Theory of elementary particles 2004/05

Homework problems 1

1. For a Dirac field, the transformation

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x),$$

where α is an arbitrary real parameter, is called a chiral transformation.

- Show that the free field Lagrangian density

$$\mathcal{L}(x) = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$$

in the limit $m \to 0$ is invariant under the chiral transformation, and that the corresponding conserved current is the axial current $J_A^\alpha \equiv \bar{\psi}(x)\gamma^\alpha\gamma_5\psi(x)$.

- Deduce the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(1-\gamma_5)\psi(x), \qquad \psi_R(x) \equiv \frac{1}{2}(1+\gamma_5)\psi(x),$$

for non-vanishing mass and show that they decouple in the limit $m \to 0$.

2. For the complex free scalar field, find

- commutation relations for field operators and annihilation and creation operators,
- the Hamiltonian density in terms of creation and annihilation operators,
- the conserved current (in terms of the creation and annihilation operators) which follows (via the Noether theorem) from the invariance of the theory under a phase transformation

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x)$$

where α is a constant phase parameter.

3. For the free Dirac field,

   - postulate commutation relations for field operators and find the corresponding commutation relations for the annihilation and creation operators,
   - show that states in the Fock space respect the Pauli exclusion principle,
   - derive the Hamiltonian density in terms of creation and annihilation operators and discuss the necessity of introducing anti-commuting fermions,
   - find the conserved current (in terms of the creation and annihilation operators) which follows (via the Noether theorem) from the invariance of the theory under a phase transformation

     $$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x)$$

     where α is a constant phase parameter.

4. From the definition of the free fermion propagator

   $$iS_F(x-x') \equiv \langle 0|\mathrm{T}\{\psi(x)\bar{\psi}(y)\}|0\rangle$$

   derive its momentum representation

   $$iS_F(p) = \frac{i}{\not{p} - m + i\varepsilon}\,.$$

5. Derive, from the very definition, the lowest-order non-vanishing S-matrix element and hence the corresponding Feynman amplitude in the momentum space for the Bhabha scattering, i.e. the process

   $$e^+(p_1, r_1) + e^-(p_2, r_2) \to e^+(p_1', s_1) + e^-(p_2', s_2)\,.$$

6. Derive, from the very definition, the lowest-order non-vanishing S-matrix element and hence the corresponding Feynman amplitude in the momentum space for the Compton scattering, i.e. the process

   $$\gamma(k, \lambda) + e^-(p, s) \to \gamma(k', \lambda') + e^-(p, s')\,.$$

7. Show, without adopting the Feynman rules, that the lowest-order Feynman amplitude for the photon self-energy (it corresponds to a diagram with two external photon lines) is given by

$$\mathcal{M} = \frac{-e^2}{(2\pi)^4} \int d^4p \mathrm{Tr}[\epsilon^*(k,r)^\mu \gamma_\mu S_F(p+k) \epsilon^\nu(k,r) \gamma_\nu S_F(p)$$

where k and $\epsilon(k,r)$ are the momentum and polarization vectors of the external photon.

8. Derive the Feynman rules for the pseudo-scalar meson theory which is defined by the Lagrangian density

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x)$$

where

$$\mathcal{L}_0 = \frac{1}{2}[\partial_\mu \phi(x) \partial^\mu \phi(x) - M^2 \phi^2(x)] + \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

describes a free real scalar field and a free fermion field, and the interaction is given by

$$\mathcal{L}_I = -ig\bar{\psi}(x)\gamma_5 \psi(x)\phi(x) \, .$$

9. Neglecting the electron mass, find the differential cross-section, $\frac{d\sigma}{d\Omega}$ in the center of mass, and the total cross-section for the muon pair production

$$e^+(p_1, r_1) + e^-(p_2, r_2) \to \mu^+(p_1', s_1) + \mu^-(p_2', s_2) \, ,$$

where Ω is the solid angle corresponding to μ^-.

10. Find the differential cross-section, $\frac{d\sigma}{d\Omega}$ in the center of mass and the total cross-section for the Bhabha scattering

$$e^+(p_1, r_1) + e^-(p_2, r_2) \to e^+(p_1', s_1) + e^-(p_2', s_2) \, ,$$

where Ω is the solid angle corresponding to μ^-. Assume the high-energy limit, $E \gg m_e$ (where E is the energy of the initial electron in the center of mass frame), and keep only the leading terms.

References

[1] F. Mandl, G. Shaw, *Quantum Field Theory.*

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