Yukawa potential

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$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \lambda\rho\varphi$$

ρ – a given source $\rho = \rho(x)$

energy-momentum tensor

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}$$

$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$$\left.\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\right|_{K-G} = \frac{\partial}{\partial(\partial^\mu\varphi)}\left(\partial^\alpha\varphi\,\partial^\beta\varphi\, g_{\alpha\beta}\right) = \partial_\mu\varphi$$

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi - \frac{m^2}{2}\varphi^2 + \lambda\rho\varphi\right)$$

Conserved 4-momentum

$$P_\mu \equiv \int d^3x\, T_{0\mu}$$

$$E = \int d^3x\, T_{00} = \int d^3x\left[\dot\varphi^2 - \left(\frac{1}{2}\dot\varphi^2 - \frac{1}{2}\sum_{i=1,2,3}\partial_i\varphi\partial_i\varphi - \frac{m^2}{2}\varphi^2 + \lambda\rho\varphi\right)\right]$$

$$E = \int d^3x\left(\frac{1}{2}\dot\varphi^2 + \frac{1}{2}\partial_i\varphi\partial_i\varphi + \frac{m^2}{2}\varphi^2 - \lambda\rho\varphi\right)$$

Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0$$

for $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu\phi_i)$

ϕ_i – a generic field

For K-G theory:

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -\frac{m^2}{2}2\varphi + \lambda\rho$$

$$\frac{\partial\mathcal{L}}{\partial_\mu\varphi} = \partial^\mu\varphi$$

$$-m^2\varphi + \lambda\rho - \partial_\mu\partial^\mu\varphi = 0$$

K-G $\quad (\Box + m^2)\varphi = \lambda\rho$

$\Box = \partial_\mu\partial^\mu$

solution $\varphi(x) = \varphi_h(x) + \varphi_p(x)$

general solution of the homogeneous equation

special solution of the inhomogeneous equation

1. $(\Box + m^2)\varphi_h = 0$

Let's look for $\varphi_h(x)$ in terms of its Fourier transform

$$\varphi_h(x) = \int\frac{d^4k}{(2\pi)^4}\varphi_h(k)e^{ikx}$$

$$\varphi_h = \varphi_h^* \Rightarrow \int d^4k\,\varphi_h(k)e^{ikx} = \int d^4k\,\varphi_h^*(k)e^{-ikx} = \int d^4k\,\varphi_h^*(-k)e^{ikx}$$

$$\varphi_h(k) = \varphi_h^*(-k)$$

$$(\Box + m^2)\varphi_h(x) = \int\frac{d^4k}{(2\pi)^4}\varphi_h(k)\left[ik_\mu ik^\mu + m^2\right]e^{ikx}$$

$$= -\int\frac{d^4k}{(2\pi)^4}(k^2 - m^2)\varphi_h(k)e^{ikx} = 0$$

for any x

$$\Downarrow$$

$$k^2 = m^2$$

$$\varphi_h(k) = f(k)\,\delta(k^2 - m^2)$$

$$\delta[\varphi(k_0)] = \sum\frac{\delta(k_0 - k_{0i})}{\quad}$$

$$\delta[g(k_0)] = \sum_i \frac{\delta(k_0 - k_{0i})}{\left|\frac{dg}{dk_0}\right|_{k_0 = k_{0i}}}$$

where $g(k_{0i}) = 0$

$$g(k_0) = k^2 - m^2 = k_0^2 - \bar{k}^2 - m^2$$
$$k_{0\pm} = \pm(\bar{k}^2 + m^2)^{1/2} \equiv \pm E_k$$
$$\delta(k^2 - m^2) = \frac{\delta(k_0 - E_k)}{2E_k} + \frac{\delta(k_0 + E_k)}{2(E_k)}$$

dropped as unphysical $k_0 > 0$

$$\varphi_h(x) = \int \frac{d^3k}{(2\pi)^4 2E_k} e^{ikx} f(k)$$

for $k = (E_k, \bar{k})$

$$\Box \varphi_h = \int \frac{d^3k}{(2\pi)^4 2E_k} \underbrace{\left(-E_k^2 + \bar{k}^2\right)}_{-\bar{k}^2 - m^2 + \bar{k}^2 = -m^2} e^{ikx} f(k)$$

$$(\Box + m^2)\varphi_h(x) = 0$$

2. $$(\Box + m^2)\varphi_p(x) = \lambda \rho(x)$$

$$\varphi_p(x) = \int \frac{d^4k}{(2\pi)^4} \hat{\varphi}_p(k) e^{ikx}$$
$$\rho(x) = \int \frac{d^4k}{(2\pi)^4} \hat{\rho}(k) e^{ikx}$$
$$\int \frac{d^4k}{(2\pi)^4} (-k^2 + m^2) \hat{\varphi}_p(k) e^{ikx} = \lambda \int \frac{d^4k}{(2\pi)^4} \hat{\rho}(k) e^{ikx}$$
$$(-k^2 + m^2)\hat{\varphi}_p(k) = \lambda \hat{\rho}(k)$$
$$\Downarrow$$
$$\hat{\varphi}_p(k) = \lambda \frac{\hat{\rho}(k)}{-k^2 + m^2}$$

the general solution

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} \Big[\underbrace{\delta(k^2 - m^2) f(k)}_{\text{plane wave}} + \underbrace{\lambda \frac{\hat{\rho}(k)}{-k^2 + m^2}}_{\text{scattered wave}} \Big] e^{ikx}$$

Assume the existence of two static localized sources:

$$\rho(x) = \delta^3(\bar{x}) + \delta^3(\bar{x} - \bar{y})$$

and find the energy corresponding to the field configuration generated by the given $\rho = \rho(x)$.

$$\rho(x) = \int \frac{d^4k}{(2\pi)^4} \hat{\rho}(k) e^{ikx}$$
$$\hat{\rho}(k) = \int d^4x \, \rho(x) e^{-ikx}$$

$$\boxed{\int d^4x \, e^{ipx} = (2\pi)^4 \delta^4(p)}$$

$$\int d^4x \, e^{-ipx} \rho(x) = \int d^4x \int \frac{d^4k}{(2\pi)^4} \hat{\rho}(k) e^{ix(k-p)} =$$
$$= \int \frac{d^4k}{(2\pi)^4} \hat{\rho}(k) (2\pi)^4 \delta^4(k-p) = \hat{\rho}(p)$$

For $\rho(x) = \delta^3(\bar{x} - \bar{y})$ we get

$$\hat{\rho}(k) = \underbrace{\int \frac{dx_0}{2\pi} e^{-ik_0x_0}}_{\delta(k_0)} 2\pi e^{+i\bar{k}\bar{y}} = e^{+i\bar{k}\bar{y}} \delta(k_0) 2\pi$$

$$\varphi_p(x) = \lambda \int \frac{d^4k}{(2\pi)^4} \frac{\hat{\rho}(k)}{-k^2 + m^2} e^{ikx} \overset{\text{for } \rho(x) = \delta^3(\bar{x} - \bar{y})}{=} \lambda \int d^4k \frac{e^{i\bar{k}\bar{y}}}{(2\pi)^3} \frac{\delta(k_0) e^{ikx}}{-k^2 + m^2} =$$
$$= \lambda \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\bar{k}(\bar{x} - \bar{y})}}{\bar{k}^2 + m^2} = \lambda \frac{e^{-mr}}{4\pi r} \quad \text{for } r = |\bar{x} - \bar{y}|$$

For $\rho(x) = \delta^3(\bar{x}) + \delta^3(\bar{x} - \bar{y})$ one gets

$$(*) \quad \varphi_p(x) = \frac{\lambda}{4\pi}\left(\frac{e^{-md}}{d} + \frac{e^{-mr}}{r}\right) \quad \text{where } d \equiv |\bar{x}|, \; r \equiv |\bar{x} - \bar{y}|$$

Let's now find the energy for the field configuration $(*)$

$$E = \int d^3x \left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\partial_i\varphi\partial_i\varphi + \frac{m^2}{2}\varphi^2 - \lambda\rho\varphi \right)\Bigg|_{\dot{\varphi}=0} =$$

E-L eq. : $(\Box + m^2)\varphi = \lambda\rho \Rightarrow (-\partial_i\partial_i + m^2)\varphi = \lambda\rho$

for $\dot{\varphi}=0$ we get

$$= \int d^3x \left[-\frac{1}{2}\varphi\underbrace{(\partial_i\partial_i - m^2)\varphi}_{= -\lambda\rho} - \lambda\rho\varphi \right]$$

$$= \int d^3x \left(-\frac{1}{2}\lambda\rho\varphi\right) = -\frac{1}{2}\lambda\left[\varphi(0) + \varphi(\vec{y})\right]$$

for $\rho(x) = \delta^3(\vec{x}) + \delta^3(\vec{x}-\vec{y})$ and $\varphi \to \varphi_p$

$$E = -\frac{\lambda}{2}\frac{\lambda}{4\pi}\left[\frac{e^{-md}}{d}\Big|_{d\to 0} + \frac{e^{-mr}}{d}\Big|_{r\to|\vec{y}|} + \frac{e^{-md}}{d}\Big|_{d\to|\vec{y}|} + \frac{e^{-mr}}{r}\Big|_{r\to 0} \right] =$$

$$= -\frac{\lambda^2}{4\pi}\frac{e^{-m|\vec{y}|}}{|\vec{y}|}$$

or

$$E = -\lambda^2 \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{y}}}{\vec{k}^2 + m^2}$$

Let's consider the case $\dot{\varphi} \neq 0$

$$E = \int d^3x \left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\partial_i\varphi\partial_i\varphi + \frac{m^2}{2}\varphi^2 - \lambda\rho\varphi \right) = \int d^3x \left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\varphi\underbrace{\partial_i\partial_i\varphi}_{\text{by parts}} + \frac{1}{2}m^2\varphi^2 - \lambda\rho\varphi \right] = \int d^3x \left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\varphi\left(\partial_0\partial_0\varphi + \cancel{m^2\varphi} - \lambda\rho\right) \cancel{+\frac{1}{2}m^2\varphi^2} \overbrace{-\lambda\rho\varphi} \right] =$$

$$(\Box + m^2)\varphi = \lambda\rho \Rightarrow \partial_0\partial_0\varphi - \partial_i\partial_i\varphi + m^2\varphi = \lambda\rho$$

$$= \int d^3x \left[\frac{1}{2}(\partial_0\varphi)^2 - \frac{1}{2}\varphi\partial_0\partial_0\varphi - \frac{1}{2}\lambda\rho\varphi \right] = \int d^3x \left[\frac{1}{2}(\partial_0\varphi\partial_0\varphi - \varphi\partial_0\partial_0\varphi) - \frac{1}{2}\lambda\rho\varphi \right]$$

$$(\Box + m^2)D(x) = \delta^4(x)$$

$$\varphi(x) = \int d^4y\, D(x-y)\lambda\rho(y) \quad \Big| \Rightarrow (\Box + m^2)_x\varphi = \int d^4y \underbrace{(\Box + m^2)_x D(x-y)}_{\delta^4(x-y)}\lambda\rho(y) = \lambda\rho(x)$$

$$E = \frac{1}{2}\int d^3x\left(\partial_0\varphi\partial_0\varphi - \varphi\partial_0\partial_0\varphi\right) - \frac{1}{2}\lambda^2\int d^3x\int d^4y\, \rho(x)D(x-y)\rho(y)$$

$$W[J] = -\frac{1}{2}\int d^4x\, d^4y\, J(x)D(x-y)J(y) \qquad W[J] = \int dx_0 \cdot E$$

Let us now consider $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu + \lambda J_\mu A^\mu$

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\partial_\nu\varphi - \eta_{\mu\nu}\mathcal{L} \Rightarrow T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\alpha)}\partial_\nu A^\alpha - \eta_{\mu\nu}\mathcal{L} = -F_{\mu\alpha}\partial_\nu A^\alpha - \eta_{\mu\nu}\mathcal{L}$$

$$-F^{\mu\nu}$$

$$\partial_\nu F^{\mu\nu} - m^2A^\mu = \lambda J^\mu \Rightarrow \begin{cases} \partial_i F^{0i} - m^2A^0 = \lambda J^0 \\ \partial_\nu F^{j\nu} - m^2A^j = \lambda J^j \end{cases} \qquad \frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \Rightarrow m^2A^\nu + \lambda J^\nu + \partial_\mu F^{\mu\nu} = 0$$

$$E = \int d^3x\, T_{00} = \int d^3x \left[-F_{0\alpha}\partial_0A^\alpha - \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu + \lambda J_\mu A^\mu \right) \right] =$$

$$= \int d^3x \left[-F_{0i}\partial_0A^i + \frac{1}{4}\underbrace{F_{0i}}_{(\partial_0A_i - \partial_iA_0)}F^{0i}\cdot 2 + \frac{1}{4}\underbrace{F_{ij}F^{ij}}_{(\partial_iA_j - \partial_jA_i)F^{ij} = 2\partial_iA_jF^{ij}} - \frac{1}{2}m^2A_\mu A^\mu - \lambda J_\mu A^\mu \right] =$$

$$= \int d^3x \left[-F_{0i}\partial_0A^i + \frac{1}{2}\partial_0A_iF^{0i} \underbrace{- \frac{1}{2}\partial_iA_0F^{0i} + \frac{1}{2}\partial_iA_jF^{ij}}_{\text{by parts}} - \frac{1}{2}m^2A_\mu A^\mu - \lambda J_\mu A^\mu \right] =$$

$$= \int d^3x \left[-\frac{1}{2}\partial_0A_iF^{0i} + \frac{1}{2}A_0\underbrace{\partial_iF^{0i}}_{= m^2A^0 + \lambda J^0} - \frac{1}{2}A_j\underbrace{\partial_iF^{ij}}_{= -m^2A^j - \lambda J^j - \partial_0F^{0j}} - \frac{1}{2}m^2A_\mu A^\mu - \lambda J_\mu A^\mu \right] =$$

$$= \int d^3x \left[-\frac{1}{2}\partial_0 A_i F^{0i} + \frac{1}{2} A_j \partial_0 F^{0j} + \frac{1}{2} m^2 \underbrace{\left(A_0 A^0 + A_j A^j \right)}_{A_\mu A^\mu} + \frac{1}{2}\lambda \underbrace{\left(A_0 J^0 + A_j J^j \right)}_{A_\mu J^\mu} - \frac{1}{2} m^2 A_\mu A^\mu - \lambda J_\mu A^\mu \right] \Rightarrow$$

$$\boxed{E = \int d^3x \left[\frac{1}{2}\left(-\partial_0 A_i + A_i \partial_0 \right) F^{0i} - \frac{1}{2}\lambda J_\mu A^\mu \right]}$$

$$\partial_\nu F^{\mu\nu} - m^2 A^\mu = \lambda J^\mu \Rightarrow \quad \left(\partial^\mu \partial^\nu - \partial_\alpha \partial^\alpha \eta^{\mu\nu} - m^2 \eta^{\mu\nu} \right) A_\nu(x) = \lambda J^\mu(x)$$

$$A_\nu(x) = \int \frac{d^4k}{(2\pi)^4} \hat{A}_\nu(k)\, e^{ikx}$$

The solution reads: $A_\nu(x) = \int d^4y\, D(x-y)_{\nu\mu} \lambda J^\mu(y)$ where $D(x-y)$ satisfies

$$\left(\partial^\mu \partial^\nu - \Box \eta^{\mu\nu} - m^2 \eta^{\mu\nu} \right)_x D(x-y)_{\nu\lambda} = \delta^4(x-y)\, \eta^\mu{}_\lambda$$

$$D(x)_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k)\, e^{ikx}$$

$$\Downarrow$$

$$D_{\mu\nu}(k) = \frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2} \quad \leftarrow \text{to be checked at the blackboard}$$

Then

$$E = \int d^3x \left[-\frac{1}{2}\left(\partial_0 A_i - A_i \partial_0 \right) F^{0i} \right] \boxed{- \frac{1}{2}\lambda^2 \int d^3x\, d^4y\, J_\mu(x)\, D^{\mu\nu}(x-y)\, J_\nu(y)}$$

see lecture notes p. 28, part 1

Assume $\quad J_\mu(x) = \left(J_0(\vec{x}), \vec{0} \right) \quad$ with $\quad J_0(\vec{x}) = \varepsilon\, \delta(\vec{x} - \vec{z})$

$$A_\mu(x) = \lambda \int d^4y \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k)\, e^{ik(x-y)} \delta^\nu_0 J_0(y) = \lambda \varepsilon \int d^4y \int \frac{d^4k}{(2\pi)^4} D_{\mu 0}(k)\, e^{ik(x-y)} \delta_3(\vec{y} - \vec{z}) =$$

$$= \lambda \varepsilon \int \frac{d^4k}{(2\pi)^4} \underbrace{\int dy^0\, e^{ik_0(x^0 - y^0)}}_{(2\pi)\,\delta(k_0)} D_{\mu 0}(k)\, e^{-i\vec{k}(\vec{x}-\vec{z})} = \lambda \varepsilon \int \frac{d^3k}{(2\pi)^3} \underbrace{D_{\mu 0}(\vec{k})}_{= \frac{-\delta_{\mu 0}}{\vec{k}^2 + m^2}} e^{i\vec{k}(\vec{x}-\vec{z})} \quad \leftarrow \delta(k_0)$$

$$A_0(x) = -\lambda \varepsilon \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{z})}}{\vec{k}^2 + m^2} \qquad A_i(x) = 0$$

- Then it goes the same way as for the Klein-Gordon field.
- Similarilly it should work for $h_{\mu\nu}$.