - \Gamma^\lambda_{\mu\kappa} \frac{\partial V_\lambda}{\partial x^\nu} + \Gamma^\lambda_{\mu\kappa} \Gamma^\sigma_{\lambda\nu} V_\sigma$$

- first and second derivative terms (of V_μ) are symmetric in (ν, κ)
- terms involving V_μ itself contain an antisymmetric part

$$V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = -V_\sigma \underbrace{\left\{ \left(\frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x^\kappa} - \underbrace{\Gamma^\lambda_{\nu\kappa} \Gamma^\sigma_{\mu\lambda}}_{\text{symmetric}} - \Gamma^\sigma_{\lambda\nu} \Gamma^\lambda_{\mu\kappa} \right) - (\nu \leftrightarrow \kappa) \right\}}_{= R^\sigma{}_{\mu\nu\kappa}} =$$

$$= -V_\sigma R^\sigma{}_{\mu\nu\kappa}$$

Similarly

$$V^{\lambda}{}_{;\nu;\kappa} - V^{\lambda}{}_{;\kappa;\nu} = V^{\sigma} R^{\lambda}{}_{\sigma\nu\kappa}$$

Analogous formulas hold for any tensor, e.g.

$$T^{\lambda}{}_{\mu;\nu;\kappa} - T^{\lambda}{}_{\mu;\kappa;\nu} = T^{\sigma}{}_{\mu} R^{\lambda}{}_{\sigma\nu\kappa} - T^{\lambda}{}_{\sigma} R^{\sigma}{}_{\mu\nu\kappa}$$

– Conclusion: if the curvature tensor vanishes ($R^{\sigma}{}_{\mu\nu\kappa} = 0$), then covariant derivatives commute, what could have been anticipated for a metric $g_{\mu\nu}$ which can be transformed into the Minkowski metric $\eta_{\mu\nu}$.

- Yang-Mills analogy:

$$F_{\mu\nu} \sim [D_\mu, D_\nu]$$

$$D_\mu \equiv \partial_\mu - ig T^a A^a_\mu$$

- if $[D_\mu, D_\nu] = 0$ then $F_{\mu\nu} = 0$, so no "electromagnetic" forces,
- however the potential A_μ can be non-zero (pure gauge potential) e.g. QED with $A_\mu = \partial_\mu \Lambda$

$$\Downarrow$$

$$\bar{E} = \bar{B} = 0 \Leftarrow F_{\mu\nu} = 0$$

In GR: $V_{\mu;\nu;\kappa} = V_{\mu;\kappa;\nu}$ is an analog of $[D_\mu, D_\nu] = 0$

it implies no gravitational forces, but $g_{\mu\nu}$ can be different from the Minkowski metric $\eta_{\mu\nu}$ (as A_μ can be non-zero even though $F_{\mu\nu} = 0$).

Algebraic properties of $R_{\lambda\mu\nu\kappa}$

It is simpler to adopt $R_{\lambda\mu\nu\kappa}$ instead of $R^{\lambda}{}_{\mu\nu\kappa}$:

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\kappa}$$

$$\frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\sigma}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\sigma}_{\nu\lambda}$$

for $\Gamma^{\sigma}_{\mu\nu}$ we use

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left\{ \frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right\}$$

Then we use the relation:

$$g_{\lambda\sigma}\frac{\partial}{\partial x^{\kappa}} g^{\sigma\rho} = - g^{\sigma\rho}\frac{\partial}{\partial x^{\kappa}} g_{\lambda\sigma} =$$

$$= - g^{\sigma\rho}\underbrace{\left(\Gamma^{\eta}_{\kappa\lambda} g_{\eta\sigma} + \Gamma^{\eta}_{\kappa\sigma} g_{\eta\lambda}\right)}_{= \frac{\partial}{\partial x^{\kappa}} g_{\lambda\sigma}}$$

This leads to (homework: to derive)

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}} \right] +$$

$$+ g_{\eta\sigma}\left(\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu} \right)$$

$\Downarrow$

i) $R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$ symmetry

ii) $R_{\lambda\mu\nu\kappa} = - R_{\mu\lambda\nu\kappa} = - R_{\lambda\mu\kappa\nu} = + R_{\mu\lambda\kappa\nu}$ antisymmetry

iii) $R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} + R_{\lambda\kappa\mu\nu} = 0$ cyclicity in (μ,ν,κ)

clear

The Ricci tensor : $R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa}$

i) $\Rightarrow$ $R_{\mu\kappa} = R_{\kappa\mu}$

ii) $\Rightarrow$ $R_{\mu\kappa}$ is the only second rank tensor that can be made of $R_{\lambda\mu\nu\kappa}$ since contractions of $R_{\lambda\mu\nu\kappa}$ with $g^{\lambda\nu}$, $g^{\lambda\mu}$ and $g^{\nu\kappa}$ give either $R_{\mu\kappa}$ or zero:

$$R_{\mu\kappa} = -g^{\lambda\nu} R_{\mu\lambda\nu\kappa} = -g^{\lambda\nu} R_{\lambda\mu\kappa\nu} = g^{\lambda\nu} R_{\mu\lambda\kappa\nu}$$

$$g^{\lambda\mu} R_{\lambda\mu\nu\kappa} = g^{\nu\kappa} R_{\lambda\mu\nu\kappa} = 0$$

ii) $\Rightarrow$ there is only one way to construct a scalar from $R_{\lambda\mu\nu\kappa}$, since

$$R \equiv g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa} = -g^{\lambda\nu} g^{\mu\kappa} R_{\mu\lambda\nu\kappa}$$

$$0 = g^{\lambda\mu} g^{\nu\kappa} R_{\lambda\mu\nu\kappa}$$ but there is also $\varepsilon^{\lambda\mu\nu\kappa}$ to form a scalar

iii) $\Rightarrow$ however the cyclicity

$$\varepsilon^{\lambda\mu\nu\kappa} R_{\lambda\mu\nu\kappa} = 0 \quad \Big| \frac{1}{\sqrt{g}}$$

The Bianchi identities

Let's choose $\underbrace{\text{a}}_{\text{at a given point } x}$ locally inertial coordinate system in which $\Gamma^{\lambda}_{\mu\nu} = 0$ (but not its derivatives), then

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^{\eta}}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right]$$

Permuting ν, κ and η cyclically we obtain the Bianchi identities:

(Homework: to derive)

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\kappa\eta;\nu} + R_{\lambda\mu\eta\nu;\kappa} = 0$$

The equations are manifestly generally covariant and they hold in locally inertial systems, therefore they are true in general.

$$g^{\lambda\nu}\left(R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\kappa\eta;\nu} + R_{\lambda\mu\eta\nu;\kappa}\right) = 0$$

$$g^{\mu\kappa}\left(R_{\mu\kappa;\eta} + R^{\nu}{}_{\mu\kappa\eta;\nu} - R_{\mu\eta;\kappa}\right) = 0 \qquad \Leftarrow g^{\lambda\nu}{}_{;\kappa} = 0$$

$$R_{;\eta} - R^{\nu}{}_{\eta;\nu} - R^{\mu}{}_{\eta;\mu} = 0 \qquad \Leftarrow$$

$$\left(R^{\mu}{}_{\eta} - \frac{1}{2}\delta^{\mu}{}_{\eta} R\right)_{;\mu} = 0$$

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0$$

Einstein's field equations

At any point X in an arbitrarily strong gravitational field we can define a locally inertial coordinate system such that

$$g_{\alpha\beta}(X) = \eta_{\alpha\beta}$$

$$\left(\frac{\partial g_{\alpha\beta}(x)}{\partial x^{\gamma}}\right)_{x=X} = 0 \qquad \Rightarrow$$

in the vicinity of X $g_{\alpha\beta}$ differ from $\eta_{\alpha\beta}$ only by terms quadratic in $(x-X)$.

In this system the gravitational field is weaker near X, therefore we expect to find a <u>linear</u> equation of motion, then reversing the coordinate transformation that made the field weak we can find the general field equations.

For a weak static field produced by a nonrelativistic mass density ρ we have obtained

$$g_{00} \cong (1 + 2\phi)$$

the Newtonian potential that satisfies the Poisson's equation $\nabla^2 \phi = 4\pi G \rho$

$$\bar{F} = -G \frac{m M}{r^2} \frac{\bar{r}}{r}$$

$$G = 6.673\,(10) \times 10^{-11} \frac{m^3}{kg\, s^2}$$

Newton's constant

In our unit system

$$\boxed{\begin{aligned} c &= 3 \cdot 10^8 \frac{m}{s} = 1 \\ \hbar &= 1{,}05 \cdot 10^{-34}\, J \cdot s = \\ &= 6{,}58 \cdot 10^{-25}\, GeV \cdot s = 1 \end{aligned}}$$

$$G = 6{,}3 \cdot 10^{-39}\, GeV^{-2} \Leftarrow$$

$$M_{Pl} \equiv \frac{1}{G^{1/2}} = 1{,}2 \cdot 10^{19}\, GeV \quad \text{– the Planck mass}$$

For nonrelativistic matter

$$T_{\alpha\beta} = \sum_n \frac{p_{n\alpha}\, p_{n\beta}}{E_n}\, \delta^{(3)}(\bar{x} - \bar{x}_n(t)) \simeq \quad \Big\{ \; p_\alpha \equiv m \frac{dx_\alpha}{d\tau}$$

$$\simeq \underset{\substack{\parallel \\ \rho}}{T_{00}} + O(\bar{v})$$

ρ – mass density

Then we have

$$\frac{1}{2} \nabla^2 g_{00} = \nabla^2 \phi = 4\pi G\, T_{00}$$

$$\nabla^2 g_{00} = 8\pi G\, T_{00} \qquad (\text{Lorentz variant})$$

$$\Downarrow \quad \underline{\text{guess}}$$

$$-G_{\alpha\beta} = 8\pi G\, T_{\alpha\beta}$$

where $G_{\alpha\beta}$ is a linear combination of $g_{\alpha\beta}$ and its first and second derivatives.

Properties of $G_{\mu\nu}$:

i) $G_{\mu\nu}$ is a tensor

ii) By assumption, $G_{\mu\nu}$ contains only terms that are either linear in second derivatives of $g_{\mu\nu}$ or quadratic in its first derivatives

iii) Since $T_{\mu\nu}$ is symmetric, so is $G_{\mu\nu}$

iv) Since $T_{\mu\nu}$ is covariantly conserved ($T^{\mu}{}_{\nu;\mu}=0$), so is $G_{\mu\nu}$:

$$G^{\mu}{}_{\nu;\mu}=0$$

v) For a weak stationary field produced by nonrelativistic matter the equation must reduce to

$$\nabla^2 g_{00} = 8\pi G T_{00}$$

therefore $-G_{00} = \nabla^2 g_{00}$ in this limit

Using the above properties we are going to determine $G_{\mu\nu}$

– we have shown that the curvature tensor $R^{\lambda}{}_{\mu\nu\kappa}$ satisfy i) and ii) (uniqueness was an element of the homework # 8).

– There are only two tensors that can be made out of $R^{\lambda}{}_{\mu\nu\kappa}$ by contracting a pair of indices:

$$R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa} \quad \text{and} \quad R = R^{\mu}{}_{\mu}$$

$$\Downarrow$$

$$G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R \quad \text{– symmetric}$$

– $$G^{\mu}{}_{\nu;\mu} = c_1 \underbrace{R^{\mu}{}_{\nu;\mu}}_{\frac{1}{2}R_{;\nu}} + c_2 g^{\mu}{}_{\nu} R_{;\mu} = \left(\tfrac{1}{2}c_1 + c_2\right) R_{;\nu}$$

$$\left(R^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu} R\right)_{;\mu} = 0$$

the Bianchi identities

- From iv) we require $G^\mu{}_{\nu;\mu} = 0$, so

either $c_2 = -\frac{1}{2} c_1$ or $R_{;\nu} = 0$ everywhere.

- $$G^\mu{}_\mu = (c_1 + 4c_2) R = -8\pi G T^\mu{}_\mu$$

$$-G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

If $R_{;\nu} = \frac{\partial R}{\partial x^\nu} = 0$ everywhere then so must $\frac{\partial T^\mu{}_\mu}{\partial x^\nu}$, so $\frac{\partial T^\mu{}_\mu}{\partial x^\nu} = 0$ but this is not the case, e.g. for nonrelativistic matter.

- so $c_2 = -\frac{1}{2} c_1$ and $G_{\mu\nu} = c_1 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$

- To determine c_1 we use the condition v).
For a nonrelativistic matter $|T_{ij}| \ll |T_{00}|$, so we consider only $|G_{ij}| \ll |G_{00}|$, so

$$R_{ij} \simeq \frac{1}{2} g_{ij} R$$

$$g_{\alpha\beta} \simeq \eta_{\alpha\beta} + O(h_{\alpha\beta}) \Longrightarrow R \simeq R_{00} - R_{ii} \simeq R_{00} + \frac{3}{2} R$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\eta_{ij} = \text{diag}(-1, -1, -1)$$

$$R \simeq -2 R_{00}$$

$$\Downarrow$$

$$G_{00} = c_1 \left(R_{00} - \frac{1}{2} \cdot (-) 2 R_{00}\right) = c_1 \, 2 R_{00}$$

To calculate R_{00} we use

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2} \left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right] + O(\Gamma^2)$$

higher order

For the static field $\frac{\partial}{\partial x^0}$ vanish, therefore

$$R_{0000} \simeq 0 \qquad R_{i0j0} \simeq \frac{1}{2}\left[0 - 0 - 0 + \frac{\partial^2 g_{00}}{\partial x^i \partial x^j}\right] = \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j}$$

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = R_{0000} - R_{i0i0} = -\frac{1}{2}\nabla^2 g_{00}$$

$$G_{00} \simeq 2C_1 (-)\frac{1}{2}\nabla^2 g_{00} = -C_1 \nabla^2 g_{00}$$

Since in the weak limit we had $\nabla^2 g_{00} = 8\pi G T_{00}$

therefore $C_1 = +1$ and $G_{\mu\nu} = +\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$

$$\Downarrow$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G\, T_{\mu\nu}$$ the Einstein field equations

$g^{\mu\nu}$ |

$$R = 8\pi G\, T^{\mu}{}_{\mu}$$

$$\Downarrow$$

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}\right)$$

- In vacuum $T_{\mu\nu} = 0$, so $R_{\mu\nu} = 0$ in empty space.
- In 4 and more dimensions $R_{\mu\nu} = 0$ <u>does not</u> imply $R_{\lambda\mu\nu\kappa} = 0$. In 2 and 3 dimensions, if $T_{\mu\nu} = 0$ then there is no gravity ($R_{\lambda\mu\nu\kappa} = 0$).
- We could modify the assumption ii) and include $g_{\mu\nu}$ (without derivatives) in the definition of $G_{\mu\nu}$, then $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + -\Lambda g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G\, T_{\mu\nu}$$

Λ: the cosmological constant

corrections to Newton's theory $\Rightarrow \Lambda \ll 1$

Coordinate conditions — gauge conditions

- $G_{\mu\nu}$ as a symmetric tensor has 10 independent components
- $G_{\mu\nu} = -8\pi G\, T_{\mu\nu} \Rightarrow$ 10 independent equations

One could expect that the Einstein equations determine 10 components of $g_{\mu\nu}$ uniquely, however this is not so!

- The Bianchi identities : $G^{\mu}{}_{\nu;\mu} = 0$
  impose 4 differential constraints, so only $G_{\mu\nu}$ has $10-4 = 6$ components

$\Downarrow$

the Einstein equation, $G_{\mu\nu} = -8\pi G\, T_{\mu\nu}$, can not determine all 10 components of $g_{\mu\nu}$. There remain four degrees of freedom.

$\Downarrow$

If $g_{\mu\nu}$ is a solution of Einstein's equation, then $g_{\mu\nu}'$, which is determined from $g_{\mu\nu}$ by a general coordinate transformation $x \to x'$, is also a solution.
It involves four arbitrary functions $x'^{\mu}(x)$.

- Electromagnetic analogy:

$$\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = -J^\beta \quad \Rightarrow \quad \Box A_\alpha - \frac{\partial^2}{\partial x^\alpha \partial x_\beta} A^\beta = -J_\alpha$$

four equations for four functions A_α, but they don't determine A_α uniquely;

$$\frac{\partial}{\partial x^\alpha}\Big| \dots \Rightarrow \frac{\partial}{\partial x^\alpha}\left\{ \Box A^\alpha - \frac{\partial^2}{\partial x_\alpha \partial x^\beta} A^\beta \right\} = 0 \qquad \left(\text{as } \frac{\partial J^\alpha}{\partial x^\alpha} = 0 \right)$$

analog of the Bianchi identities

The number of functionally independent equations is $4-1 = 3$, so there is one degree of freedom in the solution for A_α!

the freedom corresponds to gauge invariance; given any solution A_α, we can construct another one
$$A'_\alpha = A_\alpha + \frac{\partial \Lambda}{\partial x^\alpha} \quad \text{for arbitrary function } \Lambda(x).$$

In electrodynamics we remove the ambiguity by choosing a particular gauge, e.g. the Lorentz gauge:
$$\partial_\alpha A'^\alpha = 0$$
It is possible, since $A'_\alpha = A_\alpha + \dfrac{\partial \Phi}{\partial x^\alpha}$
$$\Box \Phi = -\frac{\partial A^\alpha}{\partial x^\alpha} \quad \Leftarrow \quad \partial_\alpha A'^\alpha = \partial_\alpha A^\alpha + \Box \Phi$$
↑ this determines Φ for any A_α.

In gravity the freedom of choosing the coordinate system is an analog of gauge fixing in QED.
The harmonic coordinate conditions:
$$\Gamma^\lambda \equiv g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0$$
is particularly convenient. Is such a choice always possible?

$$g'^{\mu\nu} \Big| \quad \Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\rho_{\tau\sigma} - \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}$$

$$\Gamma'^\lambda = \underbrace{\frac{\partial x'^\lambda}{\partial x^\rho} \underbrace{g'^{\mu\nu} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}}_{g^{\tau\sigma}} \Gamma^\rho_{\tau\sigma}}_{\Gamma^\rho} - g^{\rho\sigma} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}$$

$$\Gamma'^\lambda = \frac{\partial x'^\lambda}{\partial x^\rho} \Gamma^\rho - g^{\rho\sigma} \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}$$
to achieve $\Gamma'^\lambda = 0$ one needs to solve for $x'(x)$:

$\Gamma'^\lambda = 0$ in the system $x'^\mu(x)$ $\quad \Leftarrow \quad g^{\rho\sigma} \dfrac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma} = \dfrac{\partial x'^\lambda}{\partial x^\rho} \Gamma^\rho$

- Note that the condition $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}=0$ is not generally covariant as its purpose is to remove the ambiguity in the metric solution that arise due to the general covariance of the Einstein equations.

Using the affine connection expressed through the metric tensor we obtain:

$$\Gamma^{\lambda} = g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\mu\nu} g^{\lambda\kappa}\left\{\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}}\right\}$$

Recall that $g^{\lambda\kappa}\dfrac{\partial g_{\kappa\mu}}{\partial x^{\nu}} = - g_{\kappa\mu}\dfrac{\partial g^{\lambda\kappa}}{\partial x^{\nu}}$ and

$$\mathrm{Tr}\left\{M^{-1}(x)\frac{\partial}{\partial x^{\kappa}}M(x)\right\} = \frac{\partial}{\partial x^{\kappa}} \ln \mathrm{Det}\, M(x)$$

for $M(x) = g_{\mu\nu}(x)$

$$\Rightarrow \frac{1}{2} g^{\nu\mu}\frac{\partial g_{\mu\nu}}{\partial x^{\kappa}} = g^{-1/2}\frac{\partial}{\partial x^{\kappa}} g^{1/2}$$

$$\Downarrow$$

$$\Gamma^{\lambda} = - g^{-1/2}\frac{\partial}{\partial x^{\kappa}}\left(g^{\lambda\kappa} g^{1/2}\right)$$

So the harmonic coordinate condition reads

$$\frac{\partial}{\partial x^{\kappa}}\left(g^{\lambda\kappa} g^{1/2}\right) = 0$$

- ϕ is called "harmonic" if $\Box\phi = 0$

$$\Box\phi \equiv (g^{\lambda\kappa}\phi_{;\lambda})_{;\kappa}$$

$$\Box\phi = \left(g^{\lambda\kappa}\frac{\partial\phi}{\partial x^{\lambda}}\right)_{;\kappa} = g^{\lambda\kappa}\left(\frac{\partial\phi}{\partial x^{\lambda}}\right)_{;\kappa} = g^{\lambda\kappa}\left(\frac{\partial^2\phi}{\partial x^{\kappa}\partial x^{\lambda}} - \Gamma^{\eta}_{\lambda\kappa}\frac{\partial\phi}{\partial x^{\eta}}\right) =$$

$$= g^{\lambda\kappa}\frac{\partial^2\phi}{\partial x^{\kappa}\partial x^{\lambda}} - \Gamma^{\eta}\frac{\partial\phi}{\partial x^{\eta}}$$

$$\Gamma^{\eta} = 0 \Rightarrow \Box\phi = g^{\lambda\kappa}\frac{\partial^2\phi}{\partial x^{\kappa}\partial x^{\lambda}}$$

$$\Downarrow$$

$\Box x^{\mu} = 0$ "harmonic" coordinate conditions

$$\frac{\partial}{\partial x^k}\left(g^{\lambda k} g^{1/2}\right) = g^{\lambda k}\,\partial_k g^{1/2} + \left(\partial_k g^{\lambda k}\right) g^{1/2}$$

$$\Gamma^\lambda = \frac{1}{2} g^{\mu\nu} g^{\lambda k}\left(\frac{\partial g_{k\mu}}{\partial x^\nu} + \frac{\partial g_{k\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^k}\right) =$$

$$= \underbrace{-\frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^k}}_{= \; g^{-1/2}\frac{\partial}{\partial x^k} g^{1/2}} \cdot g^{\lambda k} + \underbrace{\frac{1}{2} g^{\mu\nu} (-)\, g_{k\mu} \frac{\partial g^{\lambda k}}{\partial x^\nu} \times 2}_{-\frac{1}{2} \; \frac{\partial g^{\lambda\nu}}{\partial x^\nu}}$$

$$= - g^{-1/2}\,\partial_k g^{1/2}\; g^{\lambda k} - \frac{\partial g^{\lambda\nu}}{\partial x^\nu} = - g^{-1/2}\left[g^{\lambda k}\partial_k g^{1/2} + g^{1/2}\partial_k g^{\lambda k}\right] =$$

$$= - g^{-1/2}\,\partial_k\left(g^{\lambda k} g^{1/2}\right)$$

The weak field approximation

Assume $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ and find equation of motion for the fluctuation $h_{\mu\nu}(x)$.

The curvature tensor: $R^{\lambda}{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}$

The affine connection: $\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left\{\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right\}$

The Ricci tensor: $R_{\mu\kappa} \equiv g^{\lambda\nu} R_{\lambda\mu\nu\kappa} = g^{\lambda\nu} g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\kappa} = R^{\sigma}{}_{\mu\sigma\kappa}$

$$\Gamma^{\sigma}_{\lambda\mu} = O(h) \;\rightarrow\; R^{\lambda}{}_{\mu\nu\kappa} = \underbrace{\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}}}_{=\,O(h)} + O(h^2)$$

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + O(h^2)$$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}\left[\frac{\partial h_{\rho\nu}}{\partial x^{\mu}} + \frac{\partial h_{\rho\mu}}{\partial x^{\nu}} - \frac{\partial h_{\mu\nu}}{\partial x^{\rho}}\right] + O(h^2)$$

Since we restrict ourselves to first order in h, we should use $\eta^{\mu\nu}$, not $g^{\mu\nu}$ to raise indices:

$$\eta^{\lambda\rho} h_{\rho\nu} = h^{\lambda}{}_{\nu} \qquad \eta^{\lambda\rho}\frac{\partial}{\partial x^{\rho}} = \frac{\partial}{\partial x_{\lambda}} \quad \ldots$$

Then we obtain

$$R_{\mu\nu} \simeq R^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left\{\Box h_{\mu\nu} - \frac{\partial^2 h^{\lambda}{}_{\nu}}{\partial x^{\lambda}\partial x^{\mu}} - \frac{\partial^2 h^{\lambda}{}_{\mu}}{\partial x^{\lambda}\partial x^{\nu}} + \frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\mu}\partial x^{\nu}}\right\}$$

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}\right) \quad \text{the Einstein's equation}$$

$$\Box h_{\mu\nu} - \frac{\partial^2 h^{\lambda}{}_{\nu}}{\partial x^{\lambda}\partial x^{\mu}} - \frac{\partial^2 h^{\lambda}{}_{\mu}}{\partial x^{\lambda}\partial x^{\nu}} + \frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\mu}\partial x^{\nu}} + O(h^2) = -16\pi G\left[T^{(0)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{(0)\lambda}{}_{\lambda} + O(h)\right]$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

$$R = g^{\mu\nu} R_{\mu\nu} = \eta^{\mu\nu} R^{(1)}_{\mu\nu} + O(h^2) = R^{(1)} + O(h^2)$$

$$= \frac{1}{2}\left\{ \dots \right.$$

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} + O(h^2) = -8\pi G\left(T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu\,+\dots}\right) \Big| \cdot \eta^{\mu\nu}$$

$$R^{(1)} - 2R^{(1)} + O(h^2) = -8\pi G\left(T^{(0)} + T^{(1)}_{+\dots}\right) \qquad T^{(1)} \equiv \eta^{\mu\nu} T^{(1)}_{\mu\nu}$$

$$R^{(1)} = 8\pi G\left(T^{(0)} + T^{(1)}\right) + O(h^2)$$

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Big[8\pi G\big(T^{(0)} + T^{(1)}\big)\underset{+\,O(h^2)}{}\Big] + O(h^2) = -8\pi G\left(T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu\,+\dots}\right)$$

$$R^{(1)}_{\mu\nu} = -8\pi G\left\{T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\left(T^{(0)} + T^{(1)}\right)\right\} + O(h^2)$$

$$= -8\pi G\left\{T^{(0)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{(0)} + \underbrace{\left(T^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{(1)}\right)}_{O(h)}\right\} + O(h^2)$$

if $8\pi G\left(T^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{(1)}_{\alpha\beta}\eta^{\alpha\beta}\right) = O(h^2)$ then o.k.

$$R^{(1)}_{\mu\nu} = -8\pi G\left(T^{(0)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{(0)}\right)$$

In addition we assume that $G\,T^{(0)}_{\mu} = O(h)$, then we obtain the graviton equation of motion:

$$\Box h_{\mu\nu} - \frac{\partial^2 h^{\lambda}{}_{\nu}}{\partial x^{\lambda}\partial x^{\mu}} - \frac{\partial^2 h^{\lambda}{}_{\mu}}{\partial x^{\lambda}\partial x^{\nu}} + \frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\mu}\partial x^{\nu}} = -16\pi G\,\overline{T}^{(0)}_{\mu\nu}$$

where $\overline{T}^{(0)}_{\mu\nu} \equiv T^{(0)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{(0)\lambda}{}_{\lambda}$

The same equation that we have already obtained for a spin 2 field ($\lambda = 16\pi G$)

Note that $T^{(0)}_{\mu\nu}$ satisfies ordinary conservation law

$$\frac{\partial}{\partial x^{\mu}} T^{(0)\mu}{}_{\nu} = 0$$

(as it doesn't contain $h_{\mu\nu}$)

Let's denote $\bar{T}^{(0)}_{\mu\nu}$ by $S_{\mu\nu}$, then we have

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\lambda \partial_\nu h^\lambda{}_\mu + \partial_\mu \partial_\nu h^\lambda{}_\lambda = -16\pi G S_{\mu\nu} \quad (*)$$

- Take the trace ($\eta^\mu{}_\nu$ (*)) :

$$\Box h^\mu{}_\mu - \partial_\lambda \partial_\mu h^{\lambda\mu} - \partial_\lambda \partial_\nu h^{\lambda\nu} + \Box h^\lambda{}_\lambda = -16\pi G \, S^\mu{}_\mu$$

$$\cancel{2}\underbrace{\left(\Box h^\mu{}_\mu - \partial_\lambda \partial_\mu h^{\lambda\mu}\right)} = -8\cancel{16}\pi G \, S^\mu{}_\mu$$

- Take a derivative ∂_μ :

$$\cancel{\partial_\mu \Box h^\mu{}_\nu} - \cancel{\partial_\lambda \Box h^\lambda{}_\nu} - \partial_\mu \partial_\lambda \partial_\nu h^{\lambda\mu} + \Box \partial_\nu h^\lambda{}_\lambda = -16\pi G \partial_\mu S^\mu{}_\nu$$

$$\partial_\nu \left(\Box h^\lambda{}_\lambda - \partial_\mu \partial_\lambda h^{\lambda\mu}\right) = -16\pi G \, \partial_\mu S^\mu{}_\nu$$

$$\Downarrow$$

$$\partial_\nu (-) 8\pi G \, S^\mu{}_\mu = -16\pi G \, \partial_\mu S^\mu{}_\nu$$

$$\frac{1}{2} \partial_\nu S^\mu{}_\mu = \partial_\mu S^\mu{}_\nu$$

$$\Downarrow$$

$$\frac{1}{2} \partial_\nu \left(T^{(0)\mu}{}_\mu - \frac{1}{2}\cdot 4 \, T^{(0)\lambda}{}_\lambda\right) = \partial_\mu T^{(0)\mu}{}_\nu - \frac{1}{2} \eta^\mu{}_\nu \partial_\mu T^{(0)\lambda}{}_\lambda$$

$$-\frac{1}{2} \partial_\nu T^{(0)\mu}{}_\mu = \partial_\mu T^{(0)\mu}{}_\nu - \frac{1}{2} \partial_\nu T^{(0)\mu}{}_\mu$$

$$\Downarrow$$

$$\partial_\mu T^{(0)\mu}{}_\nu = 0$$

- So we have shown that the ordinary conservation law is what is necessary for the consistency of (*).

- We have also shown that the linearized Ricci tensor satisfies Bianchi identities of the form

$$\partial_\mu R^{(1)\mu}{}_\nu = \frac{1}{2} \partial_\nu R^{(1)\mu}{}_\mu$$ since $R^{(1)}_{\mu\nu} = -8\pi G \, S_{\mu\nu}$

(*) doesn't yield a unique solutions since for any given solution we can always generate other solution by applying coordinate transformation:

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$$

We assume here that $\frac{\partial \varepsilon^\mu}{\partial x^\nu} \lesssim h_{\mu\nu}$,

since $g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\rho} g^{\lambda\rho}$ $\quad g^{\mu\nu} \simeq \eta^{\mu\nu} \cancel{+} h^{\mu\nu}$

$\hookrightarrow = \frac{\partial x^\mu}{\partial x^\rho} + \frac{\partial \varepsilon^\mu}{\partial x^\rho}$

we have

$$h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \varepsilon^\mu}{\partial x^\lambda} \eta^{\lambda\nu} - \frac{\partial \varepsilon^\nu}{\partial x^\rho} \eta^{\rho\mu}$$

If $h_{\mu\nu}$ is a solution of (*) then so is

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu \qquad (\varepsilon_\nu \equiv \eta_{\nu\mu} \varepsilon^\mu)$$

the gauge invariance of (*).

The harmonic coordinate system was defined by

$$\Gamma^\lambda \equiv g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0$$

$$\Downarrow \qquad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\rho} (\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}) + O(h^2)$$

harmonic gauge: $\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu h^\mu{}_\mu$ (*)

If $h_{\mu\nu}$ doesn't satisfy (*) then one can find $h'_{\mu\nu}$ such that it does:

$$\partial_\mu h'^\mu{}_\nu = \partial_\mu h^\mu{}_\nu - \Box \varepsilon_\nu - \partial_\mu \partial_\nu \varepsilon^\mu$$

$$\frac{1}{2} \partial_\nu h'^\mu{}_\mu = \frac{1}{2} \left(\partial_\nu h^\mu{}_\mu - \partial_\nu \partial^\mu \varepsilon_\mu - \partial_\nu \partial^\mu \varepsilon_\mu \right) = \frac{1}{2} \partial_\nu h^\mu{}_\mu - \partial_\mu \partial_\nu \varepsilon^\mu$$

$$\Downarrow$$

$$\Box \varepsilon_\nu = \partial_\mu h^\mu{}_\nu - \frac{1}{2} \partial_\nu h^\mu{}_\mu \Rightarrow h'_{\mu\nu} \text{ in the harmonic gauge}$$

$$g'^{\mu\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\lambda\rho}(x)$$

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$$

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$$

$$g^{\mu\nu}(x) = \bar{g}^{\mu\nu}(x) - h^{\mu\nu}(x) + o(h^2) \Leftarrow g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}{}_{\lambda}$$

definition of $h'^{\mu\nu}$

$$\bar{g}^{\mu\nu}(x') - h'^{\mu\nu}(x') + \ldots = (\delta^{\mu}_{\lambda} + \partial_{\lambda}\varepsilon^{\mu})(\delta^{\nu}_{\rho} + \partial_{\rho}\varepsilon^{\nu})(\bar{g}^{\lambda\rho}(x) - h^{\lambda\rho}(x) + \ldots)$$

$$h'^{\mu\nu}(x+\varepsilon) = h'^{\mu\nu}(x) + \partial_{\alpha} h^{\mu\nu}(x)\,\varepsilon^{\alpha} + \cdots$$

$$\bar{g}^{\mu\nu}(x+\varepsilon) = \bar{g}^{\mu\nu}(x) + \partial_{\alpha}\bar{g}^{\mu\nu}(x)\,\varepsilon^{\alpha} + \cdots$$

$$\bar{g}^{\mu\nu}(x) + \partial_{\alpha}\bar{g}^{\mu\nu}(x)\,\varepsilon^{\alpha} - h'^{\mu\nu}(x) \underbrace{- \partial_{\alpha} h^{\mu\nu}(x)\,\varepsilon^{\alpha}}_{o(h^2)} + \cdots =$$

$$= \bar{g}^{\mu\nu}(x) + \partial_{\lambda}\varepsilon^{\mu}\,\bar{g}^{\lambda\nu} + \partial_{\rho}\varepsilon^{\nu}\,\bar{g}^{\mu\rho} - h^{\mu\nu}(x) + \cdots$$

$$-h'^{\mu\nu}(x) = -h^{\mu\nu}(x) - \partial_{\alpha}\bar{g}^{\mu\nu}(x)\,\varepsilon^{\alpha} + \partial_{\lambda}\varepsilon^{\mu}\,\bar{g}^{\lambda\nu} + \partial_{\rho}\varepsilon^{\nu}\,\bar{g}^{\mu\rho}$$

$$h'^{\mu\nu}(x) = h^{\mu\nu}(x) + \partial_{\alpha}\bar{g}^{\mu\nu}(x)\,\varepsilon^{\alpha}(x) - \partial_{\lambda}\varepsilon^{\mu}\,\bar{g}^{\lambda\nu} - \partial_{\rho}\varepsilon^{\nu}\,\bar{g}^{\mu\rho}$$

Adopting the harmonic gauge the equation of motion for the graviton simplifies a lot:

$$\Box h_{\mu\nu} = -16\pi G\, S_{\mu\nu} \qquad \text{show}$$

A possible solution is the retarded potential

$$h_{\mu\nu}(x) = 4G \int d^3x' \, \frac{S_{\mu\nu}(t - |\bar{x} - \bar{x}'|, \bar{x}')}{|\bar{x} - \bar{x}'|} \qquad (*)$$

Homework: show that $h_{\mu\nu}(x)$ given by $(*)$ automatically satisfies the harmonic gauge condition

For a general solution one needs to add to $(*)$ any solution of the homogenous equations:

$$\left.\begin{aligned} &\Box h_{\mu\nu} = 0 \\ &\partial_\mu h^\mu{}_\nu = \tfrac{1}{2} \partial_\nu h^\mu{}_\mu \end{aligned}\right\} (*)$$

Note that $(*)$ shows that $h_{\mu\nu}(x)$ propagates with finite velocity (so with the speed of light).

<u>Plane waves</u>

$$h_{\mu\nu}(x) = e_{\mu\nu}\, e^{ik_\lambda x^\lambda} + e^*_{\mu\nu}\, e^{-ik_\lambda x^\lambda} \quad \text{– solution of } (*)$$

$$\Box h_{\mu\nu} = 0 \quad \Rightarrow \quad \boxed{k_\lambda k^\lambda = 0}$$

$$\partial_\mu h^\mu{}_\nu = \tfrac{1}{2} \partial_\nu h^\mu{}_\mu \quad \Rightarrow \quad \mathrm{Im}\left(V_\nu e^{ikx}\right) = 0$$

$$\text{for } V_\nu \equiv k_\mu e^\mu{}_\nu - \tfrac{1}{2} k_\nu e^\mu{}_\mu$$

$$\Downarrow$$

$$\boxed{k_\mu e^\mu{}_\nu = \tfrac{1}{2} k_\nu e^\mu{}_\mu} \quad \Leftarrow \quad V_\nu = 0 \qquad \left(\begin{aligned} k^\mu &= \eta^{\mu\nu} k_\nu \\ e^\mu{}_\nu &= \eta^{\mu\lambda} e_{\lambda\nu} \end{aligned} \right)$$

$$\boxed{e_{\mu\nu} = e_{\nu\mu}} \qquad e_{\mu\nu} \text{ – the polarization tensor}$$

$e_{\mu\nu} = e_{\nu\mu} \Rightarrow$ 10 independent components

$k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\lambda{}_\lambda \Rightarrow$ $e_{\mu\nu}$ has $10-4=6$ independent components

$\uparrow$ 4 constraints

How many of these 6 components of $e_{\mu\nu}$ are physical?

change of coordinates : $x^\mu \to x^\mu + \varepsilon^\mu(x)$

$$\Downarrow$$

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \varepsilon_\mu}{\partial x^\nu} - \frac{\partial \varepsilon_\nu}{\partial x^\mu}$$

suppose that

$$\varepsilon^\mu(x) = i\varepsilon^\mu e^{ik_\lambda x^\lambda} - i\varepsilon^{\mu *} e^{-ik_\lambda x^\lambda}$$

$$\Downarrow$$

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu$$ (the harmonic condition still satisfied) show

$e'_{\mu\nu}$ and $e_{\mu\nu}$ describe the same physics !

$$\Downarrow$$

4 degrees of freedom could be eliminated by choosing ε_μ $\Downarrow$

$6-4=2$ physical degrees of freedom

Example:

a wave traveling along the z-direction :

$k^1 = k^2 = 0 \qquad k^3 = k^0 \equiv k > 0$

$$k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\lambda{}_\lambda \Rightarrow \begin{cases} e_{00} + e_{30} = \frac{1}{2}(e_{00} - e_{11} - e_{22} - e_{33}) \\ \phantom{e_{00} + e_{30}} = -e_{03} - e_{33} \end{cases}$$

$$e_{01} + e_{31} = e_{02} + e_{32} = 0$$

$\Downarrow$ e_{i0} and e_{22} in terms of others six

$$e_{01} = - e_{31} \; ; \quad e_{02} = - e_{32} \; ; \quad e_{03} = - \frac{1}{2}\left(e_{00} + e_{33} \right)$$

$$e_{22} = - e_{11}$$ (six are still independent:

$e_{11}, e_{12}, e_{13}, e_{23}, e_{33}, e_{00}$)

Apply the transformation to those which are independent

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu$$

$k_0 \; k_1 \; k_2 \; k_3$

$k_\mu = (k, 0, 0, -k)$

then

$$e'_{11} = e_{11} \qquad e'_{12} = e_{12}$$
$$e'_{13} = e_{13} - k\varepsilon_1 \qquad e'_{23} = e_{23} - k\varepsilon_2$$
$$e'_{33} = e_{33} - 2k\varepsilon_3 \qquad e'_{00} = e_{00} + 2k\varepsilon_0$$

$\Downarrow$

only e_{11} and e_{12} have an absolute physical significance

Choosing

$$\varepsilon_1 = \frac{e_{13}}{k} \qquad \varepsilon_2 = \frac{e_{23}}{k} \qquad \varepsilon_3 = \frac{e_{33}}{2k} \qquad \varepsilon_0 = - \frac{e_{00}}{2k}$$

we would obtain <u>the two independent</u> degrees of freedom

$$e_{11} \neq 0 \qquad e_{12} \neq 0 \qquad e_{22} \neq 0 \; (= e_{11})$$

Let's perform a rotation around the z-axis (Lorentz transformation) :

$$R_1{}^1 = \cos\theta \qquad R_1{}^2 = \sin\theta$$
$$R_2{}^1 = -\sin\theta \qquad R_2{}^2 = \cos\theta$$
$$R_3{}^3 = R_0{}^0 = 1 \qquad \text{other } R_\mu{}^\nu = 0$$

$$e'_{\mu\nu} = R_\mu{}^\rho R_\nu{}^\sigma e_{\rho\sigma}$$ (note that k_μ is unchanged : $R_\mu{}^\nu k_\nu = k_\mu$)

$$h_{\mu\nu}(x) = e_{\mu\nu}\, e^{ikx} + e^{*}_{\mu\nu}\, e^{-ikx} \qquad e_{\mu\nu} = e_{\nu\mu}$$

$$\Box h_{\mu\nu} = 0 \qquad \& \qquad \partial_\mu h^\mu{}_\nu = \tfrac{1}{2}\partial_\nu h^\mu{}_\mu$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$k^2 = 0 \qquad\qquad k_\mu e^\mu{}_\nu = \tfrac{1}{2} k_\nu e^\mu{}_\mu$$

$$x^\mu \to x^\mu + \varepsilon^\mu(x)$$

$$L = i\varepsilon^\mu e^{ikx} - i\varepsilon^{\mu *} e^{-ik}$$

$$e_{\mu\nu} \to e'_{\mu\nu} = e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu$$

Example

$k^1 = k^2 = 0 \qquad k^3 = k^0 = k > 0 \qquad \Leftarrow \quad k^2 = 0$

e_{0i} and e_{22} in terms of $\qquad \Leftarrow \quad k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu$

$e_{11},\ e_{12},\ e_{13},\ e_{23},\ e_{33},\ e_{00}$

$\boxed{e'_{11} \neq 0,\ e'_{12} \neq 0,}\ e'_{13} = 0,\ e'_{23} = 0 \qquad \Leftarrow \quad e_{\mu\nu} \to e'_{\mu\nu} = e_{\mu\nu} + \dots$

$e'_{33} = 0,\ e_{00} = 0$

for proper choice of ε_μ

$$h'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} + \text{c.c.} =$$

$$a \equiv e'_{11} = e'_{22} \qquad b \equiv e'_{12}$$

$$= \left[a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] e^{ikx} + \text{c.c.}$$

two independent polarizations ≡ two degrees of freedom

even though a, b are complex!

Keeping all 6 independent components of $e_{\mu\nu}$ we obtain

$$e'_{\pm} = e^{\pm 2i\theta} e_{\pm}$$

$$f'_{\pm} = e^{\pm i\theta} f_{\pm}$$

$$e'_{33} = e_{33} \qquad e'_{00} = e_{00}$$

where

$$e_{\pm} = e_{11} \mp i e_{12} = -e_{22} \mp i e_{12}$$

$$f_{\pm} = e_{31} \mp i e_{32} = -e_{01} \pm i e_{02}$$

Def. A field ψ that transforms under a rotation by an angle θ around the direction of propagation as

$$\psi \to \psi' = e^{ih\theta} \psi$$

is said to have helicity h.

$$\Downarrow$$

$e_{\pm}$ has helicity ± 2

$f_{\pm}$ ± 1

e_{33} and e_{00} have helicity 0

— since $f_{\pm}$ and e_{33} and e_{00} could be removed by a gauge transformation we conclude that the graviton has helicity $= 2$.

Electrodynamic analogy :

Maxwell equation in the Lorentz gauge: $\Box A_\alpha = 0 \qquad \frac{\partial A^\alpha}{\partial x^\alpha} = 0$

$$\Updownarrow$$

$$\Box h_{\mu\nu} = 0 \qquad \frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu$$

$$A_\alpha = e_\alpha e^{ik_\beta x^\beta} + e_\alpha^* e^{-ik_\beta x^\beta} \qquad \text{with} \qquad k_\mu k^\mu = 0$$

$$k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu \longleftrightarrow k_\mu e^\mu = 0$$

- e_μ has 4 independent components in general
- $k_\mu e^\mu$ eliminates 1, so $4-1=3$ remains
- there exists the residual gauge symmetry (after fixing by $\partial_\mu A^\mu = 0$) :

$$A_\alpha \to A_\alpha' = A_\alpha + \frac{\partial \phi}{\partial x^\alpha}$$

for $\phi(x) = i\varepsilon e^{ik_\beta x^\beta} - i\varepsilon^* e^{-ik_\beta x^\beta}$ then

$$A_\alpha' = e_\alpha' e^{ik_\beta x^\beta} + e_\alpha'^* e^{-ik_\beta x^\beta}$$

with

$$e_\alpha' = e_\alpha - \varepsilon k_\alpha \Rightarrow$$ 1 extra degree of freedom can be eliminated

$\Downarrow$

2 degrees of freedom are physical

Example:

$k^\alpha = (k, 0, 0, k)$

$e_\mu k^\mu = 0 \Rightarrow e_0 = -e_3$

since $e_3' = e_3 + \varepsilon k$ then

for $\varepsilon = -\frac{e_3}{k}$ we obtain $e_3' = 0$

$\Downarrow$

e_1, e_2 remain

$$\phi = i\varepsilon e^{ik_\beta x^\beta} - i\varepsilon^* e^{-ik_\beta x^\beta}$$

$$\partial_\alpha \phi = i\varepsilon\, ik_\alpha e^{ikx} - i\varepsilon^* (-ik_\alpha) e^{-ik\cdot x} =$$

$$= -\varepsilon k_\alpha e^{ikx} - \varepsilon^* k_\alpha e^{-ikx}$$

$$A_\alpha = e_\alpha e^{ikx} + e_\alpha^* e^{-ikx}$$

$$A_\alpha' = A_\alpha + \partial_\alpha \phi = \underbrace{(e_\alpha - \varepsilon k_\alpha)}_{e_\alpha'} e^{ikx} + (e_\alpha - \varepsilon k_\alpha)^* e^{-ikx}$$

note that $\quad e_\alpha' \cdot k^\alpha = \underset{0}{\underset{\shortparallel}{e\cdot k}} - \varepsilon \underset{0}{\underset{\shortparallel}{k^2}} = 0$

Again the rotation by θ around the z-axis

$$e_\alpha' = R_\alpha{}^\beta e_\beta$$

$$\Downarrow$$

$$e_\pm' = e^{\pm i\theta} e_\pm \quad \text{for} \quad e_\pm \equiv e_1 \mp i e_2$$

$$e_3' = e_3$$

Since e_3 can be gauged away, so we obtain that the electromagnetic wave has just $3-1=2$ degrees of freedom carrying the helicity $h=1$.

Energy and momentum of gravitation

Let us adopt quasi-Minkowskian coordinate system that is defined by the requirement of $g_{\mu\nu}(x) \to \eta_{\mu\nu}$ at great distances:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

such that $h_{\mu\nu}(x) \underset{|\bar{x}|\to\infty}{\longrightarrow} 0$.

- Note that this is the case for the harmonic coordinate system, since then we have

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu} \quad \text{for} \quad S_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} T^\lambda{}_\lambda$$

$$\Downarrow$$

$$h_{\mu\nu}(x) = 4G \int d^3\bar{x}' \, \frac{S_{\mu\nu}(\bar{x}', t-|\bar{x}-\bar{x}'|)}{|\bar{x}-\bar{x}'|}$$

if $S_{\mu\nu}(\bar{x},t) \sim \delta^3(\bar{x})$ then $h_{\mu\nu} \sim \dfrac{1}{|\bar{x}|}$

- Note also that $h_{\mu\nu}(x)$ is not assumed to be small everywhere.

$$R_{\mu\kappa} = \underbrace{\frac{1}{2}\left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\kappa} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h^\lambda{}_\kappa}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x_\lambda} \right)}_{\equiv\, R^{(1)}_{\mu\kappa}} + O(h^2)$$

$h^\lambda{}_\lambda \equiv \eta^{\lambda\nu} h_{\nu\lambda}$

$\frac{\partial}{\partial x_\lambda} \equiv \eta^{\lambda\nu} \frac{\partial}{\partial x^\nu}$

$\eta^{\lambda\nu}$ for raising and lowering

Then the exact Einstein equations

$$R^{(1)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa} R^{(1)\lambda}{}_\lambda = -8\pi G \left(T_{\mu\kappa} + t_{\mu\kappa} \right)$$

where

$$t_{\mu\kappa} \equiv \frac{1}{8\pi G}\left[\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\lambda{}_\lambda \right) - \left(R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)\lambda}{}_\lambda \right) \right]$$

$-8\pi G \left(T_{\mu\kappa} + t_{\mu\kappa} \right)$ is the source for the spin 2 $h_{\mu\nu}$

$t_{\mu\kappa}$ is to be interpreted as the energy-momentum "tensor" of gravitational field itself

$$\tau^{\nu\lambda} \equiv \underbrace{\eta^{\nu\mu}\eta^{\lambda\kappa}}_{\text{note this}} \left[T_{\mu\kappa} + t_{\mu\kappa} \right]$$

is the total energy-momentum "tensor" of gravity and matter

Properties of $\tau^{\nu\lambda}$:

A) The linearized Bianchi identities

$$\frac{\partial}{\partial x^\nu}\left[R^{(1)\nu\lambda} - \frac{1}{2}\eta^{\nu\lambda} R^{(1)\mu}{}_\mu \right] \equiv 0$$

imply that

$$\frac{\partial}{\partial x^\nu} \tau^{\nu\lambda} = 0$$

Note that $T^{\nu\lambda}{}_{;\nu} = 0$, so $T^{\nu\lambda}$ is not conserved in the ordinary sense, therefore it doesn't lead to any conservation law.

Assume that we consider a finite system (contained 112 in some region of finite volume), then $\Rightarrow$

$$\frac{\partial}{\partial x^\nu} \tau^{\nu\lambda} = 0 \Rightarrow \frac{d}{dt} \int_V \tau^{0\lambda} d^3x = - \int_{\partial V} \tau^{i\lambda} n_i \, dS = 0$$

$\uparrow$ Gauss theorem

n_i: normal unit vector outward to the surface

$$p^\lambda \equiv \int_V \tau^{0\lambda} d^3x$$ is the total energy-momentum "vector" of the system

Note that for $\int_{\partial V} \tau^{i\lambda} n_i dS$ to vanish we need also

$\int_{\partial V} t^{i\lambda} n_i dS = 0$, however since 1° $h_{\mu\nu} \sim O(\frac{1}{r})$, $\partial h_{\mu\nu} \sim O(\frac{1}{r^2})$, $\partial^2 h_{\mu\nu} \sim O(\frac{1}{r^3})$

2° $t^{\nu\lambda} \sim O(h\partial^2 h, h\partial h \partial h)$

therefore

$t^{\nu\lambda} \sim O(\frac{1}{r^4})$ $\uparrow$ homework

$\Downarrow$

$$\int_{\partial V} t^{i\lambda} n_i \underbrace{dS}_{r^2 d\Omega} \to 0$$

B) $t_{\mu\nu}$, $\tau_{\mu\nu}$ are not generally covariant, but they are Lorentz covariant

C) Homework is to show that

$$R^{(1)\nu\lambda} - \frac{1}{2}\eta^{\nu\lambda} R^{(1)\mu}{}_\mu = \frac{\partial}{\partial x^\rho} Q^{\rho\nu\lambda}$$

for
$$Q^{\rho\nu\lambda} \equiv \frac{1}{2}\left\{ \frac{\partial h^\mu{}_\mu}{\partial x_\nu}\eta^{\rho\lambda} - \frac{\partial h^\mu{}_\mu}{\partial x_\rho}\eta^{\nu\lambda} - \frac{\partial h^{\mu\nu}}{\partial x^\mu}\eta^{\rho\lambda} + \frac{\partial h^{\mu\rho}}{\partial x^\mu}\eta^{\nu\lambda} + \frac{\partial h^{\nu\lambda}}{\partial x_\rho} - \frac{\partial h^{\rho\lambda}}{\partial x_\nu} \right\}$$

$\Downarrow$

$$Q^{\rho\nu\lambda} = - Q^{\nu\rho\lambda}$$

The antisymmetry was to expect since the linearized Bianchi identity holds: $\frac{\partial}{\partial x^\nu}\left(R^{(1)\nu\lambda} - \frac{1}{2}\eta^{\nu\lambda} R^{(1)\mu}{}_\mu\right) = 0$

$$P^\lambda = \int_V t^{0\lambda} d^3x \underset{\text{field equations}}{=} \frac{-1}{8\pi G}\int_V \frac{\partial Q^{\rho 0\lambda}}{\partial x^\rho} d^3x \underset{\text{antisymmetry}}{=} -\frac{1}{8\pi G}\int_V \frac{\partial Q^{i0\lambda}}{\partial x^i} d^3x \underset{\text{Gauss's theorem}}{=}$$

$$= -\frac{1}{8\pi G}\int_{\partial V} Q^{i0\lambda}\, n_i\, r^2 d\Omega$$

∂V – sphere of radius r

$r = (x^i x^i)^{1/2}$ $\qquad n_i \equiv \frac{x_i}{r}$ $\qquad d\Omega = \sin\theta\, d\theta\, d\varphi$

$$\Downarrow$$

$$P^0 = -\frac{1}{16\pi G}\int \left(\frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j}\right) n_i\, r^2 d\Omega$$

$$P^j = -\frac{1}{16\pi G}\int \left(-\frac{\partial h_{kk}}{\partial t}\delta_{ij} + \frac{\partial h_{k0}}{\partial x^k}\delta_{ij} - \frac{\partial h_{j0}}{\partial x^i} + \frac{\partial h_{ij}}{\partial t}\right) n_i\, r^2 d\Omega$$

D) (Homework) P^λ is invariant under any coordinate transforma-tion that reduces at infinity to identity, so

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x), \text{ such that } \varepsilon^\mu(x) \xrightarrow[|\vec{x}|\to\infty]{} 0 .$$

The matter action

The following equations hold for a system of n particles of mass m_n and charge e_n together with the electromagnetic field $F_{\mu\nu}(x)$ they produce:

$$\left.\begin{aligned}
&\frac{d^2 x_n^\lambda}{d\tau_n^2} + \Gamma^\lambda_{\nu\lambda}(x_n)\frac{dx_n^\nu}{d\tau_n}\frac{dx_n^\lambda}{d\tau_n} = \frac{e_n}{m_n} F^\lambda{}_\nu(x_n)\frac{dx_n^\nu}{d\tau_n} \\
&d\tau_n \equiv \left(g_{\mu\nu}(x_n)\, dx_n^\mu\, dx_n^\nu\right)^{1/2} \\
&\frac{\partial}{\partial x^\mu}\left(\sqrt{g(x)}\, F^{\mu\nu}(x)\right) = -\sum_n e_n \int \delta^4(x - x_n)\frac{dx_n^\nu}{d\tau_n}\, d\tau_n \\
&\frac{\partial}{\partial x^\lambda} F_{\mu\nu}(x) + \frac{\partial}{\partial x^\mu} F_{\nu\lambda}(x) + \frac{\partial}{\partial x^\nu} F_{\lambda\mu}(x) = 0
\end{aligned}\right\} \quad (*)$$

$$F_{\mu\nu}(x) = \frac{\partial A_\nu(x)}{\partial x^\nu} - \frac{\partial A_\mu(x)}{\partial x^\nu}$$

$$\left.\begin{matrix} A_\mu(x) \\ x_n^\nu(p) \end{matrix}\right\} \text{ dynamical variables}$$

p: a parameter that parameterises trajectories of all particles

We are going to show that the following action leads to the above equations of motion:

$$I_M = +\sum_n m_n \int_{-\infty}^{+\infty} dp \left[g_{\mu\nu}(x_n(p)) \frac{dx_n^\mu(p)}{dp}\frac{dx_n^\nu(p)}{dp}\right]^{1/2} + \qquad (**)$$

$$-\frac{1}{4}\int d^4x\, g^{1/2}(x)\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \sum_n e_n \int_{-\infty}^{+\infty} dp \frac{dx_n^\mu(p)}{dp} A_\mu(x_n(p))$$

where $g(x) \equiv -\operatorname{Det} g_{\mu\nu}$

$$F^{\mu\nu}(x) = g^{\mu\alpha}(x)\, g^{\nu\beta}(x)\, F_{\alpha\beta}(x)$$

Gravity is treated here as a fixed external field.

The principle of stationary action says that I_M will not be changed by an infinitesimal variation of the dynamical variables:

$$x^\mu(p) \to x^\mu(p) + \delta x^\mu(p)$$
$$A_\mu(x) \to A_\mu(x) + \delta A_\mu(x)$$

where
$$\delta x^\mu(p) \to 0 \qquad \text{for } |p| \to \infty$$
$$\delta A_\mu(x) \to 0 \qquad \text{for } |x^\mu| \to \infty$$

if and only if $x^\mu(p)$ and $A_\mu(x)$ obey the equations (*).

We are going to check that:

$$\delta I_M = I_M\left[x^\mu(p) + \delta x^\mu(p), A_\mu(x) + \delta A_\mu(x)\right] - I_M\left[x^\mu(p), A_\mu(x)\right] =$$
$$= \frac{1}{2}\sum_n m_n \int_{-\infty}^{+\infty} dp \left[g_{\mu\nu}(x_n(p)) \frac{dx_n^\mu(p)}{dp} \frac{dx_n^\nu(p)}{dp}\right]^{-1/2} \times$$
$$\times \left\{ 2 g_{\mu\nu}(x_n(p)) \frac{dx_n^\mu(p)}{dp} \frac{d\delta x_n^\nu(p)}{dp} + \left.\frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\right|_{x=x_n(p)} \frac{dx_n^\mu(p)}{dp} \frac{dx_n^\nu(p)}{dp} \delta x_n^\lambda(p) \right\} +$$
$$- \int d^4x \, g^{1/2}(x) F^{\mu\nu}(x) \frac{\partial}{\partial x^\mu} \delta A_\nu(x) +$$
$$+ \sum_n e_n \int_{-\infty}^{+\infty} dp \left\{ \frac{d\delta x_n^\mu(p)}{dp} A_\mu(x_n(p)) + \frac{dx_n^\mu(p)}{dp} \left.\frac{\partial A_\mu(x)}{\partial x^\lambda}\right|_{x=x_n(p)} \delta x_n^\lambda(p) + \right.$$
$$\left. + \frac{dx_n^\mu(p)}{dp} \delta A_\mu(x_n(p)) \right\}$$

We change variables of integration from p to τ_n, where τ_n defined by

$$d\tau_n = \left(g_{\mu\nu} \, dx_n^\mu \, dx_n^\nu\right)^{1/2}$$

Note that there is no Jacobian needed.

$$\delta I_M = \frac{1}{2}\sum_n m_n \int_{-\infty}^{+\infty} d\tau_n \left\{ 2 g_{\mu\lambda}(x_n) \frac{dx_n^\mu}{d\tau_n} \frac{d\,\delta x_n^\lambda}{d\tau_n} + \frac{\partial g_{\mu\nu}(x_n)}{\partial x_n^\lambda} \frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n} \delta x_n^\lambda \right\}$$

$$- \int d^4x\, g^{1/2}(x)\, F^{\mu\nu}(x) \frac{\partial}{\partial x^\mu} \delta A_\nu(x) +$$

$$+ \sum_n e_n \int_{-\infty}^{+\infty} d\tau_n \left\{ \frac{d\,\delta x_n^\mu}{d\tau_n} A_\mu(x_n) + \frac{dx_n^\mu}{d\tau_n} \frac{\partial A_\mu}{\partial x_n^\lambda} \delta x_n^\lambda + \frac{dx_n^\mu}{d\tau_n} \delta A_\mu(x_n) \right\}$$

Integrating by parts we obtain

$$\delta I_M = \sum_n \int_{-\infty}^{+\infty} d\tau_n\, g_{\mu\lambda}(x_n) \left\{ -m_n \left[\frac{d^2 x_n^\mu}{d\tau_n^2} + \Gamma^\mu_{\rho\sigma}(x_n) \frac{dx_n^\rho}{d\tau_n} \frac{dx_n^\sigma}{d\tau_n} \right] + \right.$$

using $g_{\mu\lambda}\Gamma^\mu_{\rho\sigma} = \frac{1}{2}\left(\frac{\partial g_{\sigma\lambda}}{\partial x^\rho} + \frac{\partial g_{\rho\lambda}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\lambda} \right)$

$$\left. + e_n \frac{dx_n^\rho}{d\tau_n} F^\mu{}_\rho(x_n) \right\} \delta x_n^\lambda +$$

$$+ \int d^4x \left\{ \frac{\partial}{\partial x^\mu} \left[g^{1/2}(x) F^{\mu\nu}(x) \right] + \sum_n e_n \int_{-\infty}^{+\infty} d\tau_n\, \delta^4(x - x_n) \frac{dx_n^\nu}{d\tau_n} \right\} \delta A_\nu(x)$$

$$\Downarrow$$

$\delta I_M = 0$, $\underbrace{\text{if and only if the equations } (*) \text{ are satisfied.}}_{\text{for any variations } \delta x_n^\lambda \text{ and } \delta A_\nu}$

$$-\frac{\partial g_{\sigma\lambda}}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\delta x^\lambda + \frac{1}{2}\frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\delta x^\lambda$$

$$\left(-\frac{\partial g_{\sigma\lambda}}{\partial x^\rho} + \frac{1}{2}\frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\right)\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\delta x^\lambda =$$

$$g_{\mu\lambda}\Gamma^\mu_{\rho\sigma} = \underbrace{g_{\mu\lambda}\frac{1}{2}g^{\nu\mu}}_{\frac{1}{2}\delta^\nu_\lambda}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\rho} + \frac{\partial g_{\rho\nu}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\nu}\right) =$$

$$= \frac{1}{2}\left(\frac{\partial g_{\sigma\lambda}}{\partial x^\rho} + \frac{\partial g_{\rho\lambda}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\right)$$

$$\rightarrow \underbrace{\left[-\frac{1}{2}\left(\frac{\partial g_{\sigma\lambda}}{\partial x^\rho} + \frac{\partial g_{\rho\lambda}}{\partial x^\sigma}\right) + \frac{1}{2}\frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\right]}_{-g_{\mu\lambda}\Gamma^\mu_{\rho\sigma}}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\delta x^\lambda$$

General definition of the energy-momentum tensor

Imagine $g_{\mu\nu}(x)$ to be a subject of an infinitesimal variation :

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$$

such that

$$\delta g_{\mu\nu}(x) \to 0 \quad \text{for} \quad |x^\lambda| \to \infty$$

The action I_M will change under such variation of $g_{\mu\nu}$ since so far we have treated $g_{\mu\nu}$ as an external field, therefore in general

$$\delta I_M = -\frac{1}{2}\int d^4x \; g(x)^{1/2} \; T^{\mu\nu}(x) \; \delta g_{\mu\nu}(x) + \dots$$

if $x_\mu(\tau)$ and $A_\mu(x)$ are not solutions of $(*)$

We define the energy-momentum tensor by $T^{\mu\nu}(x)$.

Example : we calculate $T^{\mu\nu}$ for the action $(**)$.

$$\delta F^{\mu\nu} = \delta\left(g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}\right) = F_{\rho\sigma}\, \delta\left(g^{\mu\rho} g^{\nu\sigma}\right) = F_{\rho\sigma}\left(g^{\mu\rho}\delta g^{\nu\sigma} + g^{\nu\sigma}\delta g^{\mu\rho}\right)$$

$$0 = \delta\left(g_{\lambda\kappa}\, g^{\kappa\sigma}\right) = g^{\kappa\sigma}\,\delta g_{\lambda\kappa} + g_{\lambda\kappa}\,\delta g^{\kappa\sigma}$$

$$\Rightarrow \delta g^{\nu\sigma} = -g^{\nu\lambda} g^{\kappa\sigma} \delta g_{\lambda\kappa}$$

$$F^{\lambda\nu} = -F^{\nu\lambda}$$

$$\delta F^{\mu\nu} = -F^{\mu\kappa} g^{\nu\lambda} \delta g_{\lambda\kappa} + F^{\nu\lambda} g^{\mu\kappa} \delta g_{\lambda\kappa}$$

As we have shown earlier

$$\delta g = g\, g^{\lambda\kappa}\, \delta g_{\lambda\kappa} \qquad g \equiv -\mathrm{Det}\, g_{\mu\nu}$$

$$\Downarrow$$

$$\delta I_M = -\frac{1}{2}\sum_n m_n \int_{-\infty}^{\infty} dp \left[g_{\mu\nu}(x_n(p)) \frac{dx_n^\mu(p)}{dp} \frac{dx_n^\nu(p)}{dp} \right]^{-1/2} \frac{dx_n^\lambda(p)}{dp} \frac{dx_n^\kappa(p)}{dp} \delta g_{\lambda\kappa}(x_n(p))$$

$$-\frac{1}{2}\int d^4x \, g^{1/2}(x) \left[\frac{1}{4} g^{\lambda\kappa}(x) F_{\mu\nu}(x) F^{\mu\nu}(x) - F_\mu{}^\lambda(x) F^{\mu\kappa}(x) \right] \delta g_{\lambda\kappa}(x)$$

$$\Downarrow$$

$$T^{\lambda\kappa}(x) = g^{-1/2}(x) \sum_n m_n \int_{-\infty}^{+\infty} d\tau_n \frac{dx_n^\lambda}{d\tau_n} \frac{dx_n^\lambda}{d\tau_n} \delta^{(4)}(x - x_n) +$$

$$\frac{1}{4} g^{\lambda\kappa}(x) F_{\mu\nu}(x) F^{\mu\nu}(x) - F_\mu{}^\lambda(x) F^{\mu\kappa}(x)$$

The definition of $T_{\mu\nu}$ is closely analogous to a similar definition of the electric current:

$$I_M = I_E + I_M'$$

$$I_E = -\frac{1}{4}\int d^4x \, g^{1/2}(x) F_{\mu\nu}(x) F^{\mu\nu}(x)$$

Consider the variation $A_\mu(x) \to A_\mu(x) + \delta A_\mu(x)$, then

$$\delta I_M' = \int d^4x \, g^{1/2}(x) J^\mu(x) \delta A_\mu(x)$$

$J^\mu(x)$ is, by definition, the electromagnetic current of the system described by the action I_M'.

Example: for the action $(**)$

$$I_M' = -\sum_n \int_{-\infty}^{+\infty} dp \left[g_{\mu\nu}(x_n(p)) \frac{dx_n^\mu(p)}{dp} \frac{dx_n^\nu(p)}{dp} \right]^{1/2} + \sum_n e_n \int_{-\infty}^{+\infty} dp \frac{dx_n^\mu(p)}{dp} A_\mu(x_n(p))$$

$$\delta I_M' = \sum_n e_n \int_{-\infty}^{+\infty} dx_n^\mu \, \delta A_\mu(x_n)$$

$$\Rightarrow \quad J^\mu(x) = g^{-1/2}(x) \sum_n e_n \int \delta^4(x - x_n) \, dx_n^\mu$$

General covariance and energy-momentum conservation

I_H is a general scalar, so $\delta I_H = 0$ is generally covariant. (xx) is an example of I_H, we shall assume that I_H is always a general scalar, then I_H will remain unchanged under a general coordinate transformation $x \to x'$:

$$d^4x \to d^4x' \qquad x_n^\mu(p) \to x_n'^\mu(p)$$

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} \qquad A_\mu(x) \to A'_\mu(x') \equiv A_\nu(x) \frac{\partial x^\nu}{\partial x'^\mu}$$

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') \equiv g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$$

- x' is just a name of integration variable, so I can relabel it as x without changing I_H (as opposed to x_n^μ, which is a dynamical (variable))
- Let's consider $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$ infinitesimal, then I_H is invariant under

$$x_n^\mu(p) \to x_n'^\mu(p) = x_n^\mu(p) + \varepsilon^\mu(x_n(p))$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - A_\nu(x)\frac{\partial \varepsilon^\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu}\varepsilon^\nu(x)$$

think about δ of μ as invariant?

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = g_{\mu\nu}(x) - g_{\mu\lambda}(x)\frac{\partial \varepsilon^\lambda(x)}{\partial x^\nu} - g_{\lambda\nu}(x)\frac{\partial \varepsilon^\lambda(x)}{\partial x^\mu} + \boxed{- \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\varepsilon^\lambda(x)}$$

See how $A'_\mu(x)$ was derived:

$$A'_\mu(x') = A_\nu(x)\frac{\partial x^\nu}{\partial x'^\mu} = A_\nu(x)\Big(\delta^\nu_\mu - \underbrace{\frac{\partial \varepsilon^\nu(x)}{\partial x'^\mu}}_{\simeq \frac{\partial \varepsilon^\nu(x)}{\partial x^\mu}}\Big) = A_\mu(x) - A_\nu(x)\frac{\partial \varepsilon^\nu(x)}{\partial x^\mu}$$

$$= A'_\mu(x+\varepsilon) = A'_\mu(x) + \underbrace{\frac{\partial A'_\mu(x)}{\partial x^\nu}}_{\simeq \frac{\partial A_\mu(x)}{\partial x^\nu}}\varepsilon^\nu(x) + \dots$$

in the leading order:

$$A'_\mu(x) = A_\mu(x) - A_\nu(x)\frac{\partial \varepsilon^\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu}\varepsilon^\nu(x)$$

Note that now the infinitesimal transformation applies to the dynamical variables only, not to the coordinates over which we integrate.

$\Downarrow$ The principle of stationary action

If the equations of motion for x_n^μ and A_μ are satisfied then the variations $\delta x_n^\mu(p) = \varepsilon^\mu(x_n(p))$ and $\delta A_\mu(x) \equiv A'_\mu(x) - A_\mu(x)$ do not change I_M

$\Downarrow$

The only change comes from $\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x)$

$\Downarrow$

$$\delta I_M = -\frac{1}{2}\int d^4x \, g^{1/2}(x)\, T^{\mu\nu}(x)\,(-)\left[g_{\mu\lambda}\frac{\partial \varepsilon^\lambda}{\partial x^\nu} + g_{\lambda\nu}\frac{\partial \varepsilon^\lambda}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\varepsilon^\lambda\right]$$

If I_M is a scalar, then $\delta I_M = 0$, so integrating by parts we obtain

$$0 = \delta I_M = -\int d^4x \left[\frac{\partial}{\partial x^\nu}\left(g^{1/2} T_\lambda{}^\nu\right) - \frac{1}{2} g^{1/2}\frac{\partial g_{\lambda\nu}}{\partial x^\lambda} T^{\mu\nu}\right]\varepsilon^\lambda$$

for any ε^λ

$\Downarrow$

$$0 = \frac{\partial}{\partial x^\nu}\left(g^{1/2} T_\lambda{}^\nu\right) - \frac{1}{2} g^{1/2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} T^{\mu\nu}$$

We have shown earlier that

$$\Gamma^\lambda_{\mu\lambda} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}\sqrt{g}$$

$$g^{1/2}\Gamma^\mu_{\mu\nu} T_\lambda{}^\nu + g^{1/2}\frac{\partial}{\partial x^\nu} T_\lambda{}^\nu - \frac{1}{2} g^{1/2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} T^{\mu\nu} =$$

$$= g^{1/2}\left[\frac{\partial}{\partial x^\nu} T_\lambda{}^\nu + \Gamma^\mu_{\mu\nu} T^\nu{}_\lambda - \frac{1}{2}\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\lambda}}_{\boxed{\Gamma^s_{\lambda\mu} g_{s\nu} + \Gamma^s_{\lambda\nu} g_{s\mu}}} T^{\mu\nu}\right] =$$

Homework

$$= g^{1/2}\left[\frac{\partial T_\lambda{}^\nu}{\partial x^\nu} + \Gamma^\mu_{\mu\nu} T^\nu{}_\lambda - \Gamma^s_{\lambda\mu} g_{s\nu} T^{\mu\nu}\right] = g^{1/2}\left(T^\nu{}_\lambda\right)_{;\nu}$$

$$0 = \delta I_M = \frac{1}{2}\int d^4x \left[-\frac{\partial}{\partial x^\nu}\left(g^{1/2} T_\lambda{}^\nu\right) - \cancel{g^{1/2} T^{\mu\nu} \frac{\partial g_{\mu\lambda}}{\partial x^\nu}} + \right.$$

$$- \frac{\partial}{\partial x^\mu}\left(g^{1/2} T^\mu{}_\lambda\right) - \cancel{g^{1/2} T^{\mu\nu} \frac{\partial g_{\lambda\nu}}{\partial x^\mu}} +$$

$$\left. + g^{1/2} T^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right] \varepsilon^\lambda =$$

$$= -\int d^4x \cancel{g^{1/2}} \left[\frac{\partial}{\partial x^\nu}\left(g^{1/2} T_\lambda{}^\nu\right) - \frac{1}{2} g^{1/2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right] \varepsilon^\lambda$$

$T^{\mu\nu}$

$$\left(T^\nu{}_\lambda\right)_{;\nu} = g_{\lambda\mu} T^{\nu\mu}{}_{;\nu} = g_{\lambda\mu}\left[\frac{\partial}{\partial x^\nu} T^{\nu\mu} + \Gamma^\nu_{\nu\rho} T^{\rho\mu} + \right.$$

$$\left. + \Gamma^\mu_{\nu\rho} T^{\nu\rho} \right] =$$

$$= g_{\lambda\mu} \frac{\partial T^{\nu\mu}}{\partial x^\nu} + \Gamma^\nu_{\nu\rho} T^\rho{}_\lambda + g_{\lambda\mu} \Gamma^\mu_{\nu\rho} T^{\nu\rho} =$$

$$= \frac{\partial}{\partial x^\nu}\underbrace{\left(g_{\lambda\mu} T^{\nu\mu}\right)}_{T^\nu{}_\lambda = T_\lambda{}^\nu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} T^{\nu\mu} + \Gamma^\nu_{\nu\rho} T^\rho{}_\lambda + g_{\lambda\mu} \Gamma^\mu_{\nu\rho} T^{\nu\rho} =$$

$$= \frac{\partial T_\lambda{}^\nu}{\partial x^\nu} + \Gamma^\mu_{\mu\nu} T^\nu{}_\lambda \underbrace{- \frac{\partial g_{\lambda\mu}}{\partial x^\nu}}_{-\Gamma^\rho_{\nu\lambda} g_{\rho\mu} - \cancel{\Gamma^\rho_{\nu\mu} g_{\rho\lambda}}} T^{\nu\mu} + \cancel{g_{\lambda\mu} \Gamma^\mu_{\nu\rho}} T^{\nu\rho} =$$

$$= \frac{\partial T_\lambda{}^\nu}{\partial x^\nu} + \Gamma^\mu_{\mu\nu} T^\nu{}_\lambda - \Gamma^\rho_{\nu\lambda} g_{\rho\mu} T^{\nu\mu}$$

$\lambda\nu$

$$(T^\nu{}_\lambda)_{;\nu} = 0$$

The energy momentum tensor is conserved in the covariant sense if and only if the matter action is a scalar.
and eq. of mot. for A_μ and x_n^μ are satisfied.

The electrodynamical analog of $(T^\nu{}_\lambda)_{;\nu} = 0$ is the covariant conservation of the electromagnetic current.

For gauge transformation

$$\delta A_\mu = \frac{\partial \varepsilon}{\partial x^\mu}$$

Then

$$0 = \delta I_M' = \int d^4x \, g^{1/2} J^\mu \frac{\partial \varepsilon}{\partial x^\mu} \underset{\text{by parts}}{=} -\int d^4x \frac{\partial}{\partial x^\nu}\left(g^{1/2} J^\nu\right) \cdot \varepsilon$$

$\Downarrow$ ε - arbitrary

$$J^\mu{}_{;\mu} = 0 \quad \Longleftrightarrow \frac{1}{g^{1/2}} \frac{\partial}{\partial x^\nu}\left(g^{1/2} J^\nu\right) = 0$$

The gravitational action

$$I = I_M + I_G$$

a candidate for the gravitational action

$$I_G \equiv \frac{1}{16\pi G} \int g^{1/2}(x) \, R(x) \, d^4x$$

$\Leftarrow$ the only scalar we can construct that contains only second derivatives of $g_{\mu\nu}$

variation of $g_{\mu\nu}(x)$: $\quad g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$

$$\delta\big(g^{1/2} \underbrace{R}_{g^{\mu\nu} R_{\mu\nu}}\big) = g^{1/2} R_{\mu\nu} \, \delta g^{\mu\nu} + R \, \delta g^{1/2} + g^{1/2} g^{\mu\nu} \, \delta R_{\mu\nu}$$

We will use the identity (Palatini identity):

$$\delta R_{\mu\nu} = (\delta \Gamma^\lambda_{\mu\lambda})_{;\nu} - (\delta \Gamma^\lambda_{\mu\nu})_{;\lambda}$$

see page 124

$\delta \Gamma^\lambda_{\mu\lambda}$ is a tensor, see p. 123

So we obtain

$$g^{1/2} g^{\mu\nu} \delta R_{\mu\nu} = g^{1/2} \left[(g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\nu})_{;\lambda} \right]$$

using the identity (derived earlier)

$$V^{\mu}{}_{;\mu} = \frac{1}{g^{1/2}} \frac{\partial}{\partial x^{\mu}} \left(g^{1/2} V^{\mu} \right)$$

we have

$$g^{1/2} g^{\mu\nu} \delta R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}} \left(g^{1/2} g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\lambda} \right) - \frac{\partial}{\partial x^{\lambda}} \left(g^{1/2} g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\nu} \right)$$

therefore $g^{1/2} g^{\mu\nu} \delta R_{\mu\nu}$ drops out upon the integration over all space.

$$\delta g^{1/2} = \frac{1}{2} g^{1/2} g^{\mu\nu} \delta g_{\mu\nu} \quad \text{and} \quad \delta g^{\mu\nu} = - g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}$$

$$\Downarrow$$

$$\delta I_G = \frac{-1}{16\pi G} \int d^4x \, g^{1/2} \left[R_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} - R \frac{1}{2} g^{\rho\sigma} \right] \delta g_{\rho\sigma}$$

$$\Downarrow$$

$$\delta (I_G + I_M) = \frac{-1}{16\pi G} \int d^4x \, g^{1/2} \left[R^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} R + 8\pi G \, T^{\rho\sigma} \right] \delta g_{\rho\sigma} = 0$$

arbitrary

$$\Downarrow$$

The Einstein equation !

$$R^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} R = - 8\pi G \, T^{\rho\sigma}$$

Gravitational fluctuations around a gravitational background

Let's consider a variation of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

such that $\delta g_{\mu\nu}$ is 'small', then the affine connection varies as well

$$\Gamma^{\lambda}_{\mu\nu} \to \Gamma^{\lambda}_{\mu\nu} + \delta\Gamma^{\lambda}_{\mu\nu}$$

for $$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\nu\mu}}{\partial x^{\alpha}} \right\}$$

$$(g^{\alpha\mu} + \delta g^{\alpha\mu})(g_{\mu\nu} + \delta g_{\mu\nu}) = \delta^{\alpha}_{\nu} + O(\delta g^2)$$

$$\Downarrow$$

$$\delta g^{\alpha\lambda} = - g^{\alpha\rho}\, \delta g_{\rho\sigma}\, g^{\sigma\lambda} + O(\delta g^2)$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = - g^{\lambda\rho}\, \delta g_{\rho\sigma}\, \Gamma^{\sigma}_{\mu\nu} + \frac{1}{2} g^{\lambda\rho}\left\{ \frac{\partial \delta g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial \delta g_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial \delta g_{\nu\mu}}{\partial x^{\rho}} \right\} =$$

$$= \frac{1}{2} g^{\lambda\rho}\left\{ \left[\frac{\partial \delta g_{\nu\rho}}{\partial x^{\mu}} - \Gamma^{\sigma}_{\nu\mu}\, \delta g_{\sigma\rho} - \Gamma^{\sigma}_{\rho\mu}\, \delta g_{\nu\sigma} \right] + \left[\frac{\partial \delta g_{\mu\rho}}{\partial x^{\nu}} - \Gamma^{\sigma}_{\mu\nu}\, \delta g_{\sigma\rho} - \Gamma^{\sigma}_{\nu\rho}\, \delta g_{\mu\sigma} \right] - \left[\frac{\partial \delta g_{\nu\mu}}{\partial x^{\rho}} - \Gamma^{\sigma}_{\nu\rho}\, \delta g_{\sigma\mu} - \Gamma^{\sigma}_{\mu\rho}\, \delta g_{\nu\sigma} \right] \right\} =$$

$$= \frac{1}{2} g^{\lambda\rho}\left\{ \delta g_{\nu\rho;\mu} + \delta g_{\mu\rho;\nu} - \delta g_{\nu\mu;\rho} \right\}$$ this is a tensor

$\delta \Gamma^{\lambda}_{\mu\nu}$ implies a change of the Ricci tensor

$$R_{\mu\kappa} \equiv g^{\lambda\nu} R_{\lambda\mu\nu\kappa} \quad \text{where}$$

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\kappa} = g_{\lambda\sigma}\left\{ \frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\sigma}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\sigma}_{\nu\eta}\right\}$$

$$R_{\mu\kappa} = R^{\nu}{}_{\mu\nu\kappa} = \frac{\partial \Gamma^{\nu}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\nu}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\nu}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\nu}_{\nu\eta}$$

Then $(\nu \to \lambda)$

$$\delta R_{\mu\kappa} = \frac{\partial \delta\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial \delta \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \delta\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} + \Gamma^{\eta}_{\mu\lambda}\delta\Gamma^{\lambda}_{\kappa\eta} +$$

$$- \delta\Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{\mu\kappa}\delta\Gamma^{\lambda}_{\lambda\eta}$$

Since $\delta\Gamma^{\lambda}_{\mu\nu}$ is a tensor, therefore $\delta R_{\mu\nu}$ is also a tensor:

$$\delta R_{\mu\kappa} = \frac{\partial \delta \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \Gamma^{\sigma}_{\mu\kappa}\delta\Gamma^{\lambda}_{\sigma\lambda} +$$

$$-\delta^{\rho}_{\lambda}\left\{ \frac{\partial \delta \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\rho}} - \Gamma^{\eta}_{\kappa\rho}\delta\Gamma^{\lambda}_{\mu\eta} - \Gamma^{\eta}_{\mu\rho}\delta\Gamma^{\lambda}_{\eta\kappa} + \right.$$

$$\left. + \Gamma^{\lambda}_{\eta\rho}\delta\Gamma^{\eta}_{\mu\kappa}\right\} = \delta\Gamma^{\lambda}_{\mu\lambda;\kappa} - \delta\Gamma^{\lambda}_{\mu\kappa;\lambda}$$

$$\delta R_{\mu\kappa} = \delta\Gamma^{\lambda}_{\mu\lambda;\kappa} - \delta\Gamma^{\lambda}_{\mu\kappa;\lambda}$$ the Palatini identity

In terms of $\delta g_{\mu\nu}$ we get

$$\delta R_{\mu\kappa} = \frac{1}{2} g^{\lambda\rho}\left\{ \delta g_{\lambda\rho;\mu;\kappa} + \delta g_{\mu\rho;\lambda;\kappa} - \delta g_{\lambda\mu;\rho;\kappa} + \right.$$

$$\left. - \delta g_{\kappa\rho;\mu;\lambda} - \delta g_{\mu\rho;\kappa;\lambda} + \delta g_{\kappa\mu;\rho;\lambda}\right\}$$

$$\delta R_{\mu\kappa} = \frac{1}{2} g^{\lambda\rho} \left\{ \delta g_{\lambda\rho;\mu;\kappa} - \delta g_{\rho\mu;\kappa;\lambda} - \delta g_{\kappa\rho;\mu;\lambda} + \delta g_{\kappa\mu;\rho;\lambda} \right\}$$

Assume that the Einstein equations are satisfied for the unperturbed metric $g_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$. Now we require that also $g_{\mu\nu} + \delta g_{\mu\nu}$ and $T_{\mu\nu} + \delta T_{\mu\nu}$ satisfy the equations :

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda} \right)$$

and

$$R_{\mu\nu} + \delta R_{\mu\nu} = -8\pi G \left\{ T_{\mu\nu} + \delta T_{\mu\nu} - \frac{1}{2} (g_{\mu\nu} + \delta g_{\mu\nu}) (T^{\lambda}{}_{\lambda} + \delta T^{\lambda}{}_{\lambda}) \right\}$$

for $\delta T^{\lambda}{}_{\lambda} = (g^{\lambda\rho} + \delta g^{\lambda\rho})(T_{\rho\lambda} + \delta T_{\rho\lambda})$

that is

$$\frac{1}{2} g^{\lambda\rho} \left\{ \delta g_{\lambda\rho;\mu;\kappa} - \delta g_{\rho\mu;\kappa;\lambda} - \delta g_{\lambda\mu;\rho;\kappa} + \delta g_{\kappa\mu;\rho;\lambda} \right\} =$$

$$= -8\pi G \left\{ \delta T_{\mu\nu} \underbrace{- \frac{1}{2} g_{\mu\nu} g^{\lambda\rho} \delta T_{\rho\lambda} \overbrace{- \frac{1}{2} g_{\mu\nu} \delta g^{\lambda\rho} T_{\rho\lambda}}}_{-\frac{1}{2} g_{\mu\nu} \delta T^{\lambda}{}_{\lambda}} - \frac{1}{2} \delta g_{\mu\nu} T^{\lambda}{}_{\lambda} \right\}$$

$$-\frac{1}{2} g_{\mu\nu} (-) g^{\lambda\sigma} \delta g_{\sigma\alpha} g^{\alpha\rho} T_{\rho\lambda} = \frac{1}{2} g_{\mu\nu} \delta g_{\lambda\eta} T^{\lambda\eta}$$

so

$$\frac{1}{2} g^{\lambda\rho} \left\{ \delta g_{\lambda\rho;\mu;\kappa} - \delta g_{\rho\mu;\kappa;\lambda} - \delta g_{\lambda\mu;\rho;\kappa} + \delta g_{\kappa\mu;\rho;\lambda} \right\} =$$

$$= -8\pi G \left\{ \delta T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\lambda\rho} \delta T_{\rho\lambda} + \frac{1}{2} g_{\mu\nu} \delta g_{\lambda\eta} T^{\lambda\eta} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda} \right\} \quad (*)$$

Linear differential, (generally covariant,) equation for $\delta g_{\mu\nu}$ (; defined with the background Γ)

The total energy-momentum tensor $T^{\nu\mu} + \delta T^{\nu\mu}$ should be covariantly conserved, so

$$0 = \left(T^{\nu\mu} + \delta T^{\nu\mu}\right)_{;\mu} = \frac{\partial}{\partial x^\mu}\left(T + \delta T\right)^{\nu\mu} + \left(\Gamma + \delta\Gamma\right)^{\nu}_{\kappa\mu}\left(T + \delta T\right)^{\kappa\mu} +$$

$$+ \left(\Gamma + \delta\Gamma\right)^{\mu}_{\kappa\mu}\left(T + \delta T\right)^{\nu\kappa} = \underbrace{T^{\nu\mu}_{\ \ ;\mu}}_{=0} +$$

$$+ \underbrace{\frac{\partial}{\partial x^\mu}\delta T^{\nu\mu} + \Gamma^{\nu}_{\kappa\mu}\delta T^{\kappa\mu} + \Gamma^{\mu}_{\kappa\mu}\delta T^{\nu\kappa}}_{=\delta T^{\nu\mu}_{\ \ ;\mu}} +$$

$$+ \delta\Gamma^{\nu}_{\kappa\mu} T^{\kappa\mu} + \delta\Gamma^{\mu}_{\kappa\mu} T^{\nu\kappa} + o(\delta^2)$$

So we get $\quad 0 = \delta T^{\nu\mu}_{\ \ ;\mu} + \delta\Gamma^{\nu}_{\kappa\mu} T^{\kappa\mu} + \delta\Gamma^{\mu}_{\kappa\mu} T^{\nu\kappa}$

Let's now distinguish physical disturbances from mere changes of the coordinate system:

$$x^\mu \to x'^\mu = x^\mu - \varepsilon^\mu(x)$$

⌊ infinitesimal field

The Einstein equations are generally covariant, therefore if $g_{\mu\nu}(x)$ is a solution for $T_{\mu\nu}(x)$, then $g'_{\mu\nu}(x)$ is also a solution for $T'_{\mu\nu}(x)$, where

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) + g_{\lambda\nu}(x)\frac{\partial\varepsilon^\lambda(x)}{\partial x^\mu} + g_{\lambda\mu}(x)\frac{\partial\varepsilon^\lambda(x)}{\partial x^\nu} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\varepsilon^\lambda(x) + o(\varepsilon^2)$$

↖ see class

The Lie derivative:

$$\Delta_\varepsilon g_{\mu\nu}(x) = g_{\lambda\nu}(x)\frac{\partial\varepsilon^\lambda(x)}{\partial x^\mu} + g_{\lambda\mu}(x)\frac{\partial\varepsilon^\lambda(x)}{\partial x^\nu} + \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\varepsilon^\lambda(x)$$

So $g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \Delta_\varepsilon g_{\mu\nu}(x)$ is a solution
(at the same location!)
of E.eq. for $T'_{\mu\nu}(x) = T_{\mu\nu}(x) + \Delta_\varepsilon T_{\mu\nu}(x)$

In covariant terms we can write

$$\Delta_\varepsilon g_{\mu\nu} \equiv \varepsilon_{\mu;\nu} + \varepsilon_{\nu;\mu} \qquad \text{see class}$$

$$\Delta_\varepsilon T_{\mu\nu} \equiv T^\lambda{}_\mu \varepsilon_{\lambda;\nu} + T^\lambda{}_\nu \varepsilon_{\lambda;\mu} + T_{\mu\nu;\lambda}\varepsilon^\lambda$$

$\Delta_\varepsilon g_{\mu\nu}$ is consistent with the general form for $\Delta_\varepsilon T_{\mu\nu}$

since $g_{\mu\nu;\lambda} = 0$.

- It is easy to show that $\delta g_{\mu\nu} = \Delta_\varepsilon g_{\mu\nu}$ is a solution of (*) for $\delta T_{\mu\nu} = \Delta_\varepsilon T_{\mu\nu}$ for any vector (infinitesimal) field $\varepsilon^\mu = \varepsilon^\mu(x)$ (see homework)

- Eq. (*) is linear, therefore given any solution $\delta g_{\mu\nu}$ one can always find another solution of the form

$$\delta g_{\mu\nu} + \Delta_\varepsilon g_{\mu\nu}$$

with the same physical content (gauge invariance!)

- $\Delta_\varepsilon g_{\mu\nu}$ contains unphysical degrees of freedom related purely to a change in the coordinate system corresponding to $\varepsilon^\mu(x)$.

For the Lie derivative:

$$\Delta_\varepsilon S \equiv S_{;\lambda}\,\varepsilon^\lambda$$

$$\Delta_\varepsilon V_\mu \equiv V^\lambda \varepsilon_{\lambda;\mu} + V_{\mu;\lambda}\,\varepsilon^\lambda$$

$$\Delta_\varepsilon V^\mu \equiv -V^\lambda \varepsilon^\mu{}_{;\lambda} + V^\mu{}_{;\lambda}\,\varepsilon^\lambda$$

$$\Delta_\varepsilon T^{\mu\nu} \equiv -T^{\lambda\nu}\varepsilon^\mu{}_{;\lambda} - T^{\mu\lambda}\varepsilon^\nu{}_{;\lambda} + T^{\mu\nu}{}_{;\lambda}\,\varepsilon^\lambda$$

$$\Delta_\varepsilon T^\mu{}_\nu \equiv -T^\lambda{}_\nu\,\varepsilon^\mu{}_{;\lambda} + T^\mu{}_\lambda\,\varepsilon^\lambda{}_{;\nu} + T^\mu{}_{\nu;\lambda}\,\varepsilon^\lambda$$

Properties:

linearity: $\Delta_\varepsilon\left[a A^\mu{}_\nu + b B^\mu{}_\nu\right] = a\,\Delta_\varepsilon A^\mu{}_\nu + b\,\Delta_\varepsilon B^\mu{}_\nu$

a, b - constants

the Leibniz rule: $\Delta_\varepsilon\left(A^\mu{}_\nu B^\lambda\right) = A^\mu{}_\nu\,\Delta_\varepsilon B^\lambda + \left(\Delta_\varepsilon A^\mu{}_\nu\right) B^\lambda$

it commutes with the contraction:

$$\delta^\lambda_\nu\,\Delta_\varepsilon T^{\mu\nu}{}_\lambda = \Delta_\varepsilon T^{\mu\lambda}{}_\lambda \equiv -T^{\nu\lambda}{}_\lambda\,\varepsilon^\mu{}_{;\nu} + T^{\mu\lambda}{}_{\lambda;\nu}\,\varepsilon^\nu$$

The Lie derivative of the energy-momentum tensor for the perfect fluid:

$$\Delta_\varepsilon T_{\mu\nu} = p\,\Delta_\varepsilon g_{\mu\nu} + g_{\mu\nu}\,\Delta_\varepsilon p + (p+\rho)\left[U_\mu\,\Delta_\varepsilon U_\nu + U_\nu\,\Delta_\varepsilon U_\mu\right] +$$
$$+ U_\mu U_\nu\left[\Delta_\varepsilon p + \Delta_\varepsilon \rho\right]$$

for $T_{\mu\nu} = p\,g_{\mu\nu} + (p+\rho)\,U_\mu U_\nu$ where $U^\mu = \dfrac{dx^\mu}{d\tau}$ for a comoving fluid element

Hydrodynamics and Hydrostatics

In the absence of gravitation: the energy-momentum for the perfect fluid is the following:

$$T^{\alpha\beta} = -p\eta^{\alpha\beta} + (p+\rho)U^{\alpha}U^{\beta}$$

for $U^0 = \gamma$, $U^i = \gamma\bar{v}$, $\gamma \equiv (1-v^2)^{-1/2}$

$\eta_{\alpha\beta}U^{\alpha}U^{\beta} = 1$ $\qquad \bar{v} \equiv \frac{d\bar{x}}{d\tau}$

In the presence of gravitation we have

$$T^{\alpha\beta} = -pg^{\alpha\beta} + (p+\rho)U^{\alpha}U^{\beta} \qquad g_{\alpha\beta}U^{\alpha}U^{\beta} = 1$$

for $U^{\alpha} = \frac{dx^{\alpha}}{d\tau}$ – local value of the four-velocity for a comoving fluid element

Note that p and ρ are defined as the pressure and density measured by an observer in a locally inertial frame that is moving with the fluid, so they are scalars.

The energy-momentum conservation gives

$$0 = T^{\mu\nu}{}_{;\nu} = (-pg^{\mu\nu})_{;\nu} + [(p+\rho)U^{\mu}U^{\nu}]_{;\nu} =$$

$$= -\frac{\partial p}{\partial x^{\nu}}g^{\mu\nu} + g^{-1/2}\frac{\partial}{\partial x^{\nu}}\left[g^{1/2}(p+\rho)U^{\mu}U^{\nu}\right] +$$

$$+ \Gamma^{\mu}_{\nu\lambda}(p+\rho)U^{\nu}U^{\lambda}$$

$$P^{\mu\nu}{}_{;\mu} = g^{-1/2}\frac{\partial}{\partial x^{\mu}}(g^{1/2}P^{\mu\nu}) + \Gamma^{\nu}_{\mu\lambda}P^{\mu\lambda}$$

Example: fluid in hydrostatic equilibrium, so $U^i = 0$, then

$$g_{\alpha\beta}U^{\alpha}U^{\beta} = 1 \Rightarrow U^0 = (g_{00})^{-1/2}$$

also $\frac{\partial g_{\mu\nu}}{\partial x^0} = \frac{\partial p}{\partial x^0} = \frac{\partial \rho}{\partial x^0} = 0$

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\}$$

So

$$\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\nu\mu} \frac{\partial g_{00}}{\partial x^{\nu}} \quad \text{and} \quad \frac{\partial}{\partial x^{\nu}} \left[(p+\rho) U^{\mu} U^{\nu} \right] = 0$$

Then from $T^{\mu\nu}{}_{;\nu} = 0$ we obtain

$$g_{\mu\lambda} \left[-\frac{\partial p}{\partial x^{\nu}} g^{\mu\nu} + g^{-1/2} \frac{\partial}{\partial x^{\nu}} \left[g^{1/2} (p+\rho) U^{\mu} U^{\nu} \right] + \Gamma^{\mu}_{\nu\lambda} (p+\rho) U^{\nu} U^{\lambda} \right] = 0$$

$$-\frac{\partial p}{\partial x^{\lambda}} + \underbrace{g^{-1/2} \frac{1}{2} g^{-1/2} \frac{\partial g}{\partial x^{\nu}} (p+\rho) U^{\mu} U^{\nu}}_{= 0} g_{\mu\lambda} + \underbrace{g_{\mu\lambda} \Gamma^{\mu}_{\nu\lambda} (p+\rho) U^{\nu} U^{\lambda}}$$

$$g_{\mu\lambda} \Gamma^{\mu}_{00} (p+\rho) U^{0} U^{0} = 0$$

$$= g_{\mu\lambda} \left(-\frac{1}{2}\right) g^{\nu\mu} \frac{\partial g_{00}}{\partial x^{\nu}} g_{00}^{-1} (p+\rho) =$$

$$= -\frac{1}{2} g_{00}^{-1} \frac{\partial g_{00}}{\partial x^{\lambda}} (p+\rho) = -\frac{\partial}{\partial x^{\lambda}} \ln (g_{00})^{1/2} (p+\rho)$$

So
$$\frac{\partial p}{\partial x^{\lambda}} = -(p+\rho) \frac{\partial}{\partial x^{\lambda}} \ln (g_{00})^{1/2}$$

- trivial for $\lambda = 0$
- for $\lambda = i$ one gets

$$\bar{\nabla} p = -(p+\rho) \bar{\nabla} \ln (g_{00})^{1/2}$$

while the ordinary non-relativistic condition for hydrostatic equilibrium reads :

$$\bar{\nabla} p = -\rho \bar{\nabla} \phi$$
ϕ — gravitational potential

Remarks:
- $\rho \to \rho + p$ general relativity effect
- since static conditions are considered, no wonder the result agrees with non-relativistic equation

- $$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \quad \text{for} \quad g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$|h_{\alpha\beta}| \ll 1$$

$\Downarrow$ (weak stationary gravitational field)

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\nabla h_{00}$$

So $g_{00} = 1 + 2\phi$ in the linear approximation.

While here we have obtained $\phi \leftrightarrow \ln(g_{00})^{1/2}$,

since $\ln(g_{00})^{1/2} = \frac{1}{2}\ln(1+2\phi) = \frac{1}{2}2\phi = \phi$

this is indeed consistent.

Back to $\vec{\nabla} p = -(p+\rho)\vec{\nabla}\ln(g_{00})^{1/2}$.

Assume that $p = p(\rho)$

(e.g. we had shown that for a high-relativistic gas $p = \frac{1}{3}\rho$), then

$$\int \frac{dp(\rho)}{p(\rho)+\rho} = -\ln(g_{00})^{1/2} + \text{const}$$

if $p(\rho) = a\rho^n$ then • for $n \neq 1$ we get

$$\frac{n}{n-1}\ln\underbrace{\left(1+a\rho^{n-1}\right)}_{\rho^{-1}(\rho+a\rho^n) = \frac{\rho+p(\rho)}{\rho}} = +\ln(g_{00})^{-1/2}\cdot c \qquad \int\frac{anx^{n-1}}{ax^n+x} = \frac{n}{n-1}\ln\left(1+ax^{n-1}\right)$$

$$\frac{\rho+p(\rho)}{\rho} = c'\left(g_{00}\right)^{(1-n)/2n}$$

• for $n = 1$

$$\rho = c'\left(g_{00}\right)^{-\frac{a+1}{2a}}$$

Remark: gravitation can never produce equilibrium in a <u>finite</u> relativistic fluid ($p = \frac{1}{3}\rho$) since then $\quad a = \frac{1}{3}$

$$\rho \propto (g_{00})^{-\frac{a+1}{2a}} = (g_{00})^{-2}$$

as $\rho \to 0$ outside of the fluid, therefore g_{00} must be divergent in that region.

Let's return to the energy-momentum conservation

$$0 = T^{\mu\nu}{}_{;\nu} = -\frac{\partial p}{\partial x^\nu} g^{\mu\nu} + g^{-1/2}\frac{\partial}{\partial x^\nu}\left[g^{1/2}(p+\rho)U^\mu U^\nu\right] + $$

$$+ \Gamma^\mu_{\nu\lambda}(p+\rho)U^\nu U^\lambda$$

and consider a special case for which $\Gamma^\mu_{tt} = 0$ with $U^t = 1$ and $U^i = 0$ for the F-R-W metric ($p = p(t)$, $\rho = \rho(t)$)

$\mu = i \qquad 0 = 0 + 0 + 0$

$$\mu = t \qquad 0 = -\frac{\partial p}{\partial t} g^{00} + \underbrace{g^{-1/2}\frac{\partial}{\partial t}\left[g^{1/2}(p+\rho)\right]}_{\frac{1}{2}\frac{1}{g}} + \overset{0}{\overset{\shortparallel}{\Gamma^t_{tt}}}(\rho+p) \Big/ g^{1/2}$$

$g^{tt} = 1 \quad g^{it} = 0 \quad g^{ij} = -R^{-2}(t)\,\tilde{g}^{ij}(x)$

$g = -\det(g_{\mu\nu}) =$

$g_{tt} = 1 \quad g_{it} = 0 \quad g_{ij} = -R^2(t)\,\tilde{g}_{ij}(x)$

$\tilde{g}_{ij} = \mathrm{diag}\left[\frac{1}{1-kr^2},\ r^2,\ r^2\sin^2\theta\right]$

$i, j = r, \theta, \varphi$

$$0 = -\frac{\partial p}{\partial t} g^{1/2} + \frac{\partial}{\partial t}\left[g^{1/2}(p+\rho)\right]$$

since $g^{1/2} \propto R^3(t)$, we finally obtain

$$\frac{dp}{dt} R^3(t) = \frac{d}{dt}\left[R^3(t)\left(p(t) + \rho(t) \right)\right]$$

- If the pressure is negligible then we get

$$\frac{d}{dt}\left(R^3 \rho\right) = 0 \qquad \rho \propto R^{-3}$$

- For high-relativistic fluid $p = \rho \frac{1}{3}$ and then

$$\frac{1}{3} \dot{\rho} R^3 = \frac{d}{dt}\left(R^3 \frac{4}{3} \rho\right) = \dot{R}^3 \frac{4}{3} \rho + R^3 \frac{4}{3} \dot{\rho}$$

$$\Downarrow$$

$$\rho \propto R^{-4}$$

The Friedmann equation

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu - p g_{\mu\nu} \qquad U^t = 1 \qquad U^i = 0$$

The Friedmann – Robertson – Walker metric:

$$d\tau^2 = dt^2 - R^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \right\}$$

the scale factor $\qquad k = 0, \pm 1$

$$g_{tt} = 1 \quad g_{it} = 0 \quad g_{ij} = -R^2(t) \, \tilde{g}_{ij}(x) \qquad x_1 = r, \; x_2 = \theta, \; x_3 = \varphi$$

Let's calculate the connection first

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right\}$$

$$\Gamma^t_{tt} = \frac{1}{2} g^{\nu t} \left\{ \underbrace{\frac{\partial g_{t\nu}}{\partial x^t}}_{0} + \underbrace{\frac{\partial g_{t\nu}}{\partial x^t}}_{0} - \underbrace{\frac{\partial g_{tt}}{\partial x^\nu}}_{0} \right\} = 0$$

$$\Gamma^t_{tj} = \frac{1}{2} g^{\nu t} \left\{ \underbrace{\frac{\partial g_{j\nu}}{\partial x^t}}_{\nu = j \,\to\, 0} + \underbrace{\frac{\partial g_{t\nu}}{\partial x^j}}_{\nu = t \,\to\, 0} - \underbrace{\frac{\partial g_{jt}}{\partial x^\nu}}_{0} \right\} = 0$$

$$\Gamma^t_{ij} = \frac{1}{2} \underbrace{g^{\nu t}}_{\nu = t} \left\{ \underbrace{\frac{\partial g_{j\nu}}{\partial x^i}}_{0} + \underbrace{\frac{\partial g_{i\nu}}{\partial x^j}}_{0} - \frac{\partial g_{ji}}{\partial x^\nu} \right\} = \frac{1}{2} + 2 R \dot{R} \, \tilde{g}_{ij} = + R \dot{R} \, \tilde{g}_{ij}.$$

$$\Gamma^i_{tt} = \frac{1}{2} g^{\nu i} \left\{ \underbrace{\frac{\partial g_{t\nu}}{\partial x^t}}_{0} + \underbrace{\frac{\partial g_{t\nu}}{\partial x^t}}_{0} - \underbrace{\frac{\partial g_{tt}}{\partial x^\nu}}_{0} \right\} = 0$$

$$\Gamma^i_{tj} = \frac{1}{2} g^{\nu i} \left\{ \frac{\partial g_{j\nu}}{\partial x^t} + \underbrace{\frac{\partial g_{t\nu}}{\partial x^j}}_{0 \, (\nu = t)} - \underbrace{\frac{\partial g_{jt}}{\partial x^\nu}}_{0} \right\} = -\frac{1}{2} R^{-2} \tilde{g}^{ki} \, (-2 R \dot{R} \, \tilde{g}_{jk}) =$$

$$= \frac{\dot{R}}{R} \delta^i_j$$

$$g^{tt} = 1, \quad g^{it} = 0, \quad g^{ij} = -R^{-2}(t) \, \tilde{g}^{ij}(x) \quad \Leftarrow \quad g^{\lambda\nu} g_{\nu\mu} = \delta^\lambda_\mu$$

$$\tilde{g}^{rr} = 1 - kr^2 \qquad \tilde{g}^{\theta\theta} = r^{-2} \qquad \tilde{g}^{\varphi\varphi} = r^{-2}\sin^{-2}\theta$$

$$\tilde{g}_{j\nu}\,\tilde{g}^{\nu i} = \delta^i_j$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{\nu i}\left\{\frac{\partial g_{k\nu}}{\partial x^j} + \frac{\partial g_{j\nu}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\nu}\right\} =$$
$\nu \to l$

$$= \frac{1}{2}\tilde{g}^{il}\left\{\frac{\partial \tilde{g}_{kl}}{\partial x^j} + \frac{\partial \tilde{g}_{jl}}{\partial x^k} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right\} = \tilde{\Gamma}^i_{jk}$$

Then the Ricci tensor reads

$$R^\lambda_{\ \mu\nu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}$$

$$R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa} = g^{\lambda\nu} g_{\lambda\sigma} R^\sigma_{\ \mu\nu\kappa} = R^\nu_{\ \mu\nu\kappa}$$

$$R_{tt} = R^\nu_{\ t\nu t} = R^t_{\ ttt} + R^r_{\ trt} + R^\theta_{\ t\theta t} + R^\varphi_{\ t\varphi t}$$

$$\boxed{\frac{\partial \Gamma^\nu_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\nu_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\nu_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\nu_{\nu\eta} = R_{\mu\kappa}}$$

$$R_{tt} = \underset{\nu\to j}{\frac{\partial \Gamma^\nu_{t\nu}}{\partial x^t}} - \underset{\overset{\parallel}{0}}{\frac{\partial \Gamma^\nu_{tt}}{\partial x^\nu}} + \underset{\eta\to j\ \nu\to i}{\Gamma^\eta_{t\nu}\Gamma^\nu_{t\eta}} - \underset{\overset{\parallel}{0}}{\Gamma^\eta_{tt}}\Gamma^\nu_{\nu\eta} =$$

$$= \frac{\partial \Gamma^j_{tj}}{\partial x^t} + \Gamma^j_{ti}\Gamma^i_{tj} = \frac{d}{dt}\left(\frac{\dot R}{R}\delta^j_j\right) + \frac{\dot R}{R}\delta^j_i\,\frac{\dot R}{R}\delta^i_j =$$

$$= \frac{\ddot R R - \dot R^2}{R^2}\cdot 3 - \frac{\dot R^2}{R^2}\cdot 3 = 3\frac{\ddot R}{R}$$

$$R_{ti} = \underset{\overset{\parallel}{0}}{\frac{\partial \Gamma^\nu_{t\nu}}{\partial x^i}} - \underset{\substack{\nu\to j\\ \overset{\parallel}{0}}}{\frac{\partial \Gamma^\nu_{ti}}{\partial x^\nu}} + \underset{\nu\to j,\ \eta\to k}{\Gamma^\eta_{t\nu}\Gamma^\nu_{i\eta}} - \underset{\eta\to j,\ \nu\to k}{\Gamma^\eta_{ti}\Gamma^\nu_{\nu\eta}} = \Gamma^k_{tj}\Gamma^j_{ik} - \Gamma^j_{ti}\Gamma^k_{kj} =$$

$$= \frac{\dot R}{R}\delta^k_j\,\Gamma^j_{ik} - \frac{\dot R}{R}\delta^j_i\,\Gamma^k_{kj} = 0$$

$$R_{ij} = \frac{\partial \Gamma^{\nu}_{i\nu}}{\partial x^j} - \frac{\partial \Gamma^{\nu}_{ij}}{\partial x^\nu} + \Gamma^{\eta}_{i\nu}\Gamma^{\nu}_{j\eta} - \Gamma^{\eta}_{ij}\Gamma^{\nu}_{\nu\eta} =$$

$\nu \to k$

$$= \frac{\partial \Gamma^{k}_{ik}}{\partial x^j} - \frac{\partial \Gamma^{t}_{ij}}{\partial x^t} - \frac{\partial \Gamma^{k}_{ij}}{\partial x^k} + \Gamma^{t}_{i\nu}\Gamma^{\nu}_{jt} + \Gamma^{k}_{i\nu}\Gamma^{\nu}_{jk} +$$

$$\underbrace{- \Gamma^{t}_{ij}\Gamma^{\nu}_{\nu t} - \Gamma^{k}_{ij}\Gamma^{\nu}_{\nu k}}_{\text{sum over } \nu :\ t,\ l}$$

$$= \underbrace{\frac{\partial \Gamma^{k}_{ik}}{\partial x^j} - \frac{\partial \Gamma^{k}_{ij}}{\partial x^k} + \Gamma^{k}_{il}\Gamma^{l}_{jk} - \Gamma^{k}_{ij}\Gamma^{l}_{lk}}_{} +$$

$$- \frac{\partial \Gamma^{t}_{ij}}{\partial x^t} + \underset{0}{\Gamma^{t}_{it}}\Gamma^{t}_{jt} + \Gamma^{t}_{il}\Gamma^{l}_{jt} + \Gamma^{k}_{it}\Gamma^{t}_{jk} +$$

$$- \Gamma^{t}_{ij}\underset{0}{\Gamma^{t}_{tt}} - \Gamma^{t}_{ij}\Gamma^{l}_{lt} - \Gamma^{k}_{ij}\underset{0}{\Gamma^{t}_{tk}} =$$

$$= \tilde{R}_{ij} - \tilde{g}_{ij}(R\ddot{R} + \dot{R}^2) + R\dot{R}\,\tilde{g}_{il}\cdot\frac{\dot{R}}{R}\delta^l_j + \frac{\dot{R}}{R}\delta^k_i R\dot{R}\,\tilde{g}_{jk}$$

$$- R\dot{R}\,\tilde{g}_{ij}\frac{\dot{R}}{R}\delta^l_l = \tilde{R}_{ij} - \tilde{g}_{ij}(R\ddot{R} + 2\dot{R}^2)$$

Homework: show that $\tilde{R}_{ij} = -2k\tilde{g}_{ij}$,

then

$$R_{ij} = -(R\ddot{R} + 2\dot{R}^2 + 2k)\,\tilde{g}_{ij}$$

The source term in the Einstein equations

$$S_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^{\alpha}{}_{\alpha}, \quad \text{for } T_{\mu\nu} = (\rho+p)U_\mu U_\nu - p g_{\mu\nu}$$

$$(T^{\alpha}{}_{\alpha} = -3p + \rho)$$

$$S_{\mu\nu} = (\rho+p)U_\mu U_\nu - \tfrac{1}{2}(\rho - p) g_{\mu\nu}$$

$U^t = 1 \quad U^i = 0 \Rightarrow S_{tt} = \tfrac{1}{2}(\rho + 3p)$

$$S_{it} = 0$$

$$S_{ij} = \tfrac{1}{2}(\rho - p) R^2 \tilde{g}_{ij}$$

The Einstein equations: $R_{\mu\nu} = -8\pi G\, S_{\mu\nu}$

(t,t): $$3\ddot{R} = -4\pi G(\rho + 3p)R$$

(i,j): $$+(R\ddot{R} + 2\dot{R}^2 + 2k) = +4\pi G(\rho - p)R^2$$

$$\Downarrow$$

$$\dot{R}^2 + k = \frac{8\pi}{3} G \rho R^2$$ the Friedmann equation

From the covariant conservation of the energy-momentum tensor ($T^{\mu\nu}{}_{;\nu} = 0$) we also have

$$\dot{p} R^3 = \frac{d}{dt}\left[R^3(\rho + p)\right]$$

$$\Downarrow$$

$$\frac{d}{dR}\left(\rho R^3\right) = -3pR^2$$

if $p = p(\rho)$ is known then can be used to determine $\rho = \rho(R)$.

The fundamental equations of dynamical cosmology are:

- the Einstein equation
- the equation of state ($p = p(\rho)$)
- the energy conservation

The Weyl transformation

string frame

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}$$

E - frame

class $\Rightarrow$ $R = \Omega^2 R' + \Delta R$

where
$$\Delta R = -\Omega^2 \left\{ 2(N-1)\,\Box' f - (N-1)(N-2)\, g'^{\mu\nu} f_\mu f_\nu \right\}$$

for $f_\nu \equiv \partial_\nu f = \partial_\nu \ln \Omega$

For $N=4$ we obtain

$$R = \Omega^2 \left\{ R' - 6\,\Box' f + 6\, g'^{\mu\nu} f_\mu f_\nu \right\}$$

remember that
$$\Box' f = (g')^{-1/2}\, \partial_\mu \left[(g')^{1/2} g'^{\mu\nu} \partial_\nu f \right]$$

$$V^\mu{}_{;\mu} = g^{-1/2}\, \partial_\mu \left(g^{1/2} V^\mu \right)$$

for $V^\mu = g^{\mu\nu} f_{;\nu} =$
$= g^{\mu\nu} f_\nu$

Let's consider the following non-minimal coupling (Brans - Dicke)

$$\mathcal{L}_{BD} = g^{1/2} \left[\frac{1}{2} F(\phi) R + \frac{1}{2} \epsilon\, g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi + \mathcal{L}_{matter} \right] \qquad (*)$$

later we will consider $F(\phi) = \xi \phi^2$

$$\mathcal{L}_1 = \frac{1}{2} g^{1/2} F(\phi) R = \frac{1}{2} \sqrt{g'}\, \Omega^{-4} F(\phi)\, \Omega^2 \left(R' - 6\,\Box' f + 6\, g'^{\mu\nu} f_\mu f_\nu \right) =$$

Weyl transformation

$$= g'^{1/2} \frac{1}{2} F(\phi)\, \Omega^{-2} \left(R' - 6\,\Box' f + 6\, g'^{\mu\nu} f_\mu f_\nu \right)$$

We can choose Ω such that

$$F(\phi)\, \Omega^{-2} = 1$$

then $\mathcal{L}_1 = g'^{1/2} \; \frac{1}{2}\left[R' - 6\Box' f + 6 g'^{\mu\nu} f_\mu f_\nu\right]$

$$= -\frac{1}{2} 6\, \partial_\mu \left(g'^{1/2} g'^{\mu\nu} \partial_\nu f \right)$$ full derivative, so drop it

$$f_\mu = \partial_\mu \ln \Omega = \partial_\mu \ln F^{1/2} = \frac{1}{2} \frac{\partial_\mu F}{F} = \frac{1}{2} \frac{F'}{F} \partial_\mu \phi$$

$$F' \equiv \frac{dF}{d\phi}$$

Then

$$\mathcal{L}_1 = g'^{1/2} \left[\frac{1}{2} R' + \frac{3}{4} \left(\frac{F'}{F}\right)^2 g'^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]$$

like a kinetic term for ϕ

Consider now the second term of (*)

$$\mathcal{L}_2 = g^{1/2} \frac{1}{2} \epsilon g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = g'^{1/2} \Omega^{-4} \frac{1}{2} \epsilon \Omega^2 g'^{\mu\nu} \partial_\mu \phi \partial_\nu \phi =$$

$$= g'^{1/2} \frac{1}{2} \frac{\epsilon}{F} g'^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

So, from $\mathcal{L}_1$ and $\mathcal{L}_2$ we get

$$g'^{1/2} \frac{1}{2} \Delta g'^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

for $\Delta = \frac{3}{2} \left(\frac{F'}{F}\right)^2 + \frac{\epsilon}{F}$

if $\Delta > 0$ then we can introduce $\sigma(x)$ such that

$$\frac{d\sigma}{d\phi} = \Delta^{1/2}$$

Then $\Delta^{1/2} \partial_\mu \phi = \frac{d\sigma}{d\phi} \partial_\mu \phi = \partial_\mu \sigma$

so that

$$g'^{1/2} \frac{1}{2} \Delta g'^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = g'^{1/2} \frac{1}{2} g'^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma$$ a canonical kinetic term for a scalar $\sigma(x)$

If $\Delta < 0$ then σ has a wrong sign kinetic term (it is a ghost). Note that $\Delta > 0$ could be achieved even if $\epsilon < 0$.

Let's focus on $F(\phi) = \xi \phi^2$, then

$$\Delta = \frac{3}{2}\left(\frac{2\phi}{\phi^2}\right)^2 + \frac{\epsilon}{\xi \phi^2} = (6 + \epsilon \xi^{-1})\phi^{-2} = \zeta^{-2}\phi^{-2}$$

$$\zeta^{-2} \equiv 6 + \epsilon \xi^{-1}$$

$$\frac{d\sigma}{d\phi} = \Delta^{1/2} = \zeta^{-1}\phi^{-1} \Rightarrow \zeta\, d\sigma = \frac{d\phi}{\phi}$$

$$\zeta\sigma = \ln\left(\frac{\phi}{\phi_0}\right) \qquad \phi = \phi_0 e^{\zeta\sigma}$$

So, finally

$$\mathcal{L}_{BD} = g'^{1/2}\left[\frac{1}{2}R' + \frac{1}{2}g'^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \mathcal{L}'_{matter}\right]$$

Remarks:

- for $\epsilon = 0$ (no kinetic term in the initial frame) $\zeta^2 = \frac{1}{6}$, so σ has a right sign energy: the scalar field ϕ acquires dynamics, as a consequence of mixing with R

- for $\epsilon = -1$ (initial ghost) and $\xi = \frac{1}{6}$, $\Delta = 0$, so no kinetic term in the E frame:

$$\mathcal{L}_{BD} = g'^{1/2}\left[\frac{1}{2}R' + \mathcal{L}'_{matter}\right]$$