8 Interaction picture (free) field operators

In Chapter 7 investigated were the conditions under which Hamiltonians of interacting particles lead to Lorentz covariant S-matrices satisfying the cluster decomposition principle. It has been found that the latter requirement can be easily met if the Hamiltonian is constructed out of the creation and annihilation operators of free one-particle states, with coefficient functions containing only a single three-momentum delta function. Lorentz invariance can in turn be ensured if the interaction operator $V^I_{\rm int}(t)$ in the Dirac picture is expressed as the space integral of an interaction Hamiltonian density $\mathcal{H}_{\rm int}(x) = \mathcal{H}_{\rm int}(t, \mathbf{x})$ which is a Lorentz scalar as in (7.101) and satisfies the local causality condition (7.102). Here we prepare the building blocks for constructing such operators $V^I_{\rm int}(t) = \int d^3\mathbf{x}\, \mathcal{H}_{\rm int}(t, \mathbf{x})$ representing interactions of various types of particles. In order to satisfy the cluster decomposition principle we will start with the creation and annihilation operators and will argue, that in order to build out of them a Lorentz scalar $\mathcal{H}_{\rm int}(x)$, they have to be combined into quantum (free) field operators. The local causality condition will then imply that these field operators must be linear combinations of both the creation and the annihilation operators (a circumstance which inevitably leads to the particle number non-conservation), that the celebrated spin-statistics connection must hold and, finally, that there must be antiparticles.

8.1 General structure

Assuming that the ground state vector $|\Omega_0\rangle$ of the free Hamiltonian H_0 (which is a sum of terms of the form (6.113) corresponding to different types of particles) is Poincaré invariant, $U_0(\Lambda, a)|\Omega_0\rangle = |\Omega_0\rangle$, the Poincaré transformation properties of the creation and annihilation operators of a particle of spin s follow directly from the corresponding properties of the one-particle states they create or annihilate (c.f. (6.111)):

$$U_0(\Lambda, a)\, a^\dagger_\sigma(\mathbf{p})\, U_0^{-1}(\Lambda, a) = e^{ia\cdot p_\Lambda} \sum_{\sigma'} a^\dagger_{\sigma'}(\mathbf{p}_\Lambda)\, D^{(s)}_{\sigma'\sigma}(W(\Lambda, p))\,,$$
$$U_0(\Lambda, a)\, a_\sigma(\mathbf{p})\, U_0^{-1}(\Lambda, a) = e^{-ia\cdot p_\Lambda} \sum_{\sigma'} a_{\sigma'}(\mathbf{p}_\Lambda)\, D^{(s)*}_{\sigma'\sigma}(W(\Lambda, p))\,. \tag{8.1}$$

The transformation properties (8.1) are complicated as they depend on the momentum arguments of the creation and annihilation operators. Therefore, constructing Lorentz scalars directly out of these objects would be a rather cumbersome task. The solution is to build first objects which have simpler Poincaré transformation properties. To this end, one therefore writes

$$\phi^{(+)}_l(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma u_l(x, \mathbf{p}, \sigma)\, a_\sigma(\mathbf{p})\,,$$
$$\phi^{(-)}_l(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma v_l(x, \mathbf{p}, \sigma)\, a^\dagger_\sigma(\mathbf{p})\,, \tag{8.2}$$

and tries to find such functions $u_l(x,\mathbf{p},\sigma)$ and $v_l(x,\mathbf{p},\sigma)$ which will lead to the relations

$$U_0(\Lambda,a)\,\phi_l^{(+)}(x)\,U_0^{-1}(\Lambda,a) = \sum_k D_{lk}(\Lambda^{-1})\,\phi_k^{(+)}(\Lambda\cdot x+a)\,,$$
$$U_0(\Lambda,a)\,\phi_l^{(-)}(x)\,U_0^{-1}(\Lambda,a) = \sum_k D_{lk}(\Lambda^{-1})\,\phi_k^{(-)}(\Lambda\cdot x+a)\,, \tag{8.3}$$

in which $D_{lk}(\Lambda)$ are constant matrices of a finite-dimensional representation of the Lorentz group (matrices of the corresponding generators will be denoted $\mathcal{J}^{\mu\nu}_{\rm rep}$ with the subscript indicating the representation). There are many different such representations, for example the trivial one with $D_{lk}(\Lambda)$ being a unit 1×1 matrix for all Λ's, or the vector representation with $D_{lk}(\Lambda)=\Lambda^\mu{}_\nu$, or various tensor and spinor representations. Once such functions $u_l(x,\mathbf{p},\sigma)$ and $v_l(x,\mathbf{p},\sigma)$ are found, terms of a Lorentz scalar $\mathcal{H}_{\rm int}(x)$ can trivially be constructed in the form

$$\mathcal{H}_{\rm int}(x) = \sum_{l_1\dots l_N;k_1\dots k_M} g_{l_1,\dots,l_N;k_1,\dots,k_M}\,\phi_{l_1}^{(-)}(x)\dots\phi_{l_N}^{(-)}(x)\phi_{k_1}^{(+)}(x)\dots\phi_{k_M}^{(+)}(x)\,, \tag{8.4}$$

with $g_{l_1,\dots,l_N;k_1,\dots,k_M}$ being the appropriate Clebsch-Gordan coefficients allowing to get a Lorentz scalar from a product of the fields $\phi_l^{(\pm)}(x)$ (or, more precisely, picking out the trivial representation from a tensor product of several Lorentz group representations corresponding to individual fields $\phi_l^{(\pm)}(x)$ in the product). Space-time derivative(s) of a field operator $\phi_l(x)$ are in (8.4) treated simply as another type of operator transforming as another matrix representation of the Lorentz group. (In other words, $\partial_{\mu_1}\dots\partial_{\mu_n}\phi_l(x)$ is in (8.4) treated as another type of operator).

Applied to $\phi_l^{(-)}(x)$, the condition (8.3) gives the relation

$$\int d\Gamma_{\mathbf{p}}\sum_\sigma v_l(x,\mathbf{p},\sigma)\,e^{ia\cdot p_\Lambda}\sum_{\sigma'} D^{(s)}_{\sigma'\sigma}(W(\Lambda,p))\,a^\dagger_{\sigma'}(\mathbf{p}_\Lambda) \tag{8.5}$$
$$= \sum_k D_{lk}(\Lambda^{-1})\int d\Gamma_{\mathbf{p}}\sum_\sigma v_k(\Lambda\cdot x+a,\mathbf{p},\sigma)\,a^\dagger_\sigma(\mathbf{p})\,.$$

Using the fact that $d\Gamma_{\mathbf{p}} = d\Gamma_{\mathbf{p}_\Lambda}$ this can be rewritten as

$$\int d\Gamma_{\mathbf{p}_\Lambda}\sum_\sigma v_l(x,\mathbf{p},\sigma)\,e^{ia\cdot p_\Lambda}\sum_{\sigma'} D^{(s)}_{\sigma'\sigma}(W(\Lambda,p))\,a^\dagger_{\sigma'}(\mathbf{p}_\Lambda)$$
$$= \sum_k D_{lk}(\Lambda^{-1})\int d\Gamma_{\mathbf{p}_\Lambda}\sum_\sigma v_k(\Lambda\cdot x+a,\mathbf{p}_\Lambda,\sigma)\,a^\dagger_\sigma(\mathbf{p}_\Lambda)\,.$$

It is easy to see (after renaming the indices $\sigma\leftrightarrow\sigma'$ in the left hand side) that this holds if

$$e^{ia\cdot p_\Lambda}\sum_{\sigma'} D^{(s)}_{\sigma\sigma'}(W(\Lambda,p))\,v_l(x,\mathbf{p},\sigma') = \sum_k D_{lk}(\Lambda^{-1})\,v_k(\Lambda\cdot x+a,\mathbf{p}_\Lambda,\sigma)\,.$$

Furthermore, using $D_{lk}(\Lambda^{-1}) = [D^{-1}(\Lambda)]_{lk}$ and

$$D^{(s)}_{\sigma\sigma'}(W) = \left[D^{(s)\dagger}(W)\right]^*_{\sigma'\sigma} = \left[D^{(s)-1}(W)\right]^*_{\sigma'\sigma}\,, \tag{8.6}$$

the basic conditions can be cast into the form

$$\begin{aligned}
\sum_{\sigma'} v_l(\Lambda\cdot x + a, \mathbf{p}_\Lambda, \sigma') D^{(s)*}_{\sigma'\sigma}(W(\Lambda, p)) &= e^{ia\cdot p_\Lambda} \sum_k D_{lk}(\Lambda)\, v_k(x, \mathbf{p}, \sigma)\,, \\
\sum_{\sigma'} u_l(\Lambda\cdot x + a, \mathbf{p}_\Lambda, \sigma') D^{(s)}_{\sigma'\sigma}(W(\Lambda, p)) &= e^{-ia\cdot p_\Lambda} \sum_k D_{lk}(\Lambda)\, u_k(x, \mathbf{p}, \sigma)\,.
\end{aligned} \tag{8.7}$$

(The second of these conditions is obtained from $\phi_l^{(+)}$ in the similar manner as the first one from $\phi_l^{(-)}$). These conditions should determine the functions u_l and v_l. To solve them, we first set in (8.7) $\Lambda = I$ so that $D_{lk}(I) = \delta_{lk}$ and $D^{(s)*}_{\sigma'\sigma}(W(\Lambda, p)) = \delta_{\sigma'\sigma}$. We get then

$$\begin{aligned}
v_l(x + a, \mathbf{p}, \sigma) &= e^{+ia\cdot p}\, v_l(x, \mathbf{p}, \sigma)\,, \\
u_l(x + a, \mathbf{p}, \sigma) &= e^{-ia\cdot p}\, u_l(x, \mathbf{p}, \sigma)\,.
\end{aligned} \tag{8.8}$$

This shows that[1]

$$v_l(x, \mathbf{p}, \sigma) = e^{+ix\cdot p}\, v_l(\mathbf{p}, \sigma)\,, \qquad u_l(x, \mathbf{p}, \sigma) = e^{-ix\cdot p}\, u_l(\mathbf{p}, \sigma)\,. \tag{8.9}$$

Note, that with this x-dependence of the u_l and v_l functions the terms of $\mathcal{H}_{\rm int}(x)$ constructed as in (8.4), when integrated over $d^3\mathbf{x}$ will have the required form (7.99) and (7.171) with (7.172). The conditions (8.7) reduce therefore to

$$\begin{aligned}
\sum_{\sigma'} v_l(\mathbf{p}_\Lambda, \sigma') D^{(s)*}_{\sigma'\sigma}(W(\Lambda, p)) &= \sum_k D_{lk}(\Lambda)\, v_k(\mathbf{p}, \sigma)\,, \\
\sum_{\sigma'} u_l(\mathbf{p}_\Lambda, \sigma') D^{(s)}_{\sigma'\sigma}(W(\Lambda, p)) &= \sum_k D_{lk}(\Lambda)\, u_k(\mathbf{p}, \sigma)\,.
\end{aligned} \tag{8.10}$$

Next, let us take the four-momentum p^μ in the right hand sides of (8.10) to be the standard four-momentum k^μ of the particle, that is $\mathbf{p} = \mathbf{k} = \mathbf{0}$ if the particle is massive and $\mathbf{p} = \mathbf{k} = (0, 0, \kappa)$ if it is massless. Furthermore, take Λ to be the standard Lorentz transformation L_q which produces q^μ out of the standard four-momentum k^μ. In this case $L_k = I$ and $W(\Lambda, k) = L^{-1}_{\Lambda\cdot k}\cdot \Lambda \cdot L_k = L^{-1}_{L_q\cdot k}\cdot L_q = L_q^{-1}\cdot L_q = I$ and the conditions (8.10) take the form[2]

$$\begin{aligned}
v_l(\mathbf{q}, \sigma) &= \sum_k D_{lk}(L_q)\, v_k(\mathbf{k}, \sigma)\,, \\
u_l(\mathbf{q}, \sigma) &= \sum_k D_{lk}(L_q)\, u_k(\mathbf{k}, \sigma)\,.
\end{aligned} \tag{8.11}$$

[1] We keep the same names for the x-dependent and x-independent functions.

[2] Notice the similarity of (8.11) with the definition (6.29) of the one-particle states $|\mathbf{p}, \sigma\rangle$.

This means that the functions $v_l(\mathbf{q},\sigma)$ and $u_l(\mathbf{q},\sigma)$ for an arbitrary momentum $\mathbf{q}$ can be obtained once they are known for the standard four-momentum (i.e. in the case of massive particles they can be reconstructed from their zero-momentum forms). It is therefore sufficient to find $v_l(\mathbf{k},\sigma)$ and $u_l(\mathbf{k},\sigma)$.

From now on massive and massless particles must be considered separately. We will discuss the case of massive particles first.

Let us investigate the transformations belonging to the little group of the standard four-momentum $k^\mu = (m, \mathbf{0})$ of a massive particle, that is, the three-dimensional rotations $\Lambda = O$. As shown in Section 6.2, in this case $W(\Lambda = O, p) = O$. The conditions (8.10) take then the simple form

$$\begin{aligned} \sum_{\sigma'} v_l(\mathbf{0},\sigma') D^{(s)*}_{\sigma'\sigma}(O) &= \sum_k D_{lk}(O)\, v_k(\mathbf{0},\sigma)\,, \\ \sum_{\sigma'} u_l(\mathbf{0},\sigma') D^{(s)}_{\sigma'\sigma}(O) &= \sum_k D_{lk}(O)\, u_k(\mathbf{0},\sigma)\,. \end{aligned} \tag{8.12}$$

It will be convenient to rewrite these conditions in terms of the generators $\mathbf{J}^{(j)}_{\sigma'\sigma}$ of the matrices $D^{(j)}_{\sigma'\sigma}(W)$ of the little group and of the generators $(\mathcal{J}^{\mu\nu})_{lk}$ of the matrix representation D_{lk} of the Lorentz group:

$$\begin{aligned} -\sum_{\sigma'} v_l(\mathbf{0},\sigma')\, \mathbf{J}^{(s)*}_{\sigma'\sigma} &= \sum_k \boldsymbol{\mathcal{J}}_{lk}\, v_k(\mathbf{0},\sigma)\,, \\ \sum_{\sigma'} u_l(\mathbf{0},\sigma')\, \mathbf{J}^{(s)}_{\sigma'\sigma} &= \sum_k \boldsymbol{\mathcal{J}}_{lk}\, u_k(\mathbf{0},\sigma)\,, \end{aligned} \tag{8.13}$$

where as usually $(\mathcal{J}^i)_{lk} = \frac{1}{2}\epsilon^{ijn}\,(\mathcal{J}^{jn})_{lk}$. These conditions mean that if the field operators $\phi^{(+)}_l(x)$ and $\phi^{(-)}_l(x)$ are to be associated with a particle of spin s, the representation $D_{lk}(\Lambda)$ according to which these fields transform must, when restricted to rotations, contain among its irreducible components (irreducible w.r.t. the rotation group) the representation $D^{(s)}$ of spin s. This follows from the fact that the functions $v_l(\mathbf{0},\sigma)$ and $u_l(\mathbf{0},\sigma)$ treated as rectangular matrices in their indices (l,σ) convolute two different finite-dimensional representations of the rotation group.[3] Since each irreducible representation of the Lorentz group contains a given representation of the rotation group at most once, the functions $u_\alpha(\mathbf{0},\sigma)$ and $v_\alpha(\mathbf{0},\sigma)$ are, for a given Lorentz group representation, determined uniquely up to a trivial rescaling.

It can be straightforwardly checked that if the conditions (8.12) are met, the functions $v_l(\mathbf{q},\sigma)$ and $u_l(\mathbf{q},\sigma)$ constructed as in (8.11) do satisfy also the more general condition

[3] In other words, $v_l(\mathbf{0},\sigma)$ and $u_l(\mathbf{0},\sigma)$ define the similarity transformations relating two representations of the rotation group. The conditions for $u_l(\mathbf{0},\sigma)$ and $v_l(\mathbf{0},\sigma)$ can hold simultaneously (are not contradictory to each other) because the representations $D^{(s)}_{\sigma'\sigma}$ and $D^{(s)*}_{\sigma'\sigma}$ of the rotation group are real if the spin s is integer and pseudo-real if it is half-integer, that is, in both cases they are unitarily equivalent to one another.

(8.10). Indeed,

$$\begin{aligned}\sum_k D_{lk}(\Lambda)\, u_k(\mathbf{q},\sigma) &= \sum_k [D(\Lambda)D(L_q)]_{lk}\, u_k(\mathbf{0},\sigma)\\ &= \sum_n D_{ln}(L_{\Lambda\cdot q}) \sum_k D_{nk}(W(\Lambda,q))\, u_k(\mathbf{0},\sigma)\\ &= \sum_n \sum_{\sigma'} D_{ln}(L_{\Lambda\cdot q})\, u_n(\mathbf{0},\sigma') D^{(s)}_{\sigma'\sigma}(W(\Lambda,q))\\ &= \sum_{\sigma'} u_l(\mathbf{q}_\Lambda,\sigma') D^{(s)}_{\sigma'\sigma}(W(\Lambda,q))\,,\end{aligned}$$

where we have inserted $I = D(L_{\Lambda\cdot q})D(L^{-1}_{\Lambda\cdot q})$, used the definition $W(\Lambda,q) = L^{-1}_{\Lambda\cdot q}\cdot\Lambda\cdot L_q$ and finally exploited the relation (8.12). The case of $v_l(\mathbf{q},\sigma)$ is similar.

8.2 Massive spin zero particles

Consider now the simplest case of spin zero particles. In this casae $D^{(s)}_{\sigma'\sigma}(W)$ are trivial 1×1 unit matrices and the generators $\mathbf{J}^{(s)}_{\sigma'\sigma}$ in (8.13) are simply zero 1×1 matrices. The conditions (8.12) or (8.13) can, therefore, be satisfied by taking the quantum field operators $\phi^{(+)}_l(x)$ and $\phi^{(-)}_l(x)$ formed from the creation and annihilation operators of a spinless particle to transform as Lorentz scalars, that is with $D_{lk}(\Lambda)$ which are trivial 1×1 unit matrices (and the generators $\mathcal{J}^{\mu\nu}_{lk}$ and, hence, also the generators $\boldsymbol{\mathcal{J}}_{lk}$ of the rotation subgroup are zero 1×1 matrices). In this case the functions $v_l(\mathbf{0},\sigma)$ and $u_l(\mathbf{0},\sigma)$ which are simply constants can be taken as equal 1 without loss of generality. Thus,

$$\phi^{(+)}(x) = \int d\Gamma_{\mathbf{p}}\, a(\mathbf{p})\, e^{-ip\cdot x}\,, \tag{8.14}$$

$$\phi^{(-)}(x) = \int d\Gamma_{\mathbf{p}}\, a^{\dagger}(\mathbf{p})\, e^{+ip\cdot x}\,. \tag{8.15}$$

We will see however, that the field operators transforming as Lorentz scalars are not the only possibility; field operators transforming as Lorentz vectors or higher tensor representations can also be associated with spinless particles (that is, can be built out of the creation and annihilation operators of spinless particles). But the physical character of the spin zero particle interaction does depend on the kind of the operator through which the creation and annihilation operators of a given particle enter $\mathcal{H}_{\rm int}$.

Neutral spinless particles

We now have to take into account the local causality condition

$$[\mathcal{H}_{\rm int}(x),\ \mathcal{H}_{\rm int}(y)] = 0 \qquad \text{for} \qquad (x-y)^2 < 0\,, \tag{8.16}$$

assuming that $\mathcal{H}_{\rm int}(x)$ is built as in (8.4) out of the field operators $\phi^{(+)}(x)$, $\phi^{(-)}(x)$ of spinless particles and possibly of other similar field operators $\phi_l^{(+)}(x)$ and $\phi_l^{(-)}(x)$ (which we are going to construct in the following sections) of other, spin zero and higher spin particles. Obviously, the interaction Hamiltonian density cannot depend only on the creation (that is on $\phi^{(-)}(x)$ only) nor only on the annihilation (that is on $\phi^{(+)}(x)$ only) operators of particles of a given type (which would be the simplest way to satisfy the local causality condition (8.16)), because it would not be Hermitian and would not lead to a unitary S-matrix. On the other hand, as we will see below, in general the (anti)commutators $[\phi_l^{(+)}(x),\ \phi_k^{(-)}(y)]$ do not vanish by itself, even if the separations of the points x and y is space-like. Therefore, the only viable possibility is to build $\mathcal{H}_{\rm int}(x)$ out of some linear combination of the $\phi^{(+)}(x)$ and $\phi^{(-)}(x)$ operators

$$\phi(x) = \alpha_+\phi^{(+)}(x) + \alpha_-\phi^{(-)}(x)\,, \tag{8.17}$$

and its Hermitian conjugation $\phi^\dagger(x)$, which have a chance to fulfill the conditions

$$[\phi(x),\ \phi(y)]_\mp = \left[\phi(x),\ \phi^\dagger(y)\right]_\mp = 0 \qquad \text{for} \qquad (x-y)^2 < 0\,, \tag{8.18}$$

either with the commutator (the $-$ sign) or with the anticommutator (the $+$ sign) - we leave for the moment undecided whether the field operators should commute or anticommute for space-like point separations. We ask therefore wheter it is possible to arrange for

$$\begin{aligned}[]
[\phi(x),\ \phi(y)]_\mp &= \alpha_+\alpha_-\left[\phi^{(+)}(x),\ \phi^{(-)}(y)\right]_\mp \\
&\quad + \alpha_-\alpha_+\left[\phi^{(-)}(x),\ \phi^{(+)}(y)\right]_\mp = 0\,, \qquad (8.19)\\
\left[\phi(x),\ \phi^\dagger(y)\right]_\mp &= |\alpha_+|^2\left[\phi^{(+)}(x),\ \phi^{(+)\dagger}(y)\right]_\mp \\
&\quad + |\alpha_-|^2\left[\phi^{(-)}(x),\ \phi^{(-)\dagger}(y)\right]_\mp = 0\,, \qquad (8.20)
\end{aligned}$$

when $(x-y)^2 < 0$. From the explicit forms (8.14), (8.15) of the operators we easily find that

$$\begin{aligned}
\left[\phi^{(+)}(x),\ \phi^{(-)}(y)\right]_\mp &= \left[\phi^{(+)}(x),\ \phi^{(+)\dagger}(y)\right]_\mp \\
&= \int d\Gamma_{\mathbf{p}}\, e^{-ip\cdot(x-y)} \equiv \Delta_+(x-y)\,,
\end{aligned} \tag{8.21}$$

where we have used the rule $\left[a(\mathbf{p}),\ a^\dagger(\mathbf{q})\right]_\mp = \delta_\Gamma^{(3)}(\mathbf{p}-\mathbf{q})$ valid for both, commutators and anticommutators. The important property of the function $\Delta_+(x-y)$ is that it is even when $(x-y)^2<0$:

$$\Delta_+(y-x) = \Delta_+(x-y) \qquad \text{when} \qquad (x-y)^2 < 0\,. \tag{8.22}$$

Indeed, since the factor $\exp(-ip\cdot(x-y))$ and the measure

$$\int d\Gamma_{\mathbf{p}} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} = \int \frac{d^4p}{(2\pi)^3}\,\delta(p^2-m^2)\,\theta(p^0) = \int d\Gamma_{\mathbf{p}_\Lambda}\,, \tag{8.23}$$

are both Lorentz invariant, the function $\Delta_+(x-y)$ is Lorentz invariant too. Therefore

$$\Delta_+(y-x) = \int d\Gamma_{\mathbf{p}_\Lambda}\, e^{-i(\Lambda p)\cdot(y-x)} = \int d\Gamma_{\mathbf{p}}\, e^{-ip\cdot(\Lambda^{-1}\cdot(y-x))}\,. \tag{8.24}$$

We have used here the equality $d\Gamma_{\mathbf{p}_\Lambda} = d\Gamma_{\mathbf{p}}$ and the fact that when $(x-y)^2 < 0$, there exists a transformation Λ^{-1} such that $\Lambda^{-1}(y-x) = -(y-x) = (x-y)$. This completes the proof. Thus, if $(x-y)^2 < 0$, in the considered case of spinless particles one obtains

$$[\phi(x),\ \phi(y)]_{\mp} = \alpha_+\alpha_-(1\mp 1)\Delta_+(x-y)\,, \tag{8.25}$$

$$\left[\phi(x),\ \phi^\dagger(y)\right]_{\mp} = (|\alpha_+|^2 \mp |\alpha_-|^2)\Delta_+(x-y)\,. \tag{8.26}$$

As is clear, both expressions vanish only if the upper sign are selected, that is, if the annihilation and creation operators of spin $s=0$ particles *commute* to $\delta^{(3)}_\Gamma(\mathbf{p}-\mathbf{q})$. Thus, the local causality condition (8.16), which ensures Lorentz covariance of the S-matrix, fixes also the statistics of spinless particles to be the Bose-Einstein one.

From (8.26) it follows that $|\alpha_+|^2 = |\alpha_-|^2$. Since the absolute scale of the field operator is not essential now, one can set[4] $\alpha_+ = \alpha_- = 1$ obtaining the field operator in the form

$$\phi(x) = \int d\Gamma_{\mathbf{p}} \left(a(\mathbf{p})\, e^{-ip\cdot x} + a^\dagger(\mathbf{p})\, e^{+ip\cdot x}\right) \equiv \phi^{(+)}(x) + \phi^{(-)}(x)\,, \tag{8.27}$$

which is Hermitian: $\phi^\dagger(x) = \phi(x)$. The operators $\phi^{(+)}(x)$ and $\phi^{(-)}(x)$ are usually called *positive and negative frequency parts* of the field operator $\phi(x)$. Since th spacetime dependence of all these operators is given by the exponentials of $p\cdot x$ with $p^2 = m^2$, the operators $\phi(x)$ and $\phi^{(\pm)}(x)$ satisfy the "Klein-Gordon equation"

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = (\partial_\mu\partial^\mu + m^2)\phi^{(\pm)}(x) = 0\,. \tag{8.28}$$

Note also, that if the terms of $\mathcal{H}_{\rm int}(x)$ are constructed as polynomials of $\phi(x)$, $\mathcal{H}_{\rm int}(x)$ cannot commute with the particle number operator $\hat{N} = \int d\Gamma_{\mathbf{p}}\, a^\dagger(\mathbf{p})a(\mathbf{p})$. The particle number conservation is therefore incompatible with the requirement of the Lorentz covariance of the S-matrix.

It will be also useful to know that the creation and annihilation operators can be expressed in terms of the field operator (8.27) with the help of the formulae

$$a(\mathbf{p}) = i\int d^3\mathbf{x}\, f^*_{\mathbf{p}}(x)\overleftrightarrow{\partial}_0\phi(x)\,, \qquad a^\dagger(\mathbf{p}) = -i\int d^3\mathbf{x}\, f_{\mathbf{p}}(x)\overleftrightarrow{\partial}_0\phi(x)\,, \tag{8.29}$$

where $f_{\mathbf{p}}(x) = e^{-ip\cdot x}$ with $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$ and $f\overleftrightarrow{\partial}_0 g \equiv f(\partial_0 g) - (\partial_0 f)g$.

[4]Redefining the phase of the one particle state $|\mathbf{p}\rangle' = e^{i\delta}|\mathbf{p}\rangle$ amounts (keeping the phase of the vacuum vector $|\Omega_0\rangle$ fixed) to changing the phase of the creation and annihilation operators $a'(\mathbf{p}) = e^{-i\delta}a(\mathbf{p})$, $a'^\dagger(\mathbf{p}) = e^{i\delta}a^\dagger(\mathbf{p})$. This allows to absorb any relative phase of α_+ and α_-. The operator (8.27) has therefore the most general form.

If the particles created and annihilated by the operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ respectively, carry a quantum number (like the electric charge or strangeness) which is to be a conserved quantity, the above construction has to be modified. With only a single type of particles, such a quantum number would be conserved only if each term of the Hamiltonian contained an equal numbers of $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ operators. In other words, $\mathcal{H}_{\rm int}(x)$ would have to commute with the operator $\hat{Q}$ of the quantum number which (among perhaps other terms) contains a term

$$\hat{Q} = \int d\Gamma_{\mathbf{p}}\, q\, a^\dagger(\mathbf{p}) a(\mathbf{p}) + \dots \tag{8.30}$$

in which q is the value of the quantity Q carried by the particles. But the operator $\hat{Q}$, which is the generator of a symmetry responsible for conservation of the quantity Q, satisfies the commutation rules

$$\begin{aligned} &[\hat{Q},\ a(\mathbf{p})] = -q\, a(\mathbf{p})\,, &&[\hat{Q},\ a^\dagger(\mathbf{p}] = +q\, a^\dagger(\mathbf{p})\,, \\ &[\hat{Q},\ \phi^{(+)}(x)] = -q\, \phi^{(+)}(x)\,, &&[\hat{Q},\ \phi^{(-)}(x)] = +q\, \phi^{(-)}(x)\,, \end{aligned} \tag{8.31}$$

which imply that $\hat{Q}$ cannot commute with $\mathcal{H}_{\rm int}(x)$ if the latter is constructed as a sum of polynomials in $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) = \phi^{(+)}(x) + \phi^{(+)\dagger}(x)$. For example $\phi(x)\phi(x)$ contains the terms $\phi^{(+)}(x)\phi^{(+)}(x)$ and $\phi^{(+)\dagger}(x)\phi^{(+)\dagger}(x)$ which do not commute with $\hat{Q}$. More generally, this is so because the commutator $[\hat{Q},\ \phi(x)]$ cannot be expressed simply in terms of $\phi(x)$ itself. The way out is to construct $\mathcal{H}_{\rm int}(x)$ out of field operators which have simple commutators with $\hat{Q}$. This in turn becomes possible if in addition to particles having the charge $+q$ there also exist particles, called antiparticles, having the same mass and Lorentz transformation properties, but carrying the charge $-q$, opposite to the charge of the particles. Available are then their creation and annihilation operators $a^{c\dagger}(\mathbf{p})$ and $a^c(\mathbf{p})$ and the corresponding operators $\phi^{c(+)}(x)$ and $\phi^{c(-)}(x)$ which satisfy the rules

$$\begin{aligned} &[\hat{Q},\ a^c(\mathbf{p})] = +q\, a^c(\mathbf{p})\,, &&[\hat{Q},\ a^{c\dagger}(\mathbf{p})] = -q\, a^{c\dagger}(\mathbf{p})\,, \\ &[\hat{Q},\ \phi^{c(+)}(x)] = +q\, \phi^{c(+)}(x)\,, &&[\hat{Q},\ \phi^{c(-)}(x)] = -q\, \phi^{c(-)}(x)\,. \end{aligned} \tag{8.32}$$

The causal quantum field operators can be then formed as the linear combinations[5]

$$\phi(x) = \phi^{(+)}(x) + \phi^{c(-)}(x)\ , \qquad \phi^\dagger(x) = \phi^{c(+)}(x) + \phi^{(-)}(x)\,, \tag{8.33}$$

(so that in this case $\phi^\dagger(x) \neq \phi(x)$), which satisfy the relations

$$[\hat{Q},\ \phi(x)] = -q\, \phi(x)\ , \qquad [\hat{Q},\ \phi^\dagger(x)] = +q\, \phi^\dagger(x)\,. \tag{8.34}$$

[5] In the remaining sections of this chapter, where the symbol $\phi^c(x)$ will no longer be used, we will change the notation and will call $\phi^{c(-)}(x)$ just $\phi^{(-)}(x)$; $\phi^{(+)}$ ($\phi^{\dagger(+)}$) will then contain the operator $a(\mathbf{p})$ ($a^c(\mathbf{p})$) and $\phi^{(-)}$ ($\phi^{\dagger(-)}$) the operator $a^{c\dagger}(\mathbf{p})$ ($a^\dagger(\mathbf{p})$).

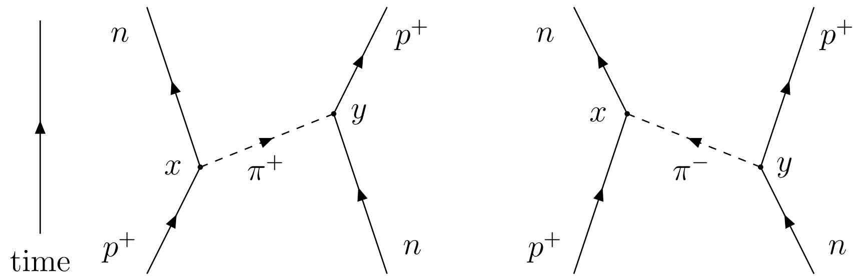

Figure 8.4: The same process seen from two different reference frames.

Each term of $\mathcal{H}_{\rm int}(x)$ commutes then with $\hat{Q}$ provided the algebraic sum of charges q_i of the field operators $\phi_{(i)}$ forming such a term vanishes:

$$
\begin{aligned}
[\phi_{(1)}\phi_{(2)}\dots\phi_{(N)},\ \hat{Q}] &= q_1\phi_{(1)}\dots\phi_{(N)} + \dots + q_N\phi_{(1)}\dots\phi_{(N)} \\
&= (q_1 + \dots + q_N)\phi_{(1)}\phi_{(2)}\dots\phi_{(N)}\,. \qquad (8.35)
\end{aligned}
$$

Explicitly

$$
\begin{aligned}
\phi(x) &= \int d\Gamma_{\mathbf{p}} \left(a(\mathbf{p})\, e^{-ip\cdot x} + a^{c\dagger}(\mathbf{p})\, e^{+ip\cdot x}\right), \\
\phi^\dagger(x) &= \int d\Gamma_{\mathbf{p}} \left(a^{c}(\mathbf{p})\, e^{-ip\cdot x} + a^{\dagger}(\mathbf{p})\, e^{+ip\cdot x}\right), \qquad (8.36)
\end{aligned}
$$

and the operators $a(\mathbf{p})$ and $a^{c\dagger}(\mathbf{p})$ ($a^c(\mathbf{p})$ and $a^\dagger(\mathbf{p})$) can be expressed in terms of $\phi(x)$ ($\phi^\dagger(x)$) with the help of the formulae similar to (8.29).

It is easy to check that the field operators (8.33) are locally causal, that is

$$
\begin{aligned}
[\phi(x),\ \phi(y)] = \left[\phi^\dagger(x),\ \phi^\dagger(y)\right] = 0\,, \\
\left[\phi(x),\ \phi^\dagger(y)\right] = 0 \qquad \text{when} \qquad (x-y)^2 < 0\,, \qquad (8.37)
\end{aligned}
$$

(the first two commutators now vanish for all values of $(x-y)^2$). Thus, the conservation of a quantum number combined with the requirement of Lorentz covariance of the S-matrix is the reason for the existence of antiparticles. Heuristically this can also be illustrated as follows (see figure 8.4). Suppose an observer sees a proton which at a space-time point x emits a positively charged pion π^+ which is at a later time (in his frame) absorbed at y by a neutron. Classically this is impossible when $(x-y)^2 < 0$, because this would mean that the velocity of π^+ was greater than c. Quantum mechanically, however, the velocity of a particle which at x^0 was at $\mathbf{x}$ cannot be precisely specified. In consequence, there is a nonzero amplitude[6] of it being absorbed at y, even if $(x-y)^2 < 0$, and this amplitude is nonnegligible if $-(x-y)^2 \lesssim \hbar^2/m^2c^2$. This seems to give rise to a paradox because if

[6]This amplitude can be identified with $\langle\Omega_0|\phi^{(+)}(y)\phi^{(-)}(x)|\Omega_0\rangle = \Delta_+(y-x)$.

$(x-y)^2 < 0$, there exists a reference frame $\mathcal{O}'$ in which $x^{0\prime} > y^{0\prime}$, that is, in which the positively charged pion is absorbed before it has been emitted! Existence of antiparticles removes the paradox because the other observer sees then the neutron emitting at $y^{0\prime}$ the *negatively* charged pion π^- which (in the frame $\mathcal{O}'$) is at $x^{0\prime}$ absorbed by the proton. This shows that to each particle carrying at least one conserved quantum number there must exist an antiparticle having the same mass[7] and Lorentz transformation properties (we do not mean parity, charge conjugation nor time reversal here!) but with opposite values of all conserved quantum numbers. Of course, some particles (like photon or π^0) do not carry any conserved charges.[8] Such particles are their own antiparticles (i.e. are essentially neutral particles).

P, T and C transformations of scalar field operators of spinless particles

Finally we discuss the transformation properties of the scalar field operators under parity, time reversal and charge conjugation. The creation and annihilation operators of spin zero particles out of which such operators are constructed satisfy the relations

$$\begin{aligned} &\mathcal{P}_0\, a^\dagger(\mathbf{p})\, \mathcal{P}_0^{-1} = \eta\, a^\dagger(-\mathbf{p})\,, \qquad && \mathcal{P}_0\, a^{c\dagger}(\mathbf{p})\, \mathcal{P}_0^{-1} = \eta^c\, a^{c\dagger}(-\mathbf{p})\,, \\ &\mathcal{T}_0\, a^\dagger(\mathbf{p})\, \mathcal{T}_0^{-1} = \zeta\, a^\dagger(-\mathbf{p})\,, && \mathcal{T}_0\, a^{c\dagger}(\mathbf{p})\, \mathcal{T}_0^{-1} = \zeta^c\, a^{c\dagger}(-\mathbf{p})\,, \\ &\mathcal{C}_0\, a^\dagger(\mathbf{p})\, \mathcal{C}_0^{-1} = \xi\, a^{c\dagger}(\mathbf{p})\,, && \mathcal{C}_0\, a^{c\dagger}(\mathbf{p})\, \mathcal{C}_0^{-1} = \xi^c\, a^{\dagger}(\mathbf{p})\,. \end{aligned} \tag{8.38}$$

In order to have a chance to build interaction Hamiltonian densities invariant under parity transformations, that is, commuting with the $\mathcal{P}_0$ operator

$$\left[V^I_{\rm int}(t),\ \mathcal{P}_0\right] = 0\,, \tag{8.39}$$

which for $V^I_{\rm int}(t) = \int d^3\mathbf{x}\, \mathcal{H}_{\rm int}(t,\mathbf{x}))$, will hold if

$$\mathcal{P}_0\, \mathcal{H}_{\rm int}(t,\mathbf{x})\, \mathcal{P}_0^{-1} = \mathcal{H}_{\rm int}(t,-\mathbf{x}) \equiv \mathcal{H}_{\rm int}(P\cdot x)\,, \tag{8.40}$$

the field operators should themselves transform simply when sandwiched between the parity operators.

Acting with $\mathcal{P}_0$ on the scalar field operator, using (8.38), we get

$$\mathcal{P}_0\, \phi(x)\, \mathcal{P}_0^{-1} = \int d\Gamma_{\mathbf{p}} \left(\eta^* a(-\mathbf{p})\, e^{-ip\cdot x} + \eta^c a^{c\dagger}(-\mathbf{p})\, e^{+ip\cdot x}\right). \tag{8.41}$$

[7] If the mass m of the particle was different than the mass m^c of its antiparticle one would not obtain (8.26) but rather

$$\left[\phi(x),\ \phi^\dagger(y)\right]_\mp = |\alpha_+|^2 \Delta_+(x-y;m) \mp |\alpha_-|^2 \Delta_+(y-x;m^c)\,,$$

and there would be no chance to get zero for $(x-y)^2 < 0$.

[8] We mean here conserved charges generating $U(1)$ symmetries; π^0 does carry isospin quantum number which is conserved in strong interactions and implementation of isospin conservation in Hamiltonians intended to model strong interactions requires introducing in addition to π^0 also π^+, π^- which are antiparticles of one another.

The operator transforms simply only if $\eta^* = \eta^c$, because then

$$\mathcal{P}_0\, \phi(x)\, \mathcal{P}_0^{-1} = \eta^* \phi(P\cdot x)\,. \tag{8.42}$$

upon using the equality $d\Gamma_{\mathbf{p}} = d\Gamma_{-\mathbf{p}}$. The relation $\eta^* = \eta^c$ implies that the intrinsic parity P of a two-particle state of a spinless particle and its antiparticle is $P = \eta\eta^c = |\eta|^2 = +1$. If a spinless particle is its own antiparticle, $\eta^c \equiv \eta$. In this case $\eta^* = \eta$, i.e. $\eta^2 = 1$, that is, $\eta = \pm 1$.

Similarly, time reversal will be a good symmetry if

$$\mathcal{T}_0\, V^I_{\text{int}}(t)\, \mathcal{T}_0^{-1} = V^I_{\text{int}}(-t)\,. \tag{8.43}$$

This is because if $[\mathcal{T}_0,\ H_0] = 0$ and $[\mathcal{T}_0,\ V_{\text{int}}] = 0$ then

$$\mathcal{T}_0\left(e^{iH_0 t}\, V_{\text{int}}\, e^{-iH_0 t}\right)\mathcal{T}_0^{-1} = e^{-iH_0 t}\, \mathcal{T}_0\, V_{\text{int}}\, \mathcal{T}_0^{-1}\, e^{iH_0 t} = V^I_{\text{int}}(-t)\,. \tag{8.44}$$

Building such $V^I_{\text{int}}(t)$ is possible if the field operators themselves transform simply when acted upon by the $\mathcal{T}_0$ operator.

Sandwiching the scalar field operator of a spinless particle between $\mathcal{T}_0$ and $\mathcal{T}_0^{-1}$, using (8.38), we get (remember that $\mathcal{T}_0$ is antilinear!)

$$\mathcal{T}_0\, \phi(x)\, \mathcal{T}_0^{-1} = \int d\Gamma_{\mathbf{p}} \left(\zeta^* a(-\mathbf{p})\, e^{+ip\cdot x} + \zeta^c a^{c\dagger}(-\mathbf{p})\, e^{-ip\cdot x}\right). \tag{8.45}$$

It transforms simply only if the equality $\zeta^* = \zeta^c$ is imposed, in which case

$$\mathcal{T}_0\, \phi(x)\, \mathcal{T}_0^{-1} = \zeta^* \phi(T\cdot x) \tag{8.46}$$

(upon using the invariance of the measure $d\Gamma_{\mathbf{p}} = d\Gamma_{-\mathbf{p}}$).

Finally, charge conjugation will be a good symmetry if

$$[V_{\text{int}},\ \mathcal{C}_0] = 0\,. \tag{8.47}$$

Using (8.38) one finds that

$$\mathcal{C}_0\, \phi(x)\, \mathcal{C}_0^{-1} = \int d\Gamma_{\mathbf{p}} \left(\xi^*\, a^c(\mathbf{p})\, e^{-ip\cdot x} + \xi^c\, a^\dagger(\mathbf{p})\, e^{+ip\cdot x}\right) = \xi^*\, \phi^\dagger(x)\,, \tag{8.48}$$

provided $\xi^* = \xi^c$. The charge conjugation parity C of a two-particle state of a spinless particle and its antiparticle is then $+1$ in the sense that

$$\mathcal{C}_0\, a^{c\dagger}(\mathbf{p}_2)\, a^\dagger(\mathbf{p}_1)|\Omega_0\rangle = \xi^c\xi\, a^\dagger(\mathbf{p}_2)\, a^{c\dagger}(\mathbf{p}_1)|\Omega_0\rangle = a^{c\dagger}(\mathbf{p}_1)\, a^\dagger(\mathbf{p}_2)|\Omega_0\rangle\,, \tag{8.49}$$

because $\xi^c = \xi^*$ and $|\xi| = 1$ because of the unitarity of $\mathcal{C}_0$. If a spin 0 particle is its own antiparticle, $\xi^c = \xi^*$ must be equal ξ and, therefore, ξ can only be equal $+1$ or -1.

8.3 Massive spin $1/2$ particles

The little group representation $D^{(1/2)}_{\sigma'\sigma}(W)$ corresponding to massive spin 1/2 particles is the spinor representation of the rotation group (or, more precisely, the fundamental representation of its universal covering group $SU(2)$). Therefore, in agreement with the general considerations, we have to find a matrix representation $D_{lk}(\Lambda)$ of the Lorentz group which, when restricted to the rotation subgroup, has the spinor representation (i.e. the fundamental representation of the $SU(2)$ group) among its irreducible components. This means that a spinor representation of the $SO(1,3)$ group is needed. In general, spinor the representations of the $SO(n,m)$ groups (or of the $O(n,m)$ groups), i.e. the representations of the universal coverings of $SO(n,m)$, called the $Spin(n,m)$ groups (the $Pin(n,m)$ groups), can be constructed with the help of the Clifford algebras. However the group $Spin(1,3)$, the universal covering of $SO(1,3)$, is isomorphic to the $SL(2,C)$ group the representations of which can be constructed also in another way. We will present both ways, as each of them has its own virtues.

Clifford algebra construction of the Lorentz group representations

We begin with the Clifford algebra construction. Recall that for infinitesimal Lorentz transformations $\Lambda^\mu{}_\nu \approx \delta^\mu{}_\nu + \omega^\mu{}_\nu$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, the representation matrices $D(\Lambda)$ can be written as

$$D(\Lambda) \approx I - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}_{\rm spin}\,, \tag{8.50}$$

with the generator matrices $\mathcal{J}^{\mu\nu}_{\rm spin}$ satisfying the relation

$$[\mathcal{J}^{\mu\nu}_{\rm spin},\ \mathcal{J}^{\lambda\rho}_{\rm spin}] = i\left(g^{\mu\rho}\mathcal{J}^{\nu\lambda}_{\rm spin} - g^{\mu\lambda}\mathcal{J}^{\nu\rho}_{\rm spin} - g^{\nu\rho}\mathcal{J}^{\mu\lambda}_{\rm spin} + g^{\nu\lambda}\mathcal{J}^{\mu\rho}_{\rm spin}\right). \tag{8.51}$$

Our task is to construct such matrix generators for the spinor represenation of the Lorentz group (for a represenation of its covering $Spin(1,3)$). To this end we introduce the Dirac matrices γ^μ with $\mu = 0, 1, \dots, d-1$ (which although will be called "matrices", should rather, for the moment be regarded as elements of an abstract Clifford algebra over the field of complex numbers; γ^μ's play the role of the algebra generators) satisfying

$$\{\gamma^\mu,\ \gamma^\nu\} = 2g^{\mu\nu}\,. \tag{8.52}$$

It is then a purely algebraic exercise to check, that the matrices

$$\mathcal{J}^{\mu\nu}_{\rm spin} = \frac{i}{4}\left[\gamma^\mu,\ \gamma^\nu\right], \tag{8.53}$$

satisfy the relation (8.51). Helpful in this can be the relation

$$\left[\mathcal{J}^{\mu\nu}_{\rm spin},\ \gamma^\lambda\right] = -i\left(g^{\mu\lambda}\gamma^\nu - g^{\nu\lambda}\gamma^\mu\right) \equiv -\left(\mathcal{J}^{\mu\nu}_{\rm vec}\right)^\lambda{}_\rho\,\gamma^\rho\,, \tag{8.54}$$

in which $\mathcal{J}^{\mu\nu}_{\text{vec}}$ is the matrix generator (D.3) of the vector representation of the Lorentz group, which can be verified by using the rules

$$
\begin{aligned}
&[AB,\ C] = A\{B,\ C\} - \{A,\ C\}B\,, \\
&[A,\ CD] = -C\{A,\ D\} + \{A,\ C\}D\,,
\end{aligned}
\tag{8.55}
$$

which together give

$$
[AB,\ CD] = -AC\{B,\ D\} + A\{B,\ C\}D - C\{A,\ D\}B + \{A,\ C\}DB\,.
$$

As can be inferred from the rule (8.54), the matrices (the Clifford algebra generators) γ^μ behave as vectors under Lorentz transformations in the sense that

$$
D(\Lambda)\gamma^\mu D(\Lambda^{-1}) = \left(\Lambda^{-1}\right)^\mu{}_\nu\,\gamma^\nu\,. \tag{8.56}
$$

As a result, the *Spin* group generators $\mathcal{J}^{\mu\nu}_{\text{spin}}$ are antisymmetric second rank tensors:

$$
D(\Lambda)\mathcal{J}^{\mu\nu}_{\text{spin}}D(\Lambda^{-1}) = \left(\Lambda^{-1}\right)^\mu{}_\lambda\left(\Lambda^{-1}\right)^\nu{}_\rho\,\mathcal{J}^{\lambda\rho}_{\text{spin}}\,. \tag{8.57}
$$

In four dimensions ($d = 4$) the parity transformation is automatically included in this formalism: the matrix $D(P)$ representing the parity transformation $P^\mu{}_\nu$, and having the property

$$
D(P)\mathcal{J}^{\mu\nu}_{\text{spin}}D(P^{-1}) = \left(P^{-1}\right)^\mu{}_\lambda\left(P^{-1}\right)^\nu{}_\rho\,\mathcal{J}^{\lambda\rho}_{\text{spin}}\,, \tag{8.58}
$$

turns out to be (up to a phase factor) $D(P) \equiv \beta = \gamma^0$. We then have

$$
D(P)\gamma^0 D(P^{-1}) = \gamma^0\,, \qquad D(P)\gamma^i D(P^{-1}) = -\gamma^i\,, \tag{8.59}
$$

that is, $D(P)\gamma^\mu D(P^{-1}) = P^\mu{}_\nu\gamma^\nu$.

As we have seen, the objects γ^μ's behave as vectors and their antisymmetrized products (8.53) are antisymmetric second rank tensors. Other antisymmetric tensors can be constructed as well:

$$
I, \qquad \gamma^\mu, \qquad \gamma^{[\mu}\gamma^{\nu]}, \qquad \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}, \qquad \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\lambda]}, \qquad \ldots \tag{8.60}
$$

Because symmetrized products can always be eliminated by using the rule (8.52), any product of linear combinations of gammas can always be written as a linear combination (with complex coefficients) of the antisymmetric tensors (8.60). The set of tensors (8.60) is linearly independent, because they all transform differently under Lorentz transformations (which include the parity transformation) that is, when sandwiched between $D(\Lambda)$ and $D(\Lambda^{-1})$ as in (8.56), and form, therefore, a basis of the Clifford algebra treated as a vector space.

In four dimensional space-time, when $\mu = 0, 1, 2, 3$, there cannot be antisymmetric tensors of rank higher than 4. As the last antisymmetric tensor one usually takes the matrix γ^5 defined as follows:

$$\gamma^5 \equiv -\frac{i}{4!}\,\epsilon_{\mu\nu\rho\lambda}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{8.61}$$

(we use $\epsilon_{0123} = -1$). Since $\epsilon_{\mu\nu\rho\lambda}$ is a tensor invariant with respect to the proper ortochronous Lorentz transformations (in $d = 4$ space-time dimensions it changes sign under P and T transformations), it is clear that γ^5 is a Lorentz scalar with respect to proper ortochroneous transformations:

$$D(\Lambda)\gamma^5 D(\Lambda^{-1}) = \gamma^5\,, \tag{8.62}$$

and changes sign under the parity transformation

$$D(P)\gamma^5 D(P^{-1}) = \beta\gamma^5\beta = -\gamma^5\,. \tag{8.63}$$

In order to construct a spinor representation of the Lorentz group, elements of the Clifford algebra have to be represented by finite dimensional matrices. To determine the minimal dimension of these matrices for $d = 4$, we note that the number of linearly independent Clifford algebra elements forming the basis in the sense discussed above is $1+4+6+4+1 = 16$. Since there can be at most p^2 linearly independent (over the field of complex number) $p \times p$ matrices, the minimal possible dimension of matrices forming a faithful representation of the objects γ^μ is $p = 4$. That representing γ's by the 4×4 matrices, called the Dirac matrices, is indeed possible is demonstrated by writing them down explicitly:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{8.64}$$

where the blocks are 2×2 matrices. (8.64) is called *Dirac* representation of the Dirac matrices. In this representation

$$\mathcal{J}^{ij}_{\rm spin} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad \mathcal{J}^{0i}_{\rm spin} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{8.65}$$

Another useful representation is the *chiral* one (called also the Weyl representation) in which

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \tag{8.66}$$

and

$$\mathcal{J}^{ij}_{\rm spin} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad \mathcal{J}^{0i}_{\rm spin} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{8.67}$$

Incidentally, the latter representation shows explicitly that if we do not include parity as symmetry transformation, the generators (8.67) of the proper ortochroneous Lorentz transformations are fully reducible. (This will become clear also from the $SL(2,C)$ construction of the spinor representation of the Lorentz group). Infinitely many other representations can be constructed by using similarity transformations $\tilde{\gamma}^\mu = X\gamma^\mu X^{-1}$ with a nonsingular 4×4 matrix X which preserve the defining relation (8.52).

The generators $\mathcal{J}^{0i}_{\rm spin}$ of boosts in (8.65) and (8.67) are not Hermitian matrices. This reflects the fact that the Lorentz group is not compact. Instead, because of the relation

$$\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu \,, \tag{8.68}$$

valid in both representations, (8.64) and (8.66), the generators are pseudo-Hermitian

$$\gamma^0\mathcal{J}^{\mu\nu\dagger}_{\rm spin}\gamma^0 = \mathcal{J}^{\mu\nu}_{\rm spin} \,, \tag{8.69}$$

which leads to the pseudounitarity of the spinor representation

$$\gamma^0 D^\dagger(\Lambda)\,\gamma^0 = D^{-1}(\Lambda)\,. \tag{8.70}$$

We now have to solve[9] the conditions (8.13)

$$\begin{aligned}
\sum_{\sigma'} u_\alpha(\mathbf{0},\sigma')\,\mathbf{J}^{(1/2)}_{\sigma'\sigma} &= \sum_\beta \boldsymbol{\mathcal{J}}^{\rm spin}_{\alpha\beta}\,u_\beta(\mathbf{0},\sigma)\,,\\
-\sum_{\sigma'} v_\alpha(\mathbf{0},\sigma')\,\mathbf{J}^{(1/2)*}_{\sigma'\sigma} &= \sum_\beta \boldsymbol{\mathcal{J}}^{\rm spin}_{\alpha\beta}\,v_\beta(\mathbf{0},\sigma)\,.
\end{aligned} \tag{8.71}$$

Here $\mathbf{J}^{(1/2)}$ are simply the familiar three Pauli matrices divided by 2 and $\mathcal{J}^i_{\rm spin} = \frac{1}{2}\epsilon^{ijk}\mathcal{J}^{jk}_{\rm spin}$. As discussed, the functions $u_\alpha(\mathbf{p},\sigma)$ and $v_\alpha(\mathbf{p},\sigma)$ with a nonzero three-momentum argument will be obtained by applying the matrix representation of the standard Lorentz transformation to $u_\alpha(\mathbf{0},\sigma)$ and $v_\alpha(\mathbf{0},\sigma)$, respectively:

$$\begin{aligned}
u_\alpha(\mathbf{p},\sigma) &= \sum_\beta D_{\alpha\beta}(L_p)\,u_\alpha(\mathbf{0},\sigma)\,,\\
v_\alpha(\mathbf{p},\sigma) &= \sum_\beta D_{\alpha\beta}(L_p)\,v_\alpha(\mathbf{0},\sigma)\,.
\end{aligned} \tag{8.72}$$

Writing the 4-component spinors $u_\alpha(\mathbf{0},\sigma)$ and $v_\alpha(\mathbf{0},\sigma)$ as

$$u_\alpha(\mathbf{0},\sigma) = \begin{pmatrix} u^+_{l\sigma}\\ u^-_{l\sigma}\end{pmatrix}, \qquad v_\alpha(\mathbf{0},\sigma) = \begin{pmatrix} v^+_{l\sigma}\\ v^-_{l\sigma}\end{pmatrix}, \qquad l=1,2 \tag{8.73}$$

(where $u^\pm_{l\sigma} \equiv u^\pm_l(\mathbf{0},\sigma)$), that is, splitting the 4-valued index α into two 2-valued ones: $\alpha = (\pm, l)$ with $l = 1,2$ (as suggested by the block-diagonal structure of the generators

[9] To respect the traditional notation for spinors, the indices lk of the matrices $D(\Lambda)$ are now renamed to $\alpha\beta$.

(8.65) and (8.67) of the generators $\mathcal{J}^{ij}_{\rm spin}$) we rewrite the first of the conditions (8.71) in the form

$$\sum_{\sigma'} u^{\pm}_{l\sigma'}\frac{1}{2}\boldsymbol{\sigma}_{\sigma'\sigma} = \sum_k \frac{1}{2}\boldsymbol{\sigma}_{lk}\, u^{\pm}_{k\sigma}\,. \tag{8.74}$$

From the Schur's lemma it then follows that the functions $u^{\pm}_{l\sigma}$ treated as 2×2 matrices in their indices $l\sigma$ must be either proportional to the 2×2 unit matrices or (the case we reject) must be zero matrices.[10] Thus, $u_\alpha(\mathbf{0},\sigma)$ must have the general form

$$u_\alpha(\mathbf{0},\sigma=\tfrac{1}{2}) = \begin{pmatrix} c_+\\ 0\\ c_-\\ 0\end{pmatrix}, \qquad u_\alpha(\mathbf{0},\sigma=-\tfrac{1}{2}) = \begin{pmatrix} 0\\ c_+\\ 0\\ c_-\end{pmatrix}, \tag{8.75}$$

with as yet unspecified constants $c_\pm$.

Similarly, the condition (8.71) on $v_\alpha(\mathbf{0},\sigma)$ takes the form

$$\sum_{\sigma'} v^{\pm}_{l\sigma'}\frac{1}{2}\left(\sigma^y\boldsymbol{\sigma}\sigma^y\right)_{\sigma'\sigma} = \sum_k \frac{1}{2}\boldsymbol{\sigma}_{lk}\, v^{\pm}_{k\sigma} \tag{8.76}$$

(where we have used the relation $-\boldsymbol{\sigma}^* = \sigma^y\boldsymbol{\sigma}\sigma^y$) or, equivalently,

$$\sum_{\sigma'} \left(v^{\pm}\sigma^y\right)_{l\sigma'}\frac{1}{2}\boldsymbol{\sigma}_{\sigma'\sigma} = \sum_k \frac{1}{2}\boldsymbol{\sigma}_{lk}\left(v^{\pm}\sigma^y\right)_{k\sigma}\,. \tag{8.77}$$

Again by the Schur's lemma $v^{\pm}\sigma^y \propto I$ and $v_\alpha(\mathbf{0},\sigma)$ can be taken in the form

$$v_\alpha(\mathbf{0},\sigma=\tfrac{1}{2}) = \begin{pmatrix} 0\\ d_+\\ 0\\ d_-\end{pmatrix}, \qquad v_\alpha(\mathbf{0},\sigma=-\tfrac{1}{2}) = \begin{pmatrix} -d_+\\ 0\\ -d_-\\ 0\end{pmatrix}, \tag{8.78}$$

with arbitrary $d_\pm$.

The relation between c_+ and c_- (d_+ and d_-) is fixed by requiring that the field operators $\psi(x)$ and $\psi^\dagger(x)$ transform under parity in a simple way (otherwise it would be difficult to construct interactions invariant under the parity operation). We will assume that the considered spin 1/2 particles, like e.g. electrons, do carry a conserved charge (the case of the neutral spin 1/2 particle will be considered below) and that there are, therefore, also the corresponding antiparticles. As it is customary, the annihilation and

[10]This is because from (8.74) it follows that the matrices $u^{\pm}_{l\sigma}$ commute with all generators of the irreducible representation of the $SU(2)$ group.

creation operators of the particles will be denoted $b(\mathbf{p},\sigma)$ and $b^\dagger(\mathbf{p},\sigma)$, while those of the corresponding antiparticles - $d(\mathbf{p},\sigma)$ and $d^\dagger(\mathbf{p},\sigma)$. These operators satisfy the relations

$$\begin{aligned} \mathcal{P}_0\, b_\sigma(\mathbf{p})\, \mathcal{P}_0^{-1} &= \eta^*\, b_\sigma(-\mathbf{p})\,, \\ \mathcal{P}_0\, d^\dagger_\sigma(\mathbf{p})\, \mathcal{P}_0^{-1} &= \eta^c\, d^\dagger_\sigma(-\mathbf{p})\,. \end{aligned} \tag{8.79}$$

Acting with the parity operator on the two parts $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$, containing respectively $b_\sigma(\mathbf{p})$ and $d^\dagger_\sigma(\mathbf{p})$, of the spinor field operator we obtain[11]

$$\begin{aligned} \mathcal{P}_0\, \psi^{(+)}_\alpha(x)\, \mathcal{P}_0^{-1} &= \eta^* \int d\Gamma_{\mathbf{p}} \sum_\sigma u_\alpha(-\mathbf{p},\sigma)\, e^{-ip\cdot(P\cdot x)}\, b_\sigma(\mathbf{p})\,, \\ \mathcal{P}_0\, \psi^{(-)}_\alpha(x)\, \mathcal{P}_0^{-1} &= \eta^c \int d\Gamma_{\mathbf{p}} \sum_\sigma v_\alpha(-\mathbf{p},\sigma)\, e^{+ip\cdot(P\cdot x)}\, d^\dagger_\sigma(\mathbf{p})\,, \end{aligned} \tag{8.80}$$

where we have already changed the integration variables $\mathbf{p} \to -\mathbf{p}$ and used $d\Gamma_{-\mathbf{p}} = d\Gamma_{\mathbf{p}}$. The functions $u(-\mathbf{p},\sigma)$ and $v(-\mathbf{p},\sigma)$ are given by[12]

$$\begin{aligned} u_\alpha(-\mathbf{p},\sigma) &= D_{\alpha\beta}(L_{P\cdot p})\, u_\beta(\mathbf{0},\sigma) = (\beta D(L_p)\beta)_{\alpha\beta}\, u_\beta(\mathbf{0},\sigma)\,, \\ v_\alpha(-\mathbf{p},\sigma) &= D_{\alpha\beta}(L_{P\cdot p})\, v_\beta(\mathbf{0},\sigma) = (\beta D(L_p)\beta)_{\alpha\beta}\, v_\beta(\mathbf{0},\sigma)\,. \end{aligned} \tag{8.81}$$

To enable constructing parity preserving interactions $V^I_{\rm int}(t)$, i.e. satisfying (8.39), the spin 1/2 particle field operators should themselves transform simply: $\mathcal{P}\,\psi^{(\pm)}(x)\,\mathcal{P}^{-1} \propto \psi^{(\pm)}(P\cdot x)$. This requires that

$$\begin{aligned} \beta\, u(\mathbf{0},\sigma) &\equiv \gamma^0\, u(\mathbf{0},\sigma = b_u\, u(\mathbf{0},\sigma)\,, \\ \beta\, v(\mathbf{0},\sigma) &\equiv \gamma^0\, v(\mathbf{0},\sigma) = b_v\, v(\mathbf{0},\sigma)\,, \end{aligned} \tag{8.82}$$

with some constants $b_u = \pm 1$ and $b_v = \pm 1$ (becausee $\beta^2 = 1$). If the relations (8.82) hold, then

$$\begin{aligned} \mathcal{P}_0\, \psi^{(+)}(x)\, \mathcal{P}_0^{-1} &= \eta^*\, b_u\, \beta\, \psi^{(+)}(P\cdot x)\,, \\ \mathcal{P}_0\, \psi^{(-)}(x)\, \mathcal{P}_0^{-1} &= \eta^c\, b_v\, \beta\, \psi^{(-)}(P\cdot x)\,. \end{aligned} \tag{8.83}$$

To proceed further one must specify the representation of the gamma matrices. Let us take the chiral representation (8.66) first. The matrix $\beta = \gamma^0$ is then off-diagonal and in order to satisfy (8.82) we must set

$$c_- = b_u\, c_+\,, \qquad\qquad d_- = b_v\, d_+\,, \tag{8.84}$$

[11] Here from the beginning we denote the operator containing $d^\dagger_\sigma(\mathbf{p})$ by $\psi^{(-)}$ instead of using for it first $\psi^{c(-)}$ and only afterwards renaming it to $\psi^{(-)}$, as we did discussing scalar field operators of spinless particles.

[12] Using the explicit form (6.46) of L_p it is easy to check that $L_{P\cdot p} = P\cdot L_p\cdot P^{-1}$. Hence, $D(L_{P\cdot p}) = D(P)D(L_p)D(P^{-1}) = \beta D(L_p)\beta$ where we have used $D(P) = \beta \equiv \gamma^0$.

in (8.75) and (8.78), respectively. We chose $b_u = 1 = -b_v$ (as will be seen shortly, local causality requires $b_v = -b_u$) and fix the normalization of the spinors u and v by choosing[13]

$$c_+ = -d_+ = \sqrt{m}\,. \tag{8.85}$$

In the Dirac representation (8.64) of the gamma matrices we find instead that for $b_u = +1$ the conditions (8.82) is satisfied if

$$c_+ = \sqrt{2m}\,, \qquad\qquad c_- = 0\,, \tag{8.86}$$

(with $b_u = -1$ one would have to set $c_+ = 0$, $c_- = \sqrt{2m}$) and for $b_v = -1$ (8.82) requires

$$d_+ = 0\,, \qquad\qquad d_- = \sqrt{2m}\,, \tag{8.87}$$

(for $b_v = +1$ one would have $d_+ = \sqrt{2m}$, $d_- = 0$). The explicit form of these functions is given in Appendix F.

With the adopted normalizations, in both representations the spinors satisfy the sum rules

$$u^\dagger(\mathbf{0},\sigma)u(\mathbf{0},\sigma') = v^\dagger(\mathbf{0},\sigma)v(\mathbf{0},\sigma') = 2m\,\delta_{\sigma\sigma'} \tag{8.88}$$

and, as it is easy to verify, also the sum rules

$$\begin{aligned}\sum_\sigma u_\alpha(\mathbf{0},\sigma)u^*_\beta(\mathbf{0},\sigma) &= 2m\left(\frac{1+b_u\beta}{2}\right)_{\alpha\beta}\,,\\ \sum_\sigma v_\alpha(\mathbf{0},\sigma)v^*_\beta(\mathbf{0},\sigma) &= 2m\left(\frac{1+b_v\beta}{2}\right)_{\alpha\beta}\,.\end{aligned} \tag{8.89}$$

To satisfy the requirement of the local causality we form the linear combination

$$\psi_\alpha(x) = \kappa_+\psi^{(+)}_\alpha(x) + \kappa_-\psi^{(-)}_\alpha(x)\,, \tag{8.90}$$

of the field operators constructed for the spin 1/2 particles carrying some conserved charges and compute

$$\begin{aligned}[\psi_\alpha(x)\ ,\psi^\dagger_\beta(y)]_\pm = |\kappa_+|^2 &\int d\Gamma_{\mathbf{p}}\sum_\sigma u_\alpha(\mathbf{p},\sigma)\,u^*_\beta(\mathbf{p},\sigma)\,e^{-ip\cdot(x-y)}\\ \pm|\kappa_-|^2 &\int d\Gamma_{\mathbf{p}}\sum_\sigma v_\alpha(\mathbf{p},\sigma)\,v^*_\beta(\mathbf{p},\sigma)\,e^{+ip\cdot(x-y)}\,,\end{aligned} \tag{8.91}$$

[13]We chose the sign convention which removes the phase factors from the action of the spinor C-matrix, see Appendix F. Choosing real zero momentum forms of the spinors $u^\dagger(\mathbf{0},\sigma)$ and $v^\dagger(\mathbf{0},\sigma)$ simplifies enormously further manipulations involving these functions.

leaving again initially undecided whether the annihilation and creation operators of the spin 1/2 particles are fermionic (+) or bosonic (−). In order to find the sums $\sum_\sigma u_\alpha u^*_\beta$ and $\sum_\sigma v_\alpha v^*_\beta$ over the spin states for nonzero $\mathbf{p}$ we write using (8.72) and (8.89):

$$\sum_\sigma u(\mathbf{p},\sigma)u^*(\mathbf{p},\sigma) = D(L_p)\sum_\sigma u(\mathbf{0},\sigma)u^*(\mathbf{0},\sigma)D^\dagger(L_p)$$
$$= m\,D(L_p)(1+b_u\beta)D^\dagger(L_p)\,.$$

Next, recalling (8.70) and (8.56), we note that

$$\begin{aligned} D(L_p)D^\dagger(L_p) &= D(L_p)\gamma^0 D^{-1}(L_p)\gamma^0 = (L_p^{-1})^0{}_\mu\gamma^\mu\beta \\ &= \gamma^\mu\,(L_p)_\mu{}^0\beta = \frac{\gamma^\mu p_\mu}{m}\,\beta\,. \end{aligned} \tag{8.92}$$

We have used here the explicit form (6.46)

$$(L_p)^0{}_0 = \frac{E_\mathbf{p}}{m}\,, \qquad (L_p)^i{}_0 = \frac{p^i}{m}\,, \tag{8.93}$$

of the standard Lorentz transformation L_p. Furthermore, using the equality

$$D(L_p)\beta D^\dagger(L_p) = D(L_p)\beta\beta D^{-1}(L_p)\beta = \beta\,, \tag{8.94}$$

we finally obtain the sum rules

$$\begin{aligned} \sum_\sigma u(\mathbf{p},\sigma)u^*(\mathbf{p},\sigma) &= (\not{p} + m\,b_u)\beta\,, \\ \sum_\sigma v(\mathbf{p},\sigma)v^*(\mathbf{p},\sigma) &= (\not{p} + m\,b_v)\beta\,, \end{aligned} \tag{8.95}$$

in which we have used the famous Feynman's slash notation:

$$\not{p} \equiv \gamma^\mu p_\mu\,. \tag{8.96}$$

The result (8.91) can be now rewritten as

$$\begin{aligned} [\psi_\alpha(x)\;,\psi^\dagger_\beta(y)]_\pm &= |\kappa_+|^2\;(i\,\not{\partial}^x + b_u m)\,\beta\int d\Gamma_\mathbf{p}\,e^{-ip\cdot(x-y)} \\ &\quad \pm|\kappa_-|^2\;(-i\,\not{\partial}^x + b_v m)\,\beta\int d\Gamma_\mathbf{p}\,e^{+ip\cdot(x-y)} \\ &= |\kappa_+|^2(i\,\not{\partial}^x + b_u\,m)\,\beta\,\Delta_+(x-y) \pm |\kappa_-|^2(-i\,\not{\partial}^x + b_v\,m)\,\beta\,\Delta_+(y-x)\,. \end{aligned} \tag{8.97}$$

Since $\Delta_+(x-y) = \Delta_+(y-x)$ for $(x-y)^2 < 0$, it is clear that in order to get zero for space-like separations of the points x and y one must set $|\kappa_+| = |\kappa_-|$. As for scalar particles one can take $\kappa_+ = \kappa_- = 1$, because any other choice of the relative phase can be reduced to this one by changing the phases of the relevant one-particle states. Moreover, the creation and annihilation operators of spin 1/2 particles must necessarily be fermionic (+) and, in addition, the (anticipated) equality $b_u = -b_v$ is necessary. Returning now to the requirement that the spin 1/2 particle field operator $\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x)$ transforms simply under space reflection, we see from (8.83) that it enforces the relation $\eta^c = -\eta^*$. This means, that spin the 1/2 particle and its antiparticle must (if η is real) have opposite intrinsic parities (note however, that this holds only for massive spin 1/2 particles and needs not be true for *massless* fermions). Without loss of generality we will take[14] $b_u = +1$ and $b_v = -1$. Thus, again the requirement of Lorentz invariance (the local causality condition (8.16)) fixes the spin-statistics connection.

Important (also from the point of view of practical calculations) relations satisfied by the functions $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ can be derived in the following way. Starting from (8.82) with $b_u = +1$ one can write

$$D(L_p)\, \beta\, D^{-1}(L_p) D(L_p)\, u(\mathbf{0}, \sigma) = D(L_p)\, u(\mathbf{0}, \sigma)\,. \tag{8.98}$$

Upon applying the rules (8.72) this can be transformed into

$$D(L_p)\, \beta\, D^{-1}(L_p)\, u(\mathbf{p}, \sigma) = u(\mathbf{p}, \sigma)\,. \tag{8.99}$$

With the help of (8.92) multiplied from the right by $\beta = \gamma^0$ we arrive at

$$\frac{\not{p}}{m}\, u(\mathbf{p}, \sigma) = u(\mathbf{p}, \sigma)\,, \qquad \frac{\not{p}}{m}\, v(\mathbf{p}, \sigma) = -v(\mathbf{p}, \sigma)\,, \tag{8.100}$$

where the second relation is obtained in a similar manner starting from (8.82) with $b_v = -1$. Thus, the spinors $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ satisfy the (momentum space) "Dirac equations"

$$\begin{aligned} (\not{p} - m)\, u(\mathbf{p}, \sigma) &= 0\,, \\ (\not{p} + m)\, v(\mathbf{p}, \sigma) &= 0\,. \end{aligned} \tag{8.101}$$

Since the spinor representation of the Lorentz group (or, more precisely, of the *Pin* group) is not unitary, the operator $\sum_\alpha \psi^\dagger_\alpha(x)\psi_\alpha(x)$ is not a Lorentz scalar. Instead, the Lorentz scalar is the operator

$$\sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \psi^\dagger_\alpha(x) (\gamma^0)_{\alpha\beta}\, \psi_\beta(x) \equiv \psi^\dagger(x)\gamma^0\psi(x)\,. \tag{8.102}$$

Similarly, objects of the form

$$\sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \psi^\dagger_\alpha(x) (\gamma^0\gamma^{\mu_1} \ldots \gamma^{\mu_r})_{\alpha\beta}\, \psi_\beta(x) \equiv \psi^\dagger(x)\gamma^0\gamma^{\mu_1} \ldots \gamma^{\mu_r}\psi(x)\,. \tag{8.103}$$

[14] The choice $b_u = -1$ and $b_v = +1$ is equivalent to using $\gamma^5\psi(x)$ as the field operator instead of $\psi(x)$; this is also a consistent possibility but we will follow the standard convention here.

transform as Lorentz tensors of the rank r. Because in objects which transform regularily the matrix γ^0 frequently stands to the right of $\psi^\dagger$ it is convenient to introduce the barred field operators and spinors:

$$\begin{aligned}\bar{\psi}_\alpha(x) &\equiv \sum_{\beta=1}^{4} \psi^\dagger_\beta(\gamma^0)_{\beta\alpha}\,,\\ \bar{u}_\alpha(\mathbf{p},\sigma) &\equiv \sum_{\beta=1}^{4} u^*_\beta(\mathbf{p},\sigma)(\gamma^0)_{\beta\alpha}\,,\\ \bar{v}_\alpha(\mathbf{p},\sigma) &\equiv \sum_{\beta=1}^{4} v^*_\beta(\mathbf{p},\sigma)(\gamma^0)_{\beta\alpha}\,.\end{aligned} \tag{8.104}$$

From (8.95) we obtain therefore two useful relations (spin sum rules)

$$\begin{aligned}\sum_\sigma u_\alpha(\mathbf{p},\sigma)\,\bar{u}_\beta(\mathbf{p},\sigma) &= (\not{p}+m)_{\alpha\beta}\,,\\ \sum_\sigma v_\alpha(\mathbf{p},\sigma)\,\bar{v}_\beta(\mathbf{p},\sigma) &= (\not{p}-m)_{\alpha\beta}\,.\end{aligned} \tag{8.105}$$

It is then easy to see that the spin 1/2 particle field operators

$$\begin{aligned}\psi_\alpha(x) &= \int d\Gamma_{\mathbf{p}} \sum_\sigma \left[u_\alpha(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,b_\sigma(\mathbf{p}) + v_\alpha(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,d^\dagger_\sigma(\mathbf{p})\right],\\ \bar{\psi}_\alpha(x) &= \int d\Gamma_{\mathbf{p}} \sum_\sigma \left[\bar{v}_\alpha(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,d_\sigma(\mathbf{p}) + \bar{u}_\alpha(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,b^\dagger_\sigma(\mathbf{p})\right],\end{aligned} \tag{8.106}$$

in addition to the equations $(\partial^\mu\partial_\mu + m^2)\psi_\alpha(x) = 0$ and $(\partial^\mu\partial_\mu + m^2)\bar{\psi}_\alpha(x) = 0$ satisfied by each of their components separately (as follows simply from the mass-shell condition $p^2 = m^2$), satisfy also the first order equations

$$\begin{aligned}(i\not{\partial} - m)\psi(x) &= 0\,,\\ \bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m) &= 0\,,\end{aligned} \tag{8.107}$$

(the symbol $\overleftarrow{\partial}$ means that the derivative acts on the object standing to its left), which are the famous Dirac (free) equations in the covariant form. Here, however, $\psi(x)$ and $\bar{\psi}(x)$ are quantum field operators and not wave functions.

Finally, using the relations

$$\begin{aligned}\bar{u}(\mathbf{p},\sigma')\gamma^0 u(\mathbf{p},\sigma) = \bar{v}(\mathbf{p},\sigma')\gamma^0 v(\mathbf{p},\sigma) &= 2E_{\mathbf{p}}\,\delta_{\sigma'\sigma}\,,\\ \bar{u}(-\mathbf{p},\sigma')\gamma^0 u(\mathbf{p},\sigma) &= 0\,,\end{aligned} \tag{8.108}$$

it is easy to check that in terms of the field operators $\psi(x)$ and $\bar{\psi}(x)$ the creation and annihilation operators can be expressed through the integrals

$$\begin{aligned}
b_\sigma(\mathbf{p}) &= \int d^3\mathbf{x}\,\bar{u}(\mathbf{p},\sigma)\,e^{ip\cdot x}\,\gamma^0\psi(x)\,,\\
d^\dagger_\sigma(\mathbf{p}) &= \int d^3\mathbf{x}\,\bar{v}(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,\gamma^0\psi(x)\,,\\
b^\dagger_\sigma(\mathbf{p}) &= \int d^3\mathbf{x}\,\bar{\psi}(x)\gamma^0 u(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,,\\
d_\sigma(\mathbf{p}) &= \int d^3\mathbf{x}\,\bar{\psi}(x)\gamma^0 v(\mathbf{p},\sigma)\,e^{ip\cdot x}\,.
\end{aligned}\tag{8.109}$$

P, T and C action on the field operators of massive charged spin $1/2$ *particles*

We now consider the transformation properties under the discrete symmetries P, T and C of the (four-component) field operators of spin 1/2 particles which are not their own antiparticles.

As far as parity is concerned, the operator $\psi(x)$ has been constructed in such a way that

$$\begin{aligned}
\mathcal{P}_0\,\psi(x)\,\mathcal{P}_0^{-1} &= \eta^*\,\beta\,\psi(P\cdot x)\,,\\
\mathcal{P}_0\,\bar{\psi}(x)\,\mathcal{P}_0^{-1} &= \eta\,\bar{\psi}(P\cdot x)\,\beta\,.
\end{aligned}\tag{8.110}$$

(The second relation follows just by taking the Hermitian conjugate of the first one and the fact that $\mathcal{P}_0^{-1} = \mathcal{P}_0^\dagger$).

Acting using (6.90) with the (antiunitary!) time reversal operator $\mathcal{T}_0$ on $\psi(x)$ one gets

$$\begin{aligned}
\mathcal{T}_0\,\psi(x)\,\mathcal{T}_0^{-1} &= \int d\Gamma_{\mathbf{p}}\sum_\sigma\Big[u^*(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,\zeta^*(-1)^{\frac{1}{2}-\sigma}b_{-\sigma}(-\mathbf{p})\\
&\qquad\qquad + v^*(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,\zeta^c(-1)^{\frac{1}{2}-\sigma}d^\dagger_{-\sigma}(-\mathbf{p})\Big]\,,\\
&= \int d\Gamma_{\mathbf{p}}\sum_\sigma\Big[\zeta^*(-1)^{\frac{1}{2}+\sigma}u^*(-\mathbf{p},-\sigma)e^{-ip\cdot(T\cdot x)}\,b_\sigma(\mathbf{p})\\
&\qquad\qquad + \zeta^c(-1)^{\frac{1}{2}+\sigma}v^*(-\mathbf{p},-\sigma)\,e^{+ip\cdot(T\cdot x)}\,d^\dagger_\sigma(\mathbf{p})\Big]\,.
\end{aligned}\tag{8.111}$$

One therefore needs to express $u^*(-\mathbf{p},-\sigma)$ and $v^*(-\mathbf{p},-\sigma)$ through $u(\mathbf{p},\sigma)$ and $v(\mathbf{p},\sigma)$. Using (F.12) and the so-called charge-conjugation matrix C given by (F.8) (it is defined by the relations (F.6)) we can write

$$\begin{aligned}
u^*(-\mathbf{p},-\sigma) &= D^*(L_{P\cdot p})\,u^*(\mathbf{0},-\sigma) = D^*(L_{P\cdot p})\,(-1)^{\frac{1}{2}-\sigma}\gamma^5 C\,u(\mathbf{0},\sigma)\,,\\
v^*(-\mathbf{p},-\sigma) &= D^*(L_{P\cdot p})\,v^*(\mathbf{0},-\sigma) = D^*(L_{P\cdot p})\,(-1)^{\frac{1}{2}-\sigma}\gamma^5 C\,v(\mathbf{0},\sigma)\,.
\end{aligned}\tag{8.112}$$

Next, (F.14) and the relations (8.81) give

$$\begin{aligned} D^*(L_{P\cdot p}) &= C\gamma^0\, D(L_{P\cdot p})\, C\gamma^0 = C\gamma^0\beta\, D(L_p)\,\beta C\gamma^0 = C\, D(L_p)\, C^{-1} \\ &= C\gamma^5\gamma^5\, D(L_p)\, C^{-1} = C\gamma^5\, D(L_p)\,\gamma^5 C^{-1} \\ &= C\gamma^5\, D(L_p)\,(C\gamma^5)^{-1} = (\gamma^5 C)\, D(L_p)\,(\gamma^5 C)^{-1}\,. \end{aligned} \tag{8.113}$$

Therefore we finally obtain from (8.112) the relations

$$\begin{aligned} u^*(-\mathbf{p},-\sigma) &= (-1)^{\frac{1}{2}-\sigma}\gamma^5 C\, u(\mathbf{p},\sigma)\,, \\ v^*(-\mathbf{p},-\sigma) &= (-1)^{\frac{1}{2}-\sigma}\gamma^5 C\, v(\mathbf{p},\sigma)\,, \end{aligned} \tag{8.114}$$

which for $\zeta^* = \zeta^c$ in (8.111) lead to the rules

$$\begin{aligned} \mathcal{T}_0\,\psi(x)\,\mathcal{T}_0^{-1} &= -\zeta^*\,\gamma^5 C\,\psi(T\!\cdot\! x) = \zeta^*\gamma^1\gamma^3\,\psi(T\!\cdot\! x)\,, \\ \mathcal{T}_0\,\bar\psi(x)\,\mathcal{T}_0^{-1} &= -\zeta\,\bar\psi(T\!\cdot\! x)\, C^\dagger\gamma^5 = \zeta\,\bar\psi(T\!\cdot\! x)\,\gamma^3\gamma^1\,. \end{aligned} \tag{8.115}$$

Action of the charge conjugation operator on the one-particle states is given by (6.97). It implies that

$$\mathcal{C}_0\,\psi(x)\,\mathcal{C}_0^{-1} = \int d\Gamma_{\mathbf{p}}\sum_\sigma \left[u(\mathbf{p},\sigma)\, e^{-ip\cdot x}\,\xi^*\, d_\sigma(\mathbf{p}) + v(\mathbf{p},\sigma)\, e^{+ip\cdot x}\,\xi^c\, b^\dagger_\sigma(\mathbf{p})\right]. \tag{8.116}$$

Setting $\xi^c = \xi^*$ and using (F.16) we get therefore

$$\begin{aligned} \mathcal{C}_0\,\psi(x)\,\mathcal{C}_0^{-1} &= \xi^* C\gamma^0\,\psi^\dagger(x) \equiv \xi^*\,\psi^c(x)\,, \\ \mathcal{C}_0\,\psi^\dagger(x)\,\mathcal{C}_0^{-1} &= \xi\ C\gamma^0\,\psi(x)\,, \end{aligned} \tag{8.117}$$

(the dagger here is to be interpreted as the Hermitian conjugation of operators and complex conjugation of c-number factors). Using the symmetry of $C\gamma^0$ in its spinor indices (see (F.8)) the rules (8.117) can be cast into the more convenient form

$$\begin{aligned} \mathcal{C}_0\,\psi(x)\,\mathcal{C}_0^{-1} &= -\xi^*\,\bar\psi(x)\, C\,, \\ \mathcal{C}_0\,\bar\psi(x)\,\mathcal{C}_0^{-1} &= \xi\, C^{-1}\,\psi(x)\,. \end{aligned} \tag{8.118}$$

As an application of these formulae we determine the charge conjugation properties of the "current" operator $J^\mu = \bar\psi\gamma^\mu\psi$:

$$\mathcal{C}_0\, J^\mu\,\mathcal{C}_0^{-1} = (C^{-1}\psi)_\alpha(\gamma^\mu)_{\alpha\beta}(-\bar\psi(x)C)_\beta = \bar\psi C(\gamma^\mu)^T C^{-1}\psi = -J^\mu\,. \tag{8.119}$$

We have used the relation $\psi_\alpha\ \bar\psi_\beta = -\bar\psi_\beta\ \psi_\alpha$ (the term arising from the anticommutator $\{\psi(x),\ \bar\psi(x)\}$, formally infinite, is proportional to $(\gamma^\mu)_{\alpha\alpha} = 0$). This shows that in quantum electrodynamics in which the interaction takes the form $\mathcal{H}_{\rm int} = eA_\mu J^\mu_{\rm EM}$ (with being A_μ the photon field operator to be constructed in Section 8.5 and $e > 0$ the electric charge), the photon C-parity is -1, i.e. we must have $\mathcal{C}_0\, A_\mu\,\mathcal{C}_0^{-1} = -A_\mu$ if charge conjugation is to be a symmetry of the electromagnetic interactions.

Neutral massive spin 1/2 particle

So far we have discussed the properties of the free field operator of spin 1/2 particles that are not their own antiparticles (there exist their antiparticles). Another possibility is a neutral spin 1/2 particle which is its own antiparticle. Such particles are called Majorana fermions. The corresponding field operator has the form

$$\psi(x) = \kappa_+ \psi^{(+)}(x) + \kappa_- \psi^{(-)}(x)\,, \tag{8.120}$$

with

$$\begin{aligned}\psi^{(+)}(x) &= \int d\Gamma_{\mathbf{p}} \sum_\sigma u(\mathbf{p},\sigma)\, e^{-ip\cdot x}\, b_\sigma(\mathbf{p})\,,\\ \psi^{(-)}(x) &= \int d\Gamma_{\mathbf{p}} \sum_\sigma v(\mathbf{p},\sigma)\, e^{+ip\cdot x}\, b^\dagger_\sigma(\mathbf{p})\,,\end{aligned} \tag{8.121}$$

where the spinors $u(\mathbf{p},\sigma)$ and $v(\mathbf{p},\sigma)$ have the same form as in the case of a charged spin 1/2 particle particle. Proceeding as in that case, it can be shown that all the three types of (anti)commutators: $[\psi(x),\ \psi^\dagger(y)]_\pm$, $[\psi(x),\ \psi(y)]_\pm$ and $[\psi^\dagger(x),\ \psi^\dagger(y)]_\pm$ vanish for $(x-y)^2 < 0$, provided neutral spin 1/2 particles are fermions, $|\kappa_+| = |\kappa_-|$ and $b_u = -b_v$. Moreover, intrinsic parities η of such a particles must be purely imaginary: $\eta = \pm i$. the factors κ_+ and κ_- introduced as in (8.90) can be both set to unity by appropriately redefining the phase of the physical one particle state. One can then take $\kappa_+ = \kappa_- = 1$. However, in applications to mixing of neutrinos it proves convenient to determine the relative phase between κ_+ and κ_- so that the mixing matrix is real if CP is a good symmetry. One then choses $\kappa_+ = 1$ and $\kappa_- = \lambda \neq 1$ ($\lambda = e^{i\chi}$ is called the creation phase factor). In this case the Majorana particle free field operator satisfies the simple relation

$$\mathcal{C}_0\, \psi\, \mathcal{C}_0^{-1} = \xi^*\, \psi\,, \tag{8.122}$$

in addition to the standard relation (valid also for Dirac fermions with $\lambda = 1$ - c.f. (8.117))

$$\mathcal{C}_0\, \psi\, \mathcal{C}_0^{-1} = \xi^* \lambda\ C\gamma^0\, \psi^\dagger, \tag{8.123}$$

where $\xi^* = \xi = \pm 1$ is the intrinsic C-parity of the Majorana particle.

The Majorana particle field operator $\psi(x)$ satisfies the equation

$$i\,\partial\!\!\!/\psi - m\, C\,\overline{\psi}^T = 0\,, \tag{8.124}$$

and its transformation properties allow to construct two additional with respect to (8.102) Lorentz scalars (they are Hermitian conjugate with respect to one another):

$$\psi^T C^{-1} \psi\,, \qquad \text{and} \qquad \bar{\psi}\, C\, \bar{\psi}^T. \tag{8.125}$$

Also in this case the operators $b_\sigma(\mathbf{p})$ and $b^\dagger_\sigma(\mathbf{p})$ can both be expressed either in terms of $\psi(x)$ or in terms of $\bar{\psi}(x)$ with the help of the formulae similar to (8.109).

8.4 Massive spin 1 particles

We now discuss the case of vector fields $V^{(+)\mu}(x)$, $V^{(-)\nu}(x)$. The matrices $D_{lk}(\Lambda)$ in (8.3) are then simply equal to $\Lambda^\mu{}_\nu$, and we ask what kind of massive particles such fields can be associated with. The basic conditions (8.13) now read

$$\begin{aligned}\sum_{\sigma'} u^\mu(\mathbf{0},\sigma')\,\mathbf{J}^{(s)}_{\sigma'\sigma} &= \sum_\nu (\boldsymbol{\mathcal{J}}_{\rm vec})^\mu{}_\nu\, u^\nu(\mathbf{0},\sigma)\,,\\ -\sum_{\sigma'} v^\mu(\mathbf{0},\sigma')\,\mathbf{J}^{(s)*}_{\sigma'\sigma} &= \sum_\nu (\boldsymbol{\mathcal{J}}_{\rm vec})^\mu{}_\nu\, v^\nu(\mathbf{0},\sigma)\,.\end{aligned} \tag{8.126}$$

We expect that s on the left hand side can be equal either 1 or 0 because the vector representation $\Lambda^\mu{}_\nu$ of the Lorentz group, when restricted to rotations decomposes into two irreducible representations of the rotation group: the vector and the scalar ones. To see this more formally, let us recall that in the vector representation of the Lorentz group the generators $\mathcal{J}^{\lambda\rho}_{\rm vec}$ take the form (D.3)

$$\left(\mathcal{J}^{\lambda\rho}_{\rm vec}\right)^\mu{}_\nu = i\left(g^{\lambda\mu}g^\rho{}_\nu - g^\lambda{}_\nu\, g^{\rho\mu}\right), \tag{8.127}$$

from which it follows that the three matrices $(\boldsymbol{\mathcal{J}}_{\rm vect})^\mu{}_\nu$ (corresponding to $\lambda\rho = 23$, 31 and 12 and generating ordinary rotations) are such that

$$(\boldsymbol{\mathcal{J}}_{\rm vec})^0{}_0 = (\boldsymbol{\mathcal{J}}_{\rm vec})^i{}_0 = (\boldsymbol{\mathcal{J}}_{\rm vec})^0{}_j = 0\,, \qquad \left(\mathcal{J}^k_{\rm vec}\right)^i{}_j = i\,\epsilon^{ikj}\,, \tag{8.128}$$

and, hence,

$$\left(\boldsymbol{\mathcal{J}}^2_{\rm vec}\right)^0{}_0 = \left(\boldsymbol{\mathcal{J}}^2_{\rm vec}\right)^i{}_0 = \left(\boldsymbol{\mathcal{J}}^2_{\rm vec}\right)^0{}_j = 0\,, \qquad \left(\boldsymbol{\mathcal{J}}^2_{\rm vec}\right)^i{}_j = 2\,\delta^i{}_j\,, \tag{8.129}$$

($\boldsymbol{\mathcal{J}}^2_{\rm vec} \equiv \mathcal{J}^x_{\rm vec}\mathcal{J}^x_{\rm vec} + \mathcal{J}^y_{\rm vec}\mathcal{J}^y_{\rm vec} + \mathcal{J}^z_{\rm vec}\mathcal{J}^z_{\rm vec}$). Acting on both sides of the conditions (8.126) with $(\boldsymbol{\mathcal{J}})^{\mu'}{}_\mu$ and applying the conditions (8.126) in the left hand side again, we find that

$$\begin{aligned}\sum_{\sigma'} u^\mu(\mathbf{0},\sigma')\,\mathbf{J}^{(s)2}_{\sigma'\sigma} &= \sum_\nu \left(\boldsymbol{\mathcal{J}}^2_{\rm vec}\right)^\mu{}_\nu\, u^\nu(\mathbf{0},\sigma)\,,\\ \sum_{\sigma'} v^\mu(\mathbf{0},\sigma')\,\mathbf{J}^{(s)*2}_{\sigma'\sigma} &= \sum_\nu \left(\boldsymbol{\mathcal{J}}^2_{\rm vec}\right)^\mu{}_\nu\, v^\nu(\mathbf{0},\sigma)\,,\end{aligned} \tag{8.130}$$

which, when combined with the results (8.129), means that

$$\begin{aligned}&\sum_{\sigma'} u^0(\mathbf{0},\sigma')\,\mathbf{J}^{(s)2}_{\sigma'\sigma} = 0\,, \qquad &&\sum_{\sigma'} u^i(\mathbf{0},\sigma')\,\mathbf{J}^{(s)2}_{\sigma'\sigma} = 2\,u^i(\mathbf{0},\sigma)\,,\\ &\sum_{\sigma'} v^0(\mathbf{0},\sigma')\,\mathbf{J}^{(s)*2}_{\sigma'\sigma} = 0\,, \qquad &&\sum_{\sigma'} v^i(\mathbf{0},\sigma')\,\mathbf{J}^{(s)*2}_{\sigma'\sigma} = 2\,v^i(\mathbf{0},\sigma)\,.\end{aligned} \tag{8.131}$$

Since $\mathbf{J}^{(s)2}_{\sigma'\sigma} = s(s+1)\,\delta_{\sigma'\sigma}$, we see that the conditions (8.126) have indeed two solutions: one is $s = 0$, for which $\mathbf{J}^{(s=0)}_{\sigma'\sigma} \equiv 0$ with $u^i(\mathbf{0}) = v^i(\mathbf{0}) = 0$, and the second one is $s = 1$ with $u^0(\mathbf{0}) = v^0(\mathbf{0}) = 0$.

Massive spin 0 particle

In the first case, using the prescription (8.11) and the explicit form of the standard Lorentz transformation (8.93), we find

$$u^\mu(\mathbf{p}) = (L_p)^\mu{}_\nu u^\nu(\mathbf{0}) \propto p^\mu\,, \qquad v^\mu(\mathbf{p}) = (L_p)^\mu{}_\nu v^\nu(\mathbf{0}) \propto p^\mu\,. \tag{8.132}$$

We can chose therefore $u^\mu(\mathbf{p}) = -i\,p^\mu$ and $v^\mu(\mathbf{p}) = i\,p^\mu$ which simply gives

$$V^{(\pm)\mu}(x) = \partial^\mu \phi^{(\pm)}(x)\,, \tag{8.133}$$

where $\phi^{(\pm)}(x)$ are the positive and negative frequency parts of the ordinary scalar field operator constructed in Section 8.2. This illustrates the remark made below the formulae (8.15) that field operators transforming as higher tensor (here vector) representations of the Lorentz group can also be associated with a spinless particle. This is also in agreement with the considerations of Section 11.5 where it will be argued that one combination of the components of an unconstrained massive vector field satisfying the appropriate classical field equation corresponds to the propagation of a scalar mode. Of course, the character of interactions of a given spinless particle does depend on which kind od operator, a scalar or a vector one, associated with it is used to build the corresponding term of the interaction Hamiltonian density $\mathcal{H}_{\rm int}(x)$: the forces mediated by a spinless particle the couplings of which to other particles (e.g. to fermions) are constructed using the vector operator will have essentially different character than the forces mediated by the same particle but represented by the scalar quantum operator.

Massive spin 1 particle

In the second case corresponding to $s = 1$ the matrices $\mathbf{J}_{\sigma'\sigma}$ have the form $(J^z_{(s=1)})_{\sigma'\sigma} = \sigma\,\delta_{\sigma'\sigma}$, and $(J^x_{(s=1)} \pm iJ^y_{(s=1)})_{\sigma'\sigma} = \delta_{\sigma'\sigma\pm1}\sqrt{(s\mp\sigma)(s\pm\sigma+1)} = \sqrt{2}\,\delta_{\sigma'\sigma\pm1}$ with $\sigma = +1, 0, -1$ and, using the formulae (8.128), we get from the conditions (8.126):

$$\sum_{\sigma'} u^i(\mathbf{0},\sigma')\,(J^z_{(s=1)})_{\sigma'\sigma} = \sigma\,u^i(\mathbf{0},\sigma) = i\sum_{j=1}^{3}\epsilon^{i3j}\,u^j(\mathbf{0},\sigma)\,, \tag{8.134}$$

from which we determine $u^i(\mathbf{0},\sigma=0) = \alpha\,\delta_{i3}$ with α being some constant which we set equal 1. The conditions

$$\sum_{\sigma'} u^i(\mathbf{0},\sigma')\,(J^x_{(s=1)} \pm iJ^y_{(s=1)})_{\sigma'\sigma=0} \equiv \sqrt{2}\,u^i(\mathbf{0},\pm1) = i\sum_{j=1}^{3}(\epsilon^{i1j} \pm i\,\epsilon^{i2j})\,u^j(\mathbf{0},\sigma=0)\,,$$

determine then the remaining two vectors. Taking into account that $u^0(\mathbf{0},\sigma) = 0$ we obtain (introducing the customary notation $\epsilon^\mu(\mathbf{0},\sigma)$ for the vector functions $u^\mu(\mathbf{0},\sigma)$)

$$\epsilon^\mu(\mathbf{0},0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \epsilon^\mu(\mathbf{0},+1) = \frac{-1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}, \quad \epsilon^\mu(\mathbf{0},-1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}. \tag{8.135}$$

In the similar way we find $v^\mu(\mathbf{0},\sigma) = \epsilon^{\mu *}(\mathbf{0},\sigma)$. For a nonzero three-momentum $\mathbf{p}$ the functions $\epsilon^\mu(\mathbf{p},\sigma)$ are given by (8.11) which here takes the form

$$\epsilon^\mu(\mathbf{p},\sigma) = (L_p)^\mu{}_\nu\, \epsilon^\nu(\mathbf{0},\sigma)\,. \tag{8.136}$$

They satisfy the condition $p_\mu \epsilon^\mu(\mathbf{p},\sigma) = 0$ (it is satisfied in the rest frame and is preserved by Lorentz transformations). The vector field operators of a massive spin $s = 1$ particle take threfore the form

$$\begin{aligned} V^{(+)}_\mu(x) &= \int d\Gamma_{\mathbf{p}} \sum_\sigma \epsilon_\mu(\mathbf{p},\sigma)\, e^{-ip\cdot x}\, a_\sigma(\mathbf{p})\,, \\ V^{(-)}_\mu(x) &= \int d\Gamma_{\mathbf{p}} \sum_\sigma \epsilon^*_\mu(\mathbf{p},\sigma)\, e^{+ip\cdot x}\, a^{c\dagger}_\sigma(\mathbf{p})\,. \end{aligned} \tag{8.137}$$

For greater generality we admit the possibility that the spin 1 particle carries some conserved charge(s) and that there exist, therefore, its antiparticle created by $a^{c\dagger}(\mathbf{p},\sigma)$. If the spin 1 particle is neutral the two terms are related by $V^{(-)}_\mu(x) = V^{(+)\dagger}_\mu(x)$. It is also useful to remember the formula

$$\sum_{\sigma=+1,0,-1} \epsilon^\mu(\mathbf{p},\sigma)\, \epsilon^{\nu *}(\mathbf{p},\sigma) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}\ , \tag{8.138}$$

(where M is the rest mass of the spin 1 particle) which can be derived either by applying the standard Lorentz transformation to its rest frame form or by noticing that the only second rank tensors available are $g^{\mu\nu}$ and $p^\mu p^\nu$ and that the sum must be the projection operator onto the subspace orthogonal to the four-vector p^μ because $\epsilon^\mu(\mathbf{0},\sigma)\, k_\mu = 0$. Of course, by construction the vectors $\epsilon^\mu(\mathbf{p},\sigma)$ satisfy

$$\epsilon^\mu(\mathbf{p},\sigma')\, \epsilon^*_\mu(\mathbf{p},\sigma) = -\delta_{\sigma'\sigma}\,. \tag{8.139}$$

The causal field operator is, as usually, given by

$$V_\mu(x) = \kappa_+ V^{(+)}_\mu(x) + \kappa_- V^{(-)}_\mu(x)\,. \tag{8.140}$$

It satisfies the Klein-Gordon equation

$$\left(\partial^\lambda \partial_\lambda + M^2\right) V^\mu(x) = 0\,, \tag{8.141}$$

which trivially follows from the mass-shell condition $p^2 = M^2$, and, in addition, the condition

$$\partial_\mu V^\mu(x) = 0\,, \tag{8.142}$$

which is the consequence of the relation $p_\mu \epsilon^\mu(\mathbf{p},\sigma) = 0$ satisfied by the polarization vectors $\epsilon^\mu(\mathbf{p},\sigma)$.

To check the local causality condition (8.16) let us consider a neutral spin 1 particle which is its own antiparticle. In this case for relevant (anti)commutators taken for $(x-y)^2<0$ are equal

$$[V^\mu(x),\ V^\nu(y)]_\mp = \kappa_+\kappa_-(1\mp 1)\left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{M^2}\right)\Delta_+(x-y)\,, \tag{8.143}$$

$$\left[V^\mu(x),\ V^{\nu\dagger}(y)\right]_\mp = (|\kappa_+|^2\mp|\kappa_-|^2)\left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{M^2}\right)\Delta_+(x-y)\,.$$

We see that they vanish only for the $(-)$ signs and $|\kappa_+|=|\kappa_-|$. Spin 1 particles must therefore be bosons. An appropriate choice of the phases of the one-particle states allows to set $\kappa_+=\kappa_-=1$. With this choice the field operator built out of the creation and annihilation operators of a neutral spin 1 particle becomes Hermitian $V^\mu(x)=V^{\mu\dagger}(x)$:

$$V_\mu(x)=\int d\Gamma_{\mathbf{p}}\sum_\sigma\left[\epsilon_\mu(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,a_\sigma(\mathbf{p})+\epsilon^*_\mu(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,a^\dagger_\sigma(\mathbf{p})\right]. \tag{8.144}$$

The creation and annihilation operators can be then expressed in terms of $V_\mu(x)$ by the formulae

$$a_\sigma(\mathbf{p}) = -i\int d^3\mathbf{x}\,\epsilon^*_\mu(\mathbf{p},\sigma)\,e^{ip\cdot x}\,\overleftrightarrow{\partial}_0 V^\mu(x)\,,$$

$$a^\dagger_\sigma(\mathbf{p}) = i\int d^3\mathbf{x}\,\epsilon_\mu(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0 V^\mu(x)\,. \tag{8.145}$$

The same statistics is obtained also for spin 1 particles which carry conserved quantum numbers and must therefore be accompanied by antiparticles of the same mass and spin. The field operator in this case is not Hermitian. By the choice of phases of the one particle states it can be cast in the form

$$V_\mu(x)=\int d\Gamma_{\mathbf{p}}\sum_\sigma\left[\epsilon_\mu(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,a_\sigma(\mathbf{p})+\epsilon^*_\mu(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,a^{c\dagger}_\sigma(\mathbf{p})\right]. \tag{8.146}$$

$V^\dagger_\mu(x)$ and the formulae expressing the creation and annihilation operators in terms of $V_\mu(x)$ and/or $V^\dagger_\mu(x)$ can be easily written down too.

P, T and C transformations of the spin 1 particle vector field operator

From the parity transformation properties of the creation and annihilation operators of a massive spin 1 particle: $\mathcal{P}_0\,a^\dagger(\mathbf{p},\sigma)\,\mathcal{P}_0^{-1}=\eta\,a^\dagger(-\mathbf{p},\sigma)$ and $\mathcal{P}_0\,a(\mathbf{p},\sigma)\,\mathcal{P}_0^{-1}=\eta^*a(-\mathbf{p},\sigma)$ it follows that the Hermitian vector vector field associated with a spin 1 neutral particle transforms as

$$\begin{aligned}\mathcal{P}_0\,V_\mu(x)\,\mathcal{P}_0^{-1} &= \int d\Gamma_{\mathbf{p}}\sum_\sigma\left[\eta^*\epsilon_\mu(\mathbf{p},\sigma)\,e^{-ip\cdot x}\,a_\sigma(-\mathbf{p})+\eta\,\epsilon^*_\mu(\mathbf{p},\sigma)\,e^{+ip\cdot x}\,a^\dagger_\sigma(-\mathbf{p})\right]\\ &= \int d\Gamma_{\mathbf{p}}\sum_\sigma\left[\eta^*\epsilon_\mu(-\mathbf{p},\sigma)\,e^{-ip\cdot(P\cdot x)}\,a_\sigma(\mathbf{p})+\eta\,\epsilon^*_\mu(-\mathbf{p},\sigma)\,e^{+ip\cdot(P\cdot x)}\,a^\dagger_\sigma(\mathbf{p})\right].\end{aligned}$$

($d\Gamma_{-\mathbf{p}} = d\Gamma_{\mathbf{p}}$). Since $L_{P\cdot p} = P\cdot L_p\cdot P$, from (8.136) and the explicit form (8.135) of $\epsilon^\mu(\mathbf{0},\sigma)$ it follows that

$$\epsilon^\mu(-\mathbf{p},\sigma) = (P\cdot L_p\cdot P)^\mu{}_\nu\,\epsilon^\nu(\mathbf{0},\sigma) = -P^\mu_{\ \nu}\,\epsilon^\nu(\mathbf{p},\sigma)\,. \tag{8.147}$$

Thus, if the operator $V^\mu(x)$ is to transform in a regular way, the equality $\eta = \eta^*$ must hold, which implies $\eta = \pm 1$. In this case

$$\mathcal{P}_0\,V^\mu(x)\,\mathcal{P}_0^{-1} = -\eta^* P^\mu_{\ \nu}\,V^\nu(P\cdot x)\,. \tag{8.148}$$

In this form the above formula remains also true for the operator of a charged spin 1 particle provided $\eta^c = \eta^*$ where η^c is the intrinsic parity of its antiparticle.

In considering the time reversal, one repeates the steps and using (for $s = 1$) the relation

$$\begin{aligned}(-1)^{1+\sigma}\epsilon^{\mu*}(-\mathbf{p},-\sigma) &= (-1)^{1+\sigma}(L_{P\cdot p})^\mu{}_\nu\,\epsilon^{\nu*}(\mathbf{0},-\sigma)\\ &= -(P\cdot L_p\cdot P)^\mu{}_\nu\,\epsilon^\nu(\mathbf{0},\sigma) = P^\mu_{\ \nu}\,\epsilon^\nu(\mathbf{p},\sigma)\,,\end{aligned} \tag{8.149}$$

finds that

$$\mathcal{T}_0\,V^\mu(x)\,\mathcal{T}_0^{-1} = \zeta^* P^\mu_{\ \nu} V^\nu(T\cdot x) \equiv -\zeta^* T^\mu_{\ \nu} V^\nu(T\cdot x)\,, \tag{8.150}$$

provided $\zeta^* = \zeta = \pm 1$, if the spin 1 particle is neutral, or $\zeta^* = \zeta^c$ in the case of a charged particle and its antiparticle.

Finally, if $\xi^c = \xi^*$ the rule

$$\mathcal{C}_0\,V^\mu(x)\,\mathcal{C}_0^{-1} = \xi^*\,V^{\mu\dagger}(x)\,, \tag{8.151}$$

is easily obtained. If the massive spin 1 particle is neutral, i.e. if it is a charge conjugation eigenstate, it must have $C \equiv \xi = \pm 1$.

8.5 Massless spin 1 particles

Finally we consider massless spin 1 particles. Recalling the transformation properties of the corresponding one-particle states we get, that the corresponding creation operator must satisfy the rule

$$U_0(\Lambda)\,a^\dagger_\lambda(\mathbf{p})\,U_0^{-1}(\Lambda) = e^{-i\lambda\theta(W(\Lambda,p))}\,a^\dagger_\lambda(\mathbf{p}_\Lambda)\,. \tag{8.152}$$

The general conditions (8.10) take in this case the form

$$\begin{aligned}u_l(\mathbf{p}_\Lambda,\lambda)\,e^{-i\lambda\theta(W(\Lambda,p))} &= \sum_k D_{lk}(\Lambda)\,u_k(\mathbf{p},\lambda)\,,\\ v_l(\mathbf{p}_\Lambda,\lambda)\,e^{i\lambda\theta(W(\Lambda,p))} &= \sum_k D_{lk}(\Lambda)\,v_k(\mathbf{p},\lambda)\,.\end{aligned} \tag{8.153}$$

The question now is what Lorentz group representation $D_{lk}(\Lambda)$ can be compatible with these conditions. Since the particle has spin $s = 1$, we could expect that the vector representation $D(\Lambda) = \Lambda$ is the smallest possible one allowing to build the causal field operators of massless spin 1 particles. However, as we will see, it is impossible to build a field operator transforming as a Lorentz vector out of the creation and annihilation operators of a massless spin 1 particle. The same is also true for the graviton - a hypothetical massless spin 2 particle and for the massless gravitino - a hypothetical massless spin 3/2 particle, both of which are necessary to construct supersymmetric theories of gravitation: it is impossible to construct out of the creation operators of these particles field operators transforming as a symmetric second rank tensor and the simplest spinor containing the spin 3/2 representation of the rotation group, respectively.

To make these statements more explicit let us try to build an operator transforming as a vector field, that is with $D(\Lambda) = \Lambda$ in (8.3), out of the creation and annihilation operators of a massless spin 1 particle. As in the case of massive particles it is sufficient to find solutions of the conditions (8.153) for the standard four vector k^μ; the functions $u^\mu(\mathbf{p}, \lambda) \equiv \epsilon^\mu(\mathbf{p}, \lambda)$ and $v^\mu(\mathbf{p}, \lambda) \equiv \tilde{\epsilon}^\mu(\mathbf{p}, \lambda)$ for arbitrary momenta $p^\mu = (|\mathbf{p}|, \mathbf{p})$ will be then given by

$$\epsilon^\mu(\mathbf{p}, \lambda) = (L_p)^\mu{}_\nu\, \epsilon^\nu(\mathbf{k}, \lambda)\,, \qquad \tilde{\epsilon}^\mu(\mathbf{p}, \lambda) = (L_p)^\mu{}_\nu\, \tilde{\epsilon}^\nu(\mathbf{k}, \lambda)\,, \tag{8.154}$$

where $k^\mu = (\kappa, 0, 0, \kappa)$ and $(L_p)^\mu{}_\nu k^\nu = p^\mu$. We consider, therefore, the relation (8.153) for $p^\mu = k^\mu$ - the standard four-momentum of a massless particle - and the Lorentz transformations $\Lambda = W$ belonging to the little group of the vector k^μ. Setting $\Lambda = W$ we get from (8.153)

$$\begin{aligned} \epsilon^\mu(\mathbf{k}, \lambda)\, e^{+i\lambda\theta} &= W^\mu{}_\nu\, \epsilon^\nu(\mathbf{k}, \lambda)\,, \\ \epsilon^{\mu*}(\mathbf{k}, \lambda)\, e^{-i\lambda\theta} &= W^\mu{}_\nu\, \epsilon^{\nu*}(\mathbf{k}, \lambda)\,, \end{aligned} \tag{8.155}$$

with $\theta \equiv \theta(W(W, k))$. (To write the second of these relations we have noticed that one can set $\tilde{\epsilon}^\mu = \epsilon^{\mu*}$.) Recall, that the little group transformations $W^\mu{}_\nu$ are of the form

$$W^\mu{}_\nu = S^\mu{}_\lambda(\alpha, \beta)\, O^\lambda{}_\nu(\theta_W, \mathbf{e}_z)\,, \tag{8.156}$$

where $O(\theta_W, \mathbf{e}_z)$ is the (active) rotation around the z-axis by the angle θ_W and $S(\alpha, \beta)$ is the matrix

$$S^\mu{}_\lambda(\alpha, \beta) = \begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix}, \tag{8.157}$$

the parameters α and β of which are arbitrary, while $\zeta \equiv (\alpha^2 + \beta^2)/2$. In the particular case considered here $W(W, k) = W$ and the angle θ in (8.155) is just the same as θ_W. Therefore, for $\alpha = \beta = 0$ we obtain from (8.155) the condition

$$\epsilon^\mu(\mathbf{k}, \lambda)\, e^{-i\lambda\theta} = O^\mu{}_\nu(\theta, \mathbf{e}_z)\, \epsilon^\nu(\mathbf{k}, \lambda)\,, \tag{8.158}$$

while for $\theta = 0$ satisfying the relation (8.155) requires

$$\epsilon^{\mu}(\mathbf{k}, \lambda) = S^{\mu}{}_{\nu}(\alpha, \beta)\, \epsilon^{\nu}(\mathbf{k}, \lambda)\,. \tag{8.159}$$

Since

$$O^{\mu}{}_{\nu}(\theta, \mathbf{e}_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8.160}$$

the solution of (8.158) is

$$\epsilon^{\mu}(\mathbf{k}, \lambda = \pm 1) = \mp \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}, \tag{8.161}$$

(the overall signs have been chosen so as to comply with (8.135) for $\lambda = \pm 1$) but then the condition (8.159) can be satisfied only if $\alpha \pm i\beta = 0$ that is, only for $\alpha = \beta = 0$. In the general case acting with $W^{\mu}{}_{\nu}$ on the polarization vectors (8.161) we get

$$\begin{aligned} W^{\mu}{}_{\nu}(\alpha, \beta, \theta)\, \epsilon^{\nu}(\mathbf{k}, \pm 1) &= e^{\pm i\theta} S^{\mu}{}_{\nu}(\alpha, \beta)\, \epsilon^{\nu}(\mathbf{k}, \pm 1) \\ &= \mp \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \pm i\beta \\ 1 \\ \pm i \\ \alpha \pm i\beta \end{pmatrix} e^{\pm i\theta} = e^{\pm i\theta} \left[\epsilon^{\mu}(\mathbf{k}, \pm 1) \mp \frac{\alpha \pm i\beta}{\sqrt{2}} \frac{k^{\mu}}{|\mathbf{k}|} \right]. \end{aligned} \tag{8.162}$$

It follows, that with the annihilation and creation operators of a massless spin 1 particle the basic conditions (8.12) cannot be fulfilled for the vector representation $D(\Lambda) = \Lambda$.

We can nevertheless try to build quantum field operators with one Lorenz vector index

$$\begin{aligned} A^{(+)}_{\mu}(x) &= \int d\Gamma_{\mathbf{p}} \sum_{\lambda} \epsilon_{\mu}(\mathbf{p}, \lambda)\, e^{-ip\cdot x}\, a_{\lambda}(\mathbf{p})\,, \\ A^{(-)}_{\mu}(x) &= \int d\Gamma_{\mathbf{p}} \sum_{\lambda} \epsilon^{*}_{\mu}(\mathbf{p}, \lambda)\, e^{+ip\cdot x}\, a^{\dagger}_{\lambda}(\mathbf{p})\,, \end{aligned} \tag{8.163}$$

(we have assumed that, like the photon, the considered spin 1 particle is neutral), where

$$\epsilon^{\mu}(\mathbf{p}, \lambda) = (L_p)^{\mu}{}_{\nu} \epsilon^{\nu}(\mathbf{k}, \lambda) \qquad \text{with} \qquad L_p = R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}) \cdot B_z\!\left(\frac{|\mathbf{p}|}{|\mathbf{k}|}\right). \tag{8.164}$$

Because of its matrix structure (see (6.53)) the boost B_z in the above formula acts on the polarization vectors $\epsilon^{\nu}(\mathbf{k}, \lambda)$ given in (8.161) as the unit matrix. Hence,

$$\epsilon^{\mu}(\mathbf{p}, \lambda) = (R_{\hat{\mathbf{z}}}(\hat{\mathbf{p}}))^{\mu}{}_{\nu}\, \epsilon^{\nu}(\mathbf{k}, \lambda)\,. \tag{8.165}$$

It follows, that $\epsilon^0(\mathbf{p},\lambda) = 0$ in any Lorentz frame. This again shows that $\epsilon^\mu(\mathbf{p},\lambda)$ are not true four-vectors and, therefore, that the field operators (8.163) cannot transform as four-vectors either.

It is also easy to derive the formula for the sum over polarizations. For the standard four-momentum k^μ using (8.161) we get

$$\sum_{\lambda=\pm 1} \epsilon^i(\mathbf{k},\lambda)\,\epsilon^{j*}(\mathbf{k},\lambda) = \delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}\;, \tag{8.166}$$

and using (8.165) we obtain

$$\sum_{\lambda=\pm 1} \epsilon^i(\mathbf{p},\lambda)\,\epsilon^{j*}(\mathbf{p},\lambda) = \delta^{ij} - \frac{p^i p^j}{|\mathbf{p}|^2}\;, \tag{8.167}$$

Furthermore, it is easy to see that $k_\mu\epsilon^\mu(\mathbf{k},\lambda) = 0$. This is preserved by Lorentz transformations, so it remains true in any Lorentz frame:

$$p_\mu\epsilon^\mu(\mathbf{p},\lambda) = 0\,, \tag{8.168}$$

and, because $\epsilon^0(\mathbf{p},\lambda) = 0$, also

$$\mathbf{p}\cdot\boldsymbol{\epsilon}(\mathbf{p},\lambda) = 0\,. \tag{8.169}$$

Of course, by construction, i.e. by virtue of (8.161) and (8.165)

$$\epsilon^*_\mu(\mathbf{p},\lambda')\,\epsilon^\mu(\mathbf{p},\lambda) = -\boldsymbol{\epsilon}^*(\mathbf{p},\lambda')\cdot\boldsymbol{\epsilon}(\mathbf{p},\lambda) = -\delta_{\lambda'\lambda}\,. \tag{8.170}$$

From (8.168) and (8.169) it follows that the photon field operators (8.163) satisfy the following two conditions

$$\partial_\nu\partial^\nu A^{(\pm)}_\mu(x) = 0\,, \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\mathbf{A}^{(\pm)}(x) = 0\,, \tag{8.171}$$

and, moreover, $A^{(\pm)}_0(x) \equiv 0$. It is also easy to check that

$$\begin{aligned} a_\lambda(\mathbf{p}) &= -i\int d^3\mathbf{x}\,\epsilon^*_\mu(\mathbf{p},\lambda)\,e^{ip\cdot x}\,\overleftrightarrow{\partial}_0 A^\mu(x)\,,\\ a^\dagger_\lambda(\mathbf{p}) &= i\int d^3\mathbf{x}\,\epsilon_\mu(\mathbf{p},\lambda)\,e^{-ip\cdot x}\,\overleftrightarrow{\partial}_0 A^\mu(x)\,. \end{aligned} \tag{8.172}$$

We can then ask how the field operators (8.163) transform under the action of $U_0(\Lambda)$. Recalling the transformation rules of the massless spin 1 particle creation and annihilation operators we find

$$\begin{aligned} U_0(\Lambda)\,A^{(+)\mu}(x)\,U_0^{-1}(\Lambda) &= \int d\Gamma_{\mathbf{p}}\sum_\lambda \epsilon^\mu(\mathbf{p},\lambda)\,e^{-ip\cdot x}e^{i\lambda\theta_W}\,a_\lambda(\mathbf{p}_\Lambda)\\ &= \int d\Gamma_{\mathbf{p}_\Lambda}\sum_\lambda \epsilon^\mu(\mathbf{p},\lambda)\,e^{i\lambda\theta_W}\,e^{-i(\Lambda\cdot p)\cdot(\Lambda\cdot x)}\,a_\lambda(\mathbf{p}_\Lambda)\,. \end{aligned} \tag{8.173}$$

If the conditions (8.155) were satisfied, as were the conditions (8.12) for massive particles, we would have $\Lambda^{\mu}{}_{\nu}\epsilon^{\nu}(\mathbf{p},\lambda) = e^{-i\lambda\theta_W}\epsilon^{\mu}(\mathbf{p}_\Lambda,\lambda)$ that is, $e^{i\lambda\theta_W}\epsilon^{\mu}(\mathbf{p},\lambda) = (\Lambda^{-1})^{\mu}{}_{\nu}\epsilon^{\nu}(\mathbf{p}_\Lambda,\lambda)$, and, as a result, $U_0(\Lambda)A^{(+)\mu}(x)U_0^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\nu}A^{(+)\nu}(\Lambda\cdot x)$. Here however, performing the necessary steps and using for $D_{nk}(W)u_k(\mathbf{k},\lambda) \equiv W^{\mu}{}_{\nu}\epsilon^{\nu}(\mathbf{k},\lambda)$ the result (8.162), we find

$$\begin{aligned}\Lambda^{\mu}{}_{\nu}\epsilon^{\nu}(\mathbf{p},\pm1) &= (L_{\Lambda\cdot p})^{\mu}{}_{\nu}\left[\epsilon^{\nu}(\mathbf{k},\pm1) \mp \frac{\alpha\pm i\beta}{\sqrt{2}}\frac{k^{\nu}}{|\mathbf{k}|}\right]e^{\mp i\theta} \\ &= \epsilon^{\mu}(\mathbf{p}_\Lambda,\pm1)\,e^{\mp i\theta} \mp \frac{\alpha\pm i\beta}{\sqrt{2}}\frac{(\Lambda\cdot p)^{\mu}}{|\mathbf{k}|}\,e^{\mp i\theta}\,, \qquad (8.174)\end{aligned}$$

where the parameters θ, α and β depend on $W(\Lambda,p)$. In other words, we get

$$\epsilon^{\mu}(\mathbf{p},\lambda)\,e^{i\lambda\theta} = (\Lambda^{-1})^{\mu}{}_{\nu}\,\epsilon^{\nu}(\mathbf{p}_\Lambda,\lambda) - \frac{\lambda\alpha+i\beta}{\sqrt{2}}\frac{p^{\mu}}{|\mathbf{k}|}\,, \qquad (8.175)$$

where $|\mathbf{k}| \equiv \kappa$, the length of the spatial component of $L_p^{-1}\cdot p$, is fixed by the convention used to define massless particle states. Therefore for the constructed operator, we obtain the following transformation rule:

$$U_0(\Lambda)\,A^{(\pm)\mu}(x)\,U_0^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\nu}A^{(\pm)\nu}(\Lambda\cdot x) + \partial^{\mu}\Theta^{(\pm)}(x,\Lambda)\,, \qquad (8.176)$$

in which $\Theta^{(\pm)}$ are some scalar operators.

The result (8.176) shows that out of the creation and annihilation operators of a massless spin 1 particle it is easy to construct a field operator which transforms as a decent second rank tensor. For this it is sufficient to take the combination

$$F^{(\pm)}_{\mu\nu}(x) \equiv \partial_{\mu}A^{(\pm)}_{\nu}(x) - \partial_{\nu}A^{(\pm)}_{\mu}(x)\,, \qquad (8.177)$$

which, as follows from (8.176), transforms as

$$U_0(\Lambda)\,F^{\mu\nu(\pm)}(x)\,U_0^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\rho}(\Lambda^{-1})^{\nu}{}_{\lambda}F^{\rho\lambda(\pm)}(\Lambda\cdot x)\,. \qquad (8.178)$$

Combining the positive and negative frequency operators into the causal quantum field

$$F_{\mu\nu}(x) = \kappa_+F^{(+)}_{\mu\nu}(x) + \kappa_-F^{(-)}_{\mu\nu}(x)\,, \qquad (8.179)$$

and computing the (anti)commutators

$$\begin{aligned}&[F_{\mu\nu}(x),\ F_{\rho\lambda}(y)]_{\pm}\ , \\ &\left[F_{\mu\nu}(x),\ F^{\dagger}_{\rho\lambda}(y)\right]_{\pm}\ , \qquad (8.180)\end{aligned}$$

we find that they vanish for $(x-y)^2 < 0$ only if the creation and annihilation operators of massless spin 1 particles are bosonic and only for $|\kappa_+| = |\kappa_-|$. The phases of κ_+ and

κ_- can be made equal by appropriately choosing the phase of the one particle states. We can then set $\kappa_+ = \kappa_- = 1$.

As to the field operator

$$\begin{aligned} A_\mu(x) &= A_\mu^{(+)}(x) + A_\mu^{(-)}(x) \\ &= \int d\Gamma_{\mathbf{p}} \sum_\lambda \left[\epsilon_\mu(\mathbf{p},\lambda)\, e^{-ip\cdot x}\, a_\lambda(\mathbf{p}) + \epsilon_\mu^*(\mathbf{p},\lambda)\, e^{+ip\cdot x}\, a_\lambda^\dagger(\mathbf{p})\right], \end{aligned} \tag{8.181}$$

it can be checked, that it *does not* satisfy the local causality requirement, i.e. in general the commutator of two such field operators taken at two space-time points x and y *does not* vanish for $(x-y)^2 < 0$. From the point of view of the quantum theory of electromagnetic field this can be interpreted as the manifestation of the unphysical character of the four-potential A_μ which is not measurable even classically. As can be checked, only the *equal time* commutators of these operators vanish:

$$\begin{aligned} &[A^\mu(t,\mathbf{x}), \quad A^\nu(t,\mathbf{y})] = 0\,, \\ &\left[A^\mu(t,\mathbf{x}),\ A^{\nu\dagger}(t,\mathbf{y})\right] = 0\,. \end{aligned} \tag{8.182}$$

Nonvanishing of the commutator $[A_\mu(x),\ A_\nu(y)]$ for $(x-y)^2 < 0$ and the inhomogeneous Lorentz transformation rule of $A_\mu(x)$ are at the origin of the appearance of noncovariant terms in the propagator of massless spin 1 particles which will be derived in Section 9.6. These two factors jeopardize Lorentz covariance of the S-matrix. Thus, if one insist on the use of the operator $A_\mu(x)$ to construct interactions of a massless spin 1 particle, the interaction Hamiltonian must have a very special form if such interactions are to produce a covariant S-matrix. It will turn out, that such interactions can indeed be built and the resulting Hamiltonians, when viewed from a different perspective, will correspond to quantized classical field theories possessing gauge invariance.

One could of course build interaction of massless spin 1 particles (photons, gluons) by using only the tensor field operator $F_{\mu\nu}(x)$ which has regular Lorentz transformation properties. However, as will be demonstrated, it is possible to build interactions leading to Lorenz covariant S-matrices also using the field operator $A_\mu(x)$ and it turns out this is precisely the possibility which Nature chooses.[15] This follows from the fact that observed electromagnetic interactions have a long-range character which, roughly speaking, requires that the matrix elements of the interaction Hamiltonian do not vanish too fast with decreasing momenta of the interacting particles.[16] This is not so in the case of interactions

[15]Moreover, with $F_{\mu\nu}(x)$ alone it is impossible to write down *renormalizable* interactions of photons, that is, such that it is possible to remove all infinities (appearing in higher orders of the Dyson perturbative expansion) from the S-matrix elements by redefining a finite number of coupling constants. This circumstance in itself is, however, no longer very compelling.

[16]This is only a necessary condition. In QCD the gluon field operator couples to the quark current in a way superficially similar to the way the photon operator does, yet, no long-range forces are produced: the gluon interactions get screened at large distances as a result of the infrared slavery (confinement of quarks and gluons).

constructed using $F_{\mu\nu}(x)$ - such interactions, because of the derivatives involved in $F_{\mu\nu}(x)$, have matrix elements vanishing too fast. Long-range forces leading to Lorentz covariant S-matrix and mediated by spin 1 particles require a gauge invariant theory. Similarly, long-range forces mediated by massless spin 2 particles (gravitons) and giving rise to a Lorentz covariant S-matrix require that the interaction of these particles are built from the field operator $h_{\mu\nu}(x)$ transforming as a second rank symmetric tensor only up to such gauge transformations which in the general theory of relativity are related to general coordinate transformations. In this sense the Einstein's principle of equivalence, which forms the physical basis of General Relativity, seems to find its roots in the quantum field theory.

8.6 The $SL(2,C)$ construction of the Lorentz group representations

The Clifford algebra method used in section 8.3 is very general. It allows to construct representations of the universal covering of any $SO(n,m)$ group (i.e. in any number of space-time dimensions). To construct representations of the $SO(1,3)$ group (the Lorentz group in 4 dimensions) one can however directly use the fact that its universal covering group $Spin(1,3)$ is isomorphic to $SL(2,C)$ - the group of complex 2×2 matrices of unit determinant, similarly as the $Spin(0,3)$ group is isomorphic to $SU(2)$. The relation of an $SL(2,C)$ matrix M to the Lorentz transformation Λ corresponding to it was discussed in section 6.2 - see the formulae (6.73)-(6.75).

The $SL(2,C)$ group has four obvious two-dimensional representations. Let M be an element of $SL(2,C)$, i.e. a 2×2 matrix of unit determinant. Then

$$M\,, \qquad \left(M^\dagger\right)^{-1}, \qquad \left(M^T\right)^{-1}, \qquad \text{and} \qquad M^*\,, \tag{8.183}$$

are all faithful representations[17] of $SL(2,C)$. Let us introduce objects (two-component spinors), which will be called fields, transforming according to these representations:

$$\lambda'_\alpha(x') = M_\alpha{}^\beta\lambda_\beta(x)\,, \qquad \lambda'^\alpha(x') = \left(M^{T-1}\right)^\alpha{}_\beta\,\lambda^\beta(x)\,, \tag{8.184}$$

$$\bar\chi'^{\dot\alpha}(x') = \left(M^{\dagger-1}\right)^{\dot\alpha}{}_{\dot\beta}\,\bar\chi^{\dot\beta}(x)\,, \qquad \bar\chi'_{\dot\alpha}(x') = \left(M^*\right)_{\dot\alpha}{}^{\dot\beta}\,\bar\chi_{\dot\beta}(x)\,, \tag{8.185}$$

where $x' = \Lambda\cdot x$. It is obvious that if λ_α (λ^α) transforms through a matrix $M_\beta{}^\alpha$ (through the matrix $(M^{T-1})^\beta{}_\alpha$), then its complex conjugate (or Hermitian conjugate in case these objects are field operators) transforms through the matrix $(M^*)_{\dot\beta}{}^{\dot\alpha}$ (through $(M^{\dagger-1})^{\dot\beta}{}_{\dot\alpha}$).

[17]The representation r of a group G is (in the mathematical sense) a mapping $r:\ G\to \mathrm{End}(V)$, i.e. a mapping of G into a space of linear endomorphisms of a vector space V, such that $r(g_1)\,r(g_2) = r(g_1{\cdot}g_2)$ etc., where $g_i\in G$. In the case at hand we have $g = M\in SL(2,C)$ and its representations: $r_1(M) = M$, $r_2(M) = (M^T)^{-1}$, $r_3(M) = M^*$ and $r_4(M) = (M^\dagger)^{-1}$. A representation is said to be faithful if $g_1\neq g_2$ implies $r(g_1)\neq (g_2)$.

It is customary to denote fields transforming as complex conjugates by bars and to dot their indices. Therefore we write

$$(\lambda_\alpha)^* \equiv \bar{\lambda}_{\dot{\alpha}}\,, \qquad (\lambda^\alpha)^* \equiv \bar{\lambda}^{\dot{\alpha}}\,. \tag{8.186}$$

The four different representations (8.183) are pairwise unitarily equivalent one to another. To make this explicit, let us introduce the antisymmetric matrices $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and $\bar{\epsilon}^{\dot{\alpha}\dot{\beta}}$, $\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}$

$$\epsilon^{12} = \epsilon_{12} = -1\,, \qquad \bar{\epsilon}^{\dot{1}\dot{2}} = \bar{\epsilon}_{\dot{1}\dot{2}} = 1\,, \tag{8.187}$$

so that

$$\epsilon^{\alpha\beta}\,\epsilon_{\gamma\beta} = \delta^{\alpha}{}_{\gamma}\,, \qquad \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\,\bar{\epsilon}^{\dot{\gamma}\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\gamma}}\,. \tag{8.188}$$

The 2×2 matrices $\epsilon^{\alpha\beta}$, etc, are invariant tensors of $SL(2,C)$:

$$M_\alpha{}^\delta M_\beta{}^\gamma \epsilon_{\gamma\delta} = \epsilon_{\alpha\beta}\,, \qquad \bar{\epsilon}_{\dot{\delta}\dot{\gamma}}\,(M^{\dagger-1})^{\dot{\gamma}}{}_{\dot{\alpha}}(M^{\dagger-1})^{\dot{\delta}}{}_{\dot{\beta}} = \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\,. \tag{8.189}$$

It is easy to check that the relations[18]

$$\begin{aligned} \lambda^\alpha &= \epsilon^{\alpha\beta}\lambda_\beta\,, \qquad & \lambda_\alpha &= \lambda^\beta\epsilon_{\beta\alpha}\,, \\ \bar{\chi}_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}\,, \qquad & \bar{\chi}^{\dot{\alpha}} &= \bar{\chi}_{\dot{\beta}}\,\epsilon^{\dot{\beta}\dot{\alpha}}\,, \end{aligned} \tag{8.190}$$

show that the representation M is unitarily equivalent to M^{T-1} while M^* is unitarily equivalent to $M^{\dagger-1}$.

If λ_α and φ_α are two anticommuting (fermionic) two-component fields, the product

$$\lambda\varphi \equiv \lambda^\alpha\varphi_\alpha = \varphi^\alpha\lambda_\alpha = \epsilon^{\alpha\beta}\lambda_\beta\varphi_\alpha = \lambda^\beta\varphi^\alpha\epsilon_{\alpha\beta}\,. \tag{8.191}$$

is Lorentz (or $SL(2,C)$) invariant. A similar invariant constructed out of $\bar{\chi}^{\dot{\alpha}}$ and $\bar{\psi}^{\dot{\alpha}}$ reads

$$\bar{\chi}\bar{\psi} \equiv \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\chi}_{\dot{\beta}}\bar{\psi}_{\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}\dot{\beta}} = \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}}\bar{\psi}^{\dot{\alpha}}\,. \tag{8.192}$$

Notice that to write down these invariant for the same field operator ($\varphi = \lambda$ or $\bar{\psi} = \bar{\chi}$) we have to assume they are anticommuting (independently of the argument following from the local causality); if we want to write down a classical Lagrangian for spinor fields using the invariant $\lambda^\alpha\lambda_\alpha$ or $\bar{\chi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$ we have to assume that these classical fields are not ordinary c-number fields (for which $\lambda^\alpha\lambda_\alpha = \bar{\chi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = 0$ by antisymmetry of the ϵ tensors)

[18] In the literature one encounters many different conventions for the ϵ tensors and different rules for raising and lowering spinor indices. The weak point of our notation is that $(\epsilon_{\gamma\delta})^* = -\bar{\epsilon}_{\dot{\gamma}\dot{\beta}}$. (It is for this reason that the bar over the dotted epsilon tensor has been introduced). The equalities $\epsilon^{12} = -\bar{\epsilon}^{\dot{1}\dot{2}}$ and $\epsilon_{12} = -\bar{\epsilon}_{\dot{1}\dot{2}}$ are necessary if the relations: $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$ and $\bar{\lambda}_{\dot{\alpha}} = \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\beta}}$ are to be consistent with $\bar{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^*$ and $\bar{\lambda}^{\dot{\alpha}} = (\lambda^\alpha)^*$.

but Grassmann, anticommuting (like the fermionic field operators) objects (see section 11.8 for a more formal definition of such fields).

Let us now consider the generators $\mathcal{J}^{\mu\nu}$ of a matrix representation of the Lorentz group (or, more precisely, of the $SL(2,C)$ group) and form the following complex combinations:

$$\boldsymbol{\mathcal{N}}_{\pm} = \frac{1}{2}\left(\boldsymbol{\mathcal{J}} \pm i\boldsymbol{\mathcal{K}}\right), \tag{8.193}$$

of the generators $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{K}}$ defined in (6.20). Using the commutation rules (6.19) it is easy to check that

$$\left[\mathcal{N}^i_{\pm},\ \mathcal{N}^j_{\pm}\right] = i\epsilon^{ijk}\mathcal{N}^k_{\pm} \qquad \text{and} \qquad \left[\mathcal{N}^i_{\pm},\ \mathcal{N}^j_{\mp}\right] = 0\,. \tag{8.194}$$

Any element $D(M)$ of a matrix representation of the $SL(2,C)$ group takes then the form[19]

$$D(M) = e^{-i\omega_{0i}\mathcal{K}^i - \frac{i}{2}\omega_{ij}\mathcal{J}^{ij}} \equiv e^{-i\xi^i\mathcal{K}^i - i\eta^i\mathcal{J}^i} = e^{-i\zeta^i\mathcal{N}^i_+ - i\zeta^{i*}\mathcal{N}^i_-}\,, \tag{8.195}$$

where $\zeta^i = \eta^i - i\xi^i$. Superficially it looks as if $SL(2,C)$ were a direct product of two complexified $SU(2)$ groups, but this is not exactly so, because the parameters of the subgroup generated by $\boldsymbol{\mathcal{N}}_-$ are complex conjugate of the parameters of the subgroup generated by $\boldsymbol{\mathcal{N}}_+$.

There is however nothing which could prevent using this decomposition for building spinor representations $r(\Lambda)$ of the Lorentz group, i.e. a matrix representation of $SL(2,C)$. Two simplest realizations of the commutation rules (8.194) are obvious:

$$\begin{cases} r_1(\mathcal{N}^i_+) = \frac{1}{2}\sigma^i \\ \\ r_1(\mathcal{N}^i_-) = 0 \end{cases}, \qquad \begin{cases} r_2(\mathcal{N}^i_+) = 0 \\ \\ r_2(\mathcal{N}^i_-) = \frac{1}{2}\sigma^i \end{cases}, \tag{8.196}$$

We denote these representations as $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively. By taking tensor products of these two representations one can build higher representations denoted as (s_+,s_-), where $s_+(s_++1)$ and $s_-(s_-+1)$ are the eigenvalues of the operators

$$\boldsymbol{\mathcal{N}}^2_+ \equiv \sum_{i=1}^{3}\mathcal{N}^i_+\mathcal{N}^i_+\,, \qquad \boldsymbol{\mathcal{N}}^2_- \equiv \sum_{i=1}^{3}\mathcal{N}^i_-\mathcal{N}^i_-\,, \tag{8.197}$$

on such representations.

We now denote

$$\begin{aligned} r_1\left(e^{-i\xi^i\mathcal{K}^i - i\eta^i\mathcal{J}^i}\right) &= e^{-\frac{i}{2}\sigma^i(\eta^i - i\xi^i)} \equiv M\,, \\ r_2\left(e^{-i\xi^i\mathcal{K}^i - i\eta^i\mathcal{J}^i}\right) &= e^{-\frac{i}{2}\sigma^i(\eta^i + i\xi^i)} \equiv M^{\dagger -1}\,, \end{aligned} \tag{8.198}$$

[19] $D(M)$ will be one of the four possible (8.183) two-dimensional representations of $SL(2,C)$.

and identify fields denoted by $\lambda_\alpha(x)$ as furnishing the $(\frac{1}{2},0)$ representation and those denoted by $\bar{\chi}^{\dot{\alpha}}$ as furnishing the $(0,\frac{1}{2})$ representation of $SL(2,C)$. The two representations are in fact identical when restricted to ordinary rotations ($\eta^i \neq 0$, $\xi^i = 0$) and differ only in how they realize boosts. It is also clear that they are precisely the two irreducible representations to which decomposes the Weyl (chiral) representation (8.67) obtained using the Clifford algebra construction when parity is not included as symmetry operation.

Before we proceed further we introduce a useful (covariant) notation. Let us introduce first the two sets of matrices

$$\sigma^\mu \equiv (I, \sigma^i)\,, \qquad \bar{\sigma}^\mu \equiv (I, -\sigma^i)\,. \tag{8.199}$$

It is easy to check that

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2g^{\mu\nu} = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu\,. \tag{8.200}$$

The representations of the generators $\mathcal{J}^{\mu\nu}$ in the two $SL(2,C)$ representations can be then written as

$$\begin{aligned} r_1(\mathcal{J}^{\mu\nu}) &= \frac{1}{2}\,(\sigma^{\mu\nu})_\alpha^{\ \beta} \equiv \frac{i}{4}\,(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta} \qquad (\frac{1}{2},0) \quad \text{rep.} \\ r_2(\mathcal{J}^{\mu\nu}) &= \frac{1}{2}\,(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \equiv \frac{i}{4}\,(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}_{\ \dot{\beta}} \qquad (0,\frac{1}{2}) \quad \text{rep.} \end{aligned} \tag{8.201}$$

This implies that the spinor indices of the matrices σ^μ and $\bar{\sigma}^\mu$ (8.199) should be written with the following placement of the indices

$$(\sigma^\mu)_{\alpha\dot{\beta}}\,, \qquad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\,.$$

With all this cosmetics the transformation rules of the fields λ_α and $\bar{\chi}^{\dot{\alpha}}$ read

$$\begin{aligned} \lambda'_\alpha(x') &= \left(e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_\alpha^{\ \beta} \lambda_\beta(x)\,, \\ \bar{\chi}'^{\dot{\alpha}}(x') &= \left(e^{-\frac{i}{4}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}\right)^{\dot{\alpha}}_{\ \dot{\beta}} \bar{\chi}^{\dot{\beta}}(x)\,. \end{aligned} \tag{8.202}$$

The fields λ_α and $\bar{\chi}^{\dot{\alpha}}$ are called *left-* and *right-chiral* Weyl (two-component) spinors, respectively. The Dirac spinors ψ (transforming according to the representation in which the generators $\mathcal{J}^{\mu\nu}$ are represented through the matrices γ^μ (8.53)) transform as the direct sum $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ of the two two-dimensional representations

$$\psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}\,, \qquad \bar{\psi} = \left(\chi^\alpha,\ \bar{\lambda}_{\dot{\alpha}}\right)\,. \tag{8.203}$$

The matrices σ^μ and $\bar{\sigma}^\mu$ satisfy the relations

$$\begin{aligned} \Lambda^\nu_{\ \mu}\, M\, \sigma^\mu\, M^\dagger &= \sigma^\nu\,, \\ \Lambda^\nu_{\ \mu}\, M^{\dagger -1}\, \bar{\sigma}^\mu\, M^\dagger &= \bar{\sigma}^\nu\,, \end{aligned} \tag{8.204}$$

which show that they are invariant tensors of $SL(2,C)$ or, in other words, they are Clebsch-Gordan coefficients connecting the tensor product $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ of the two chiral representations with the ordinary vector $(\frac{1}{2},\frac{1}{2})$ representation.

From (8.198) it is clear that, when restricted to rotations, the representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of $SL(2,C)$ contain spin 1/2 representations, so they can serve for constructing field operators of spin 1/2 particles. Let us first return to the construction of field operators out of the creation and annihilation operators of a massive spin 1/2 particle which are not its own antiparticle. Following the standard procedure we construct the field operators $\lambda_\alpha(x)$ or $\bar{\chi}^{\dot\alpha}$

$$\lambda_\alpha(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma \left[u_\alpha(\mathbf{p},\sigma)\, e^{-ipx}\, b_\sigma(\mathbf{p}) + v_\alpha(\mathbf{p},\sigma)\, e^{+ipx}\, d^\dagger_\sigma(\mathbf{p})\right], \tag{8.205}$$

$$\bar{\chi}^{\dot\alpha}(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma \left[\bar{u}^{\dot\alpha}(\mathbf{p},\sigma)\, e^{-ipx}\, b_\sigma(\mathbf{p}) + \bar{v}^{\dot\alpha}(\mathbf{p},\sigma)\, e^{+ipx}\, d^\dagger_\sigma(\mathbf{p})\right], \tag{8.206}$$

where the two-component functions[20] $u_\alpha(\mathbf{p},\sigma)$ and $v_\alpha(\mathbf{p},\sigma)$, or $\bar{u}^{\dot\alpha}(\mathbf{p},\sigma)$ and $\bar{v}^{\dot\alpha}(\mathbf{p},\sigma)$ are recovered from their standard momentum forms $u_\alpha(\mathbf{0},\sigma)$ and $v_\alpha(\mathbf{0},\sigma)$, or $\bar{u}^{\dot\alpha}(\mathbf{0},\sigma)$ and $\bar{v}^{\dot\alpha}(\mathbf{0},\sigma)$ with the help of

$$\begin{aligned} u_\alpha(\mathbf{p},\sigma) &= [M(L_p)]_\alpha^{\ \beta}\, u_\beta(\mathbf{0},\sigma)\,, \\ v_\alpha(\mathbf{p},\sigma) &= [M(L_p)]_\alpha^{\ \beta}\, v_\beta(\mathbf{0},\sigma)\,, \end{aligned} \tag{8.207}$$

and

$$\begin{aligned} \bar{u}^{\dot\alpha}(\mathbf{p},\sigma) &= \left[M^{\dagger-1}(L_p)\right]^{\dot\alpha}_{\ \dot\beta}\, \bar{u}^{\dot\beta}(\mathbf{0},\sigma)\,, \\ \bar{v}^{\dot\alpha}(\mathbf{p},\sigma) &= \left[M^{\dagger-1}(L_p)\right]^{\dot\alpha}_{\ \dot\beta}\, \bar{v}^{\dot\beta}(\mathbf{0},\sigma)\,, \end{aligned} \tag{8.208}$$

(see the formula (8.11)) with L_p given by (6.49) and (6.46). In turn, for the standard momentum the functions u_α, v_α or $\bar{u}^{\dot\alpha}$, $\bar{v}^{\dot\alpha}$ must satisfy the relations (8.13) which here take the form

$$\begin{aligned} \sum_{\sigma'} u_\alpha(\mathbf{0},\sigma') J^{(1/2)z}_{\sigma'\sigma} &= \sum_\beta \frac{1}{2}\left(\sigma^{12}\right)_\alpha^{\ \beta} u_\beta(\mathbf{0},\sigma)\,, \\ -\sum_{\sigma'} v_\alpha(\mathbf{0},\sigma') J^{(1/2)z*}_{\sigma'\sigma} &= \sum_\beta \frac{1}{2}\left(\sigma^{12}\right)_\alpha^{\ \beta} v_\beta(\mathbf{0},\sigma)\,, \end{aligned} \tag{8.209}$$

and

$$\begin{aligned} \sum_{\sigma'} \bar{u}^{\dot\alpha}(\mathbf{0},\sigma') J^{(1/2)z}_{\sigma'\sigma} &= \sum_{\dot\beta} \frac{1}{2}\left(\bar{\sigma}^{12}\right)^{\dot\alpha}_{\ \dot\beta} \bar{u}^{\dot\beta}(\mathbf{0},\sigma)\,, \\ -\sum_{\sigma'} \bar{v}^{\dot\alpha}(\mathbf{0},\sigma') J^{(1/2)z*}_{\sigma'\sigma} &= \sum_\beta \frac{1}{2}\left(\bar{\sigma}^{12}\right)^{\dot\alpha}_{\ \dot\beta} \bar{v}^{\dot\beta}(\mathbf{0},\sigma)\,, \end{aligned} \tag{8.210}$$

[20]In this approach barred symbols mean only the $SL(2,C)$ transformation properties and should not be confused with barred symbols used in section 8.3.

and similar relations with ($z \to x$, $12 \to 23$) and ($z \to y$, $12 \to 31$). Since $\sigma^{12} = \bar{\sigma}^{12} \equiv \sigma^z$ etc. the conditions for u_α and $\bar{u}^{\dot{\alpha}}$ are precisely the conditions (8.74) for u^+ and u^- (but written with different labels) whereas the conditions for v_α and $\bar{v}^{\dot{\alpha}}$ are identical with the conditions (8.76) for v^+ and v^-. Therefore we can immediately write down the solutions

$$\begin{aligned} u_\alpha(\mathbf{0}, \sigma = \frac{1}{2}) &= \sqrt{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad & v_\alpha(\mathbf{0}, \sigma = \frac{1}{2}) &= \sqrt{m}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ u_\alpha(\mathbf{0}, \sigma = -\frac{1}{2}) &= \sqrt{m}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad & v_\alpha(\mathbf{0}, \sigma = -\frac{1}{2}) &= \sqrt{m}\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \end{aligned} \tag{8.211}$$

and exactly the same solutions for $\bar{u}^{\dot{\alpha}}(\mathbf{0}, \sigma)$ and $\bar{v}^{\dot{\alpha}}(\mathbf{0}, \sigma)$. Thus, in the case of massive spin 1/2 particles we get essentially nothing new: we can build interaction Hamiltonians using only $\lambda_\alpha(x)$ given by (8.205), and its Hermitian conjugate

$$(\lambda^\dagger)_{\dot{\alpha}}(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma \left[(u^*)_{\dot{\alpha}}(\mathbf{p}, \sigma)\, e^{+ipx}\, b^\dagger_\sigma(\mathbf{p}) + (v^*)_{\dot{\alpha}}(\mathbf{p}, \sigma)\, e^{-ipx}\, d_\sigma(\mathbf{p}) \right],$$

or only $\bar{\chi}^{\dot{\alpha}}(x)$ given by (8.206) and its Hermitian conjugate

$$(\bar{\chi}^\dagger)^{\alpha}(x) = \int d\Gamma_{\mathbf{p}} \sum_\sigma \left[(\bar{u}^*)^{\alpha}(\mathbf{p}, \sigma)\, e^{+ipx}\, b^\dagger_\sigma(\mathbf{p}) + (\bar{v}^*)^{\alpha}(\mathbf{p}, \sigma)\, e^{-ipx}\, d_\sigma(\mathbf{p}) \right],$$

or both. However, in the interaction Hamiltonians we can always get the same using only the Dirac spinor operator ψ because, as it is easy to see, in the chiral representation of the Dirac matrices (8.66)

$$\begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} = \frac{1 - \gamma^5}{2}\, \psi_\alpha\,, \qquad \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \frac{1 + \gamma^5}{2}\, \psi_\alpha\,, \tag{8.212}$$

and

$$(\chi^\alpha,\ 0) = \bar{\psi}\, \frac{1 - \gamma^5}{2}\,, \qquad \left(0\ , \bar{\lambda}_{\dot{\alpha}}\right) = \bar{\psi}\, \frac{1 + \gamma^5}{2}\,, \tag{8.213}$$

so that

$$\psi_\alpha = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \qquad \text{and} \qquad \bar{\psi}\psi = \chi^\alpha \lambda_\alpha + \bar{\lambda}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}\,, \tag{8.214}$$

etc. It follows that in the chiral representation (8.66) the Dirac matrices are just

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{8.215}$$

where σ^μ and $\bar{\sigma}^\mu$ are defined in (8.199). If a spin 1/2 particle is massive *and* carries a conserved charge, we must use both operators, $\lambda_\alpha(x)$ and $\bar{\chi}^{\dot{\alpha}}(x)$ (and their Hermitian conjugates) to construct interactions, because with $\lambda_\alpha(x)$ alone ($\bar{\chi}^{\dot{\alpha}}(x)$ alone) it is impossible to form a Lorentz invariant scalar like $\bar{\psi}\psi$ (see (8.214)) commuting with the charge operator which could be coupled to a neutral spinless particle. This is, however, possible if the spin 1/2 particle is neutral (a Majorana fermion) by writing simply $\lambda^\alpha\lambda_\alpha$ and $\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$; one can equivalently use the four-component Majorana field operator

$$\psi_\alpha = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} . \tag{8.216}$$

In any case, if the interaction is to preserve parity, both operators $\lambda_\alpha(x)$ and $\bar{\chi}^{\dot{\alpha}}(x)$ have to be used.

Things change when we discuss massless spin 1/2 particles. In this case we write the field operators again in the form (8.205) and (8.206) and to obtain the functions $u_\alpha(\mathbf{p},\sigma)$, $v_\alpha(\mathbf{p},\sigma)$, $\bar{u}^{\dot{\alpha}}(\mathbf{p},\sigma)$ and $\bar{v}^{\dot{\alpha}}(\mathbf{p},\sigma)$ we apply the formulae (8.207) and (8.208) to their standard four-vector ($k^\mu = (1,0,0,1)$) forms but now with L_p given by (6.71). To find the the functions $u_\alpha(\mathbf{k},\sigma)$, $v_\alpha(\mathbf{k},\sigma)$, $\bar{u}^{\dot{\alpha}}(\mathbf{k},\sigma)$ and $\bar{v}^{\dot{\alpha}}(\mathbf{k},\sigma)$ we use the relation (8.10) with $\mathbf{p} = \mathbf{k}$ (i.e. we specify it to the case of the standard four-vector) and $\Lambda = W$

$$D(W) = e^{-i\alpha\mathcal{A}-i\beta\mathcal{B}}e^{-i\theta\mathcal{J}^z} , \tag{8.217}$$

where α, β and θ parameterize the little group of the standard four vector - see (6.61). The explicit matrix forms of the little group generators are (see (8.198)):

$$\begin{aligned} \mathcal{A} = \mathcal{K}^x - \mathcal{J}^y &= -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \\ \mathcal{B} = \mathcal{K}^y + \mathcal{J}^x &= -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{8.218}$$

in the $(\frac{1}{2},0)$ representation and

$$\begin{aligned} \mathcal{A} = \mathcal{K}^x - \mathcal{J}^y &= \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \\ \mathcal{B} = \mathcal{K}^y + \mathcal{J}^x &= \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{8.219}$$

in the $(0,\frac{1}{2})$ representation. The matrix form of $\mathcal{J}^z$ is the same both in the $(\frac{1}{2},0)$ and in the $(0,\frac{1}{2})$ representations:

$$\mathcal{J}^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{8.220}$$

The functions $u_\alpha(\mathbf{k},\sigma)$ and $v_\alpha(\mathbf{k},\sigma)$ must satisfy

$$\begin{aligned} u(\mathbf{k},\sigma)\, e^{-i\sigma\theta} &= e^{-i\alpha\mathcal{A}-i\beta\mathcal{B}}\, e^{-i\theta\mathcal{J}^z} u(\mathbf{k},\sigma) , \\ v(\mathbf{k},\sigma)\, e^{i\sigma\theta} &= e^{-i\alpha\mathcal{A}-i\beta\mathcal{B}}\, e^{-i\theta\mathcal{J}^z} v(\mathbf{k},\sigma) , \end{aligned} \tag{8.221}$$

with $\mathcal{A}$ and $\mathcal{B}$ given by (8.218). In contrast to the case of massless spin 1 particles considered in Section 8.5, here, for spin $s = \frac{1}{2}$, the solution does exist but only for $\sigma = -\frac{1}{2}$ for $u_\alpha(\mathbf{k}, \sigma)$ (and not for $\sigma = +\frac{1}{2}$) and only for $\sigma = +\frac{1}{2}$ for $v_\alpha(\mathbf{k}, \sigma)$. Similarly, $\bar{u}^{\dot{\alpha}}(\mathbf{k}, \sigma)$ and $\bar{v}^{\dot{\alpha}}(\mathbf{k}, \sigma)$ must satisfy

$$\begin{aligned} \bar{u}(\mathbf{k}, \sigma)\, e^{-i\sigma\theta} &= e^{-i\alpha\mathcal{A} - i\beta\mathcal{B}}\, e^{-i\theta\mathcal{J}^z}\, \bar{u}_\kappa(\mathbf{k}, \sigma)\,, \\ \bar{v}_\kappa(\mathbf{k}, \sigma)\, e^{i\sigma\theta} &= e^{-i\alpha\mathcal{A} - i\beta\mathcal{B}}\, e^{-i\theta\mathcal{J}^z}\, \bar{v}_\kappa(\mathbf{k}, \sigma)\,, \end{aligned} \tag{8.222}$$

with $\mathcal{A}$ and $\mathcal{B}$ given by (8.219) and the solution exists only for $\sigma = +\frac{1}{2}$ for $\bar{u}^{\dot{\alpha}}(\mathbf{k}, \sigma)$ and only for $\sigma = -\frac{1}{2}$ for $\bar{v}^{\dot{\alpha}}(\mathbf{k}, \sigma)$. We conclude that operators associated with massless spin 1/2 particles have the form

$$\lambda_\alpha(x) = \int d\Gamma_{\mathbf{p}} \left[u_\alpha(\mathbf{p}, -\tfrac{1}{2})\, e^{-ipx}\, b_{-\frac{1}{2}}(\mathbf{p}) + v_\alpha(\mathbf{p}, +\tfrac{1}{2})\, e^{+ipx}\, d^\dagger_{+\frac{1}{2}}(\mathbf{p}) \right], \tag{8.223}$$

$$\bar{\chi}^{\dot{\alpha}}(x) = \int d\Gamma_{\mathbf{p}} \left[\bar{u}^{\dot{\alpha}}(\mathbf{p}, +\tfrac{1}{2})\, e^{-ipx}\, b_{+\frac{1}{2}}(\mathbf{p}) + \bar{v}^{\dot{\alpha}}(\mathbf{p}, -\tfrac{1}{2})\, e^{+ipx}\, d^\dagger_{-\frac{1}{2}}(\mathbf{p}) \right], \tag{8.224}$$

i.e. the *left-chiral* operator $\lambda_\alpha(x)$ annihilates only *lefthanded* (i.e. of helicity $-1/2$) particles and creates only *righthanded* antiparticles whereas the *right-chiral* operator $\bar{\chi}^{\dot{\alpha}}(x)$ annihilates only *righthanded* (of helicity 1/2) particles and creates only *lefthanded* antiparticles.

8.7 Interacting and *in* and *out* field operators

The operators constructed in this section are called *free-field operators*. In the approach to quantum field theory based on quantum mechanics of relativistic particles these operators appear naturally and, since the interaction $V^I_{\rm int}(t)$ is built out of them, one obtains directly a working expansion for S-matrix elements formulated in terms of Feynman diagrams and Feynman rules (Chapter 9). In the general structure of quantum field theory the free-field operators play the role of the interaction picture operators and should always be carefuly distinguished from the Heisenberg (picture) operators also called *interacting* operators. Hamiltonians obtained by canonically quantizing classical fields (see Chapter 11) are naturally expressed through the latter type of operators (Heisenberg picture operators can also be introduced in the approach based on quantum mechanics of relativistic particles - see Section 13) and only upon the procedure called transition to the interaction picture (section 11.9) become expressed in terms of the interaction picture ones. Moreover, as will become clear, there is a considerable freedom (exploited for removing infinities from Green's functions) in chosing normalization of the Heisenberg picture operators but independently of the adopted normalization, for constructing perturbative expansion of off-shell Green's functions the Heisenberg picture operators taken at $x^0 = 0$ are always equated (see the discussion leading to the formula (13.10)) to the free field operators of the corresponding type (i.e. having the same Lorentz transformation properties) constructed in this section.

Since the *in* and *out* state-vectors $|\alpha_+\rangle$ (the eigevectors of H), $|\alpha_-\rangle$ are in one-to-one correspondence with the H_0 (or $\tilde{H}_0$) eigenvectors $|\alpha_0\rangle$ representing free partcles and have the same transformation properties with respect to the Poincaré symmetry group (and discrete operations P, C, and T, if H preserves these symmetries), it is possible to build *in* and *out* field operators $\phi_{\rm in}(x)$, $\psi_{\rm in}(x)$, $V^{\mu}_{\rm in}(x)$ etc., replacing everywhere the free particle creation and annihilation operators by the corresponding *in* and *out* creation and annihilation operators. The *in* and *out* field operators constructed in this way play the role in the LSZ reduction formula (to be disscussed in Section 13.4) which allows to extract S-matrix elements from the appropriate off-shell Green's functions.

Appendix F Four-component spinors

Here we collect some useful formulae for four-component spinors. We will work in both, the Dirac (8.65) and the chiral (8.67) representations of the Dirac matrices γ^μ. We choose the intrinsic parities defined by (8.82) to be $b_u = +1$ and $b_v = -1$.

In the chiral representation (8.67) we have $c_+ = b_u c_-$, $d_- = b_v d_+$. We choose then $c_+ = c_- = \sqrt{m}$ and $d_+ = -d_- = -\sqrt{m}$. (This choice of sign is motivated by the charge conjugation convention.) Thus,

$$
\begin{aligned}
u_\alpha(\mathbf{0}, \sigma = \tfrac{1}{2}) &= \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad & u_\alpha(\mathbf{0}, \sigma = -\tfrac{1}{2}) &= \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \\
v_\alpha(\mathbf{0}, \sigma = \tfrac{1}{2}) &= \sqrt{m}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}, \qquad & v_\alpha(\mathbf{0}, \sigma = -\tfrac{1}{2}) &= \sqrt{m}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}.
\end{aligned}
\tag{F.1}
$$

In the Dirac representation (8.65) we choose $c_+ = \sqrt{2m}$, $c_- = 0$ and $d_+ = 0$, $d_- = \sqrt{2m}$ so that

$$
\begin{aligned}
u_\alpha(\mathbf{0}, \sigma = \tfrac{1}{2}) &= \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad & u_\alpha(\mathbf{0}, \sigma = -\tfrac{1}{2}) &= \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \\
v_\alpha(\mathbf{0}, \sigma = \tfrac{1}{2}) &= \sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \qquad & v_\alpha(\mathbf{0}, \sigma = -\tfrac{1}{2}) &= \sqrt{2m}\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}.
\end{aligned}
\tag{F.2}
$$

In both we have

$$
\begin{aligned}
u^\dagger(\mathbf{0}, \sigma)\cdot u(\mathbf{0}, \sigma') = v^\dagger(\mathbf{0}, \sigma)\cdot v(\mathbf{0}, \sigma') &= 2m\,\delta_{\sigma\sigma'}\,, \\
u^\dagger(\mathbf{0}, \sigma)\cdot v(\mathbf{0}, \sigma') = v^\dagger(\mathbf{0}, \sigma)\cdot u(\mathbf{0}, \sigma') &= 0\,,
\end{aligned}
\tag{F.3}
$$

and

$$
\begin{aligned}
\bar{u}(\mathbf{0}, \sigma)\cdot u(\mathbf{0}, \sigma') &= 2m\,\delta_{\sigma\sigma'}\,, \\
\bar{v}(\mathbf{0}, \sigma)\cdot v(\mathbf{0}, \sigma') &= -2m\,\delta_{\sigma\sigma'}\,,
\end{aligned}
\tag{F.4}
$$

and finally

$$
\sum_\sigma u_\alpha(\mathbf{0}, \sigma)u^*_\beta(\mathbf{0}, \sigma) = 2m\left(\frac{1+\beta}{2}\right)_{\alpha\beta},
$$

$$\sum_{\sigma} v_\alpha(\mathbf{0},\sigma) v^*_\beta(\mathbf{0},\sigma) = 2m \left(\frac{1-\beta}{2}\right)_{\alpha\beta} . \tag{F.5}$$

The spinor C-matrix[1] C s defined by the relations

$$C\gamma^0\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu , \qquad \text{or} \qquad C\gamma^{\mu T} C^{-1} = -\gamma^\mu . \tag{F.6}$$

Explicitly $C = i\gamma^2\gamma^0$, so that

$$C = -C^T = -C^\dagger = -C^{-1} , \tag{F.7}$$

and $(C\gamma^0)^{-1} = C\gamma^0$. The product $C\gamma^0$ has the same form in the chiral and Dirac representations:

$$C\gamma^0 = i\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} . \tag{F.8}$$

With the choice of the factors $c_\pm$ and $d_\pm$ made in (8.85), (8.86) and (8.87), in both representations, (8.65) and (8.67), of the Dirac matrices

$$\begin{aligned} C\gamma^0\, u^*(\mathbf{0},\sigma) &= v(\mathbf{0},\sigma) , \\ C\gamma^0\, v^*(\mathbf{0},\sigma) &= u(\mathbf{0},\sigma) . \end{aligned} \tag{F.9}$$

It is also easy to check that

$$C\,\gamma^5\, C^{-1} = \gamma^5 , \qquad \text{i.e.} \quad [C,\ \gamma^5] = 0 , \tag{F.10}$$

and that the matrix $\gamma^5 C = C\gamma^5 = -\gamma^1\gamma^3$ which has the same form in both representations of the gamma matrices

$$\gamma^5 C = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} , \tag{F.11}$$

has the property (independent of the gamma matrices representation)

$$\begin{aligned} \gamma^5 C\, u(\mathbf{0},-\sigma) &= (-1)^{\frac{1}{2}+\sigma} u^*(\mathbf{0},\sigma) , \\ \gamma^5 C\, v(\mathbf{0},-\sigma) &= (-1)^{\frac{1}{2}+\sigma} v^*(\mathbf{0},\sigma) . \end{aligned} \tag{F.12}$$

[1]It is known as the charge conjugation matrix. This name which originates from the role it played in the old theory of the Dirac equation is misleading in the present context and will not be used.

For nonzero $\mathbf{p}$ the functions $u^*(\mathbf{p},\sigma)$ and $v^*(\mathbf{p},\sigma)$ are given by

$$\begin{aligned} u^*(\mathbf{p},\sigma) &= D^*(L_p)\, u^*(\mathbf{0},\sigma)\,, \\ v^*(\mathbf{p},\sigma) &= D^*(L_p)\, v^*(\mathbf{0},\sigma)\,. \end{aligned} \tag{F.13}$$

Since

$$D^*(L_p) = e^{\frac{1}{8}\omega_{\mu\nu}[\gamma^\mu,\ \gamma^\nu]^*} = C\gamma^0 e^{\frac{1}{8}\omega_{\mu\nu}[\gamma^\mu,\ \gamma^\nu]} C\gamma^0 = C\gamma^0 D(L_p) C\gamma^0\,, \tag{F.14}$$

we one gets

$$\begin{aligned} u^*(\mathbf{p},\sigma) &= C\gamma^0\, v(\mathbf{p},\sigma)\,, \\ v^*(\mathbf{p},\sigma) &= C\gamma^0\, u(\mathbf{p},\sigma)\,, \end{aligned} \tag{F.15}$$

or, because γ^0 is symmetric,

$$\begin{aligned} u(\mathbf{p},\sigma) &= C\gamma^0\, v^*(\mathbf{p},\sigma) = C\, \bar{v}^T(\mathbf{p},\sigma) \equiv -\bar{v}(\mathbf{p},\sigma)C\,, \\ v(\mathbf{p},\sigma) &= C\gamma^0\, u^*(\mathbf{p},\sigma) = C\, \bar{u}^T(\mathbf{p},\sigma) \equiv -\bar{u}(\mathbf{p},\sigma)C\,. \end{aligned} \tag{F.16}$$

The barred spinors can, in turn, be written as

$$\begin{aligned} \bar{u}(\mathbf{p},\sigma) &= -v^T(\mathbf{p},\sigma)\, C^{-1} \equiv C^{-1} v(\mathbf{p},\sigma)\,, \\ \bar{v}(\mathbf{p},\sigma) &= -u^T(\mathbf{p},\sigma)\, C^{-1} \equiv C^{-1} u(\mathbf{p},\sigma)\,. \end{aligned} \tag{F.17}$$