9 S-matrix and Feynman rules

We are now going to show how the formula (7.63) derived within the time-dependent formalism leads to covariant Feynman rules. These rules allow to organize perturbative computations of S-matrix elements in a transparent way, maintaining (in relativistic theories) manifest Lorentz invariance at every stage.[1]

We will begin by constructing simple interaction Hamiltonians of spin 0 and spin 1/2 particles represented by the simplest field operators constructed in Sections 8.2 and 8.3. Before formulating covariant Feynman rules which apply to interactions of such particles, what is a straightforward task, we will compute in the lowest order the amplitude (the S-matrix element) of the muon decay by applying the formula (7.63) directly to the realistic (effective) Hamiltonian of leptonic weak interactions. This example - although somewhat at odds with the general principles on which the theory of interacting relativistic particles developed here and in Chapter 7 is based (as all particles represented by the states $|\alpha_0\rangle$ related to the states $|\alpha_\pm\rangle$ must be absolutely stable within this approach, consistent treatment of decay processes requires investigating resonances of the partial wave amplitudes corresponding to appropriate reactions in which only stable particles take part, instead of computing their rates using S-matrix elements; however in the first order of the perturbative expansion the S-matrix approach does not lead to any troubles) - nicely illustrates how the "wave function" factors u_l and v_l in the notation of Chapter 8 (u, $\bar{u}$, v and $\bar{v}$ in the case of spin $\frac{1}{2}$ fermions) appear in the computations of rates of processes involving fermions in their initial and/or final state(s) and how they are associated with the external lines (representing particles in the initial and in the final state) of Feynman diagrams. After discussing these wave function factors in full generality and explaining the origin of Feynman propagators, general Feynman rules will be presented. Formulation of covariant Feynman rules pertaining to interactions of massive and massless spin 1 particles (and to all particles represented in the interaction Hamiltonian density by operators involving derivatives, like e.g. the operator (8.132) representing a spinless particle) is more complicated and will be be discussed in separate sections.

In the last section of this chapter identified will be an important condition which the interaction V_{int} (or $\mathcal{H}_{\text{int}}$) must satisfy if the postulates of Section 7.3, on which the method of direct computation of S-matrix elements with the help of the formula (7.63) is based, are to be respected. Relaxing this condition is possible but requires going beyond the formula (7.63) and reformulating the whole approach to the computation of S-matrix elements (that is to base it on the LSZ prescription discussed in Chapter 13). Still, within the perturbative expansion this will amount to a relatively small modification of the Feynman rules presented in this section.

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[1]The alternative approach to S-matrix elements computation consists of taking the interaction Hamiltonians $\mathcal{H}_{\text{int}}$ given in this section at $t = 0$ and in applying to them the "old-fashioned" time-ordered perturbative expansion based on the formulae (7.37) and (7.62) Lorentz invariance is then not manifest at intermediate steps because of the occurrence of "energy denominators".

9.1 Simple interactions

From the results of Chapter 8 it is clear that to obtain interaction Hamiltonian densities $\mathcal{H}_{\text{int}}(x)$ leading to covariant S-matrices satisfying the cluster decomposition principle one has to form Lorentz scalar densities out of field operators transforming as irreducible representations of the Lorentz group. An additional constraint is the requirement that the resulting Hamiltonians must be Hermitian - a condition which is necessary for unitarity of the S-matrix. Finally, $\mathcal{H}_{\text{int}}$ must be such that the *in* and *out* states of the full Hamiltonian H are in the one-to-one correspondence with the free particle states $|\alpha_0\rangle$ (which in particular implies - see Section 7.3 - that H and H_0 have the same spectra). This last requirement will be addressed in Section 9.7.

Let us consider first a theory of interacting massive (or massless) neutral spinless particles of one kind. The simplest interactions which can be built out of the single Hermitian field operator $\varphi(x) = \varphi^\dagger(x)$:

$$\varphi(x) = \int d\Gamma_{\mathbf{p}} \left(a(\mathbf{p}) e^{-ip\cdot x} + a^\dagger(\mathbf{p}) e^{ip\cdot x} \right), \tag{9.1}$$

are of the form[2]

$$\mathcal{H}_{\text{int}}(x) = \frac{g}{3!} \varphi^3(x) + \frac{\lambda}{4!} \varphi^4(x) + \frac{h}{5!} \varphi^5(x) + \frac{f}{6!} \varphi^6(x) + \dots \tag{9.2}$$

with real (Hermiticity!) constants (called *coupling constants*) g, λ, h etc. In principle $\mathcal{H}_{\text{int}}$ can consist of a finite or an infinite number of such terms. Each interaction term in (9.2) is obviously by itself a Lorentz scalar and, because $[\varphi(x), \varphi(y)] = 0$ for $(x-y)^2 < 0$, the interaction (9.2) is locally causal (i.e. satisfies the local causality requirement (7.109)). The interaction $V_{\text{int}}^I(t)$ entering the formula (7.63) is given by $V_{\text{int}}^I(t) = \int d^3\mathbf{x} \, \mathcal{H}_{\text{int}}(t, \mathbf{x})$. Since the free part H_0 of the full Hamiltonian must be (in order that the particles long before and long after the interaction behave as free particles with the appropriate Lorentz transformation properties) of the form

$$H_0 = \int d\Gamma_{\mathbf{p}} \, E(\mathbf{p}, m) \, a^\dagger(\mathbf{p}) a(\mathbf{p}), \tag{9.3}$$

with $E(\mathbf{p}, m) = \sqrt{\mathbf{p}^2 + m^2}$, where m is the mass of the considered particles, the operator (9.1) satisfies the relation

$$\varphi(t, \mathbf{x}) = e^{iH_0 t} \, \varphi(0, \mathbf{x}) \, e^{-iH_0 t}, \tag{9.4}$$

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[2] $\mathcal{H}_{\text{int}}$ can also have terms like

$$\frac{f'}{4} \left(\varphi(x) \partial_\mu \varphi(x) \right) \left(\varphi(x) \partial^\mu \varphi(x) \right)$$

in which spin zero particles are represented by vector operators $\partial_\mu \varphi(x)$; discussion of the Feynman rules for such interactions will be given in Section 9.5.

and the interaction operator $V_{\text{int}}^I(t)$ has the necessary property (7.106)

$$V_{\text{int}}^I(t) = e^{iH_0 t}\, V_{\text{int}}(0)\, e^{-iH_0 t} \equiv e^{iH_0 t}\, V_{\text{int}}\, e^{-iH_0 t}\,. \tag{9.5}$$

Thus, $V_{\text{int}}^I(t) = \int d^3\mathbf{x}\,\mathcal{H}_{\text{int}}(t, \mathbf{x})$ fulfills all the conditions formulated in Section 7.5 necessary for producing a Lorentz covariant S-matrix.

At this point a couple of remarks is in order. Firstly, the formula (9.3) and the adopted normalization of the $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ operators fix completely the normalization of the field operator $\varphi(x)$; any change in its overall scale (i.e. the arbitrariness in the real constants $\alpha_+ = \alpha_-$ in the formulae like (8.17)) can be absorbed in the coupling constants like g, λ, h etc. in (9.2). In general, the free field operators (operators in the interaction picture) constructed in Chapter 8 are normalized in such a way that at $t = 0$ they coincide with the field operators which will be obtained by quantizing the corresponding classical fields (possessing canonically normalized kinetic terms in the Lagrangian) in which procedure their scale is fixed by the canonical commutation relations (see Chapter 11). In fact, as said in the introduction to Chapter 7, relativistic quantum theories of particle interactions which are constructed here by adding interaction terms V_{int} to H_0's can also be obtained by quantizing classical field theories defined by appropriate Lorentz scalar Lagrangian densities. We follow here a different approach (in which the underlying "ontology" are particles) partly in order to show that any relativistic quantum mechanics of a finite number of types of particles[3] must necessarily take the form of a quantum theory of fields and partly to display the connection of the relativistic field theory with the nonrelativistic many-body theory formulated in the language of the second quantization. The approach to the relativistic field theory based on quantization of classical fields has many advantages (see Chapters 11, 16) but has the disadvantage, that it is much more abstract as far as fermions are concerned.

Secondly, in mathematically oriented textbooks interactions are usually built out of normal ordered products like $:\varphi^4(x):$ (of the interaction picture field operators[4]) in which (see Section 5.9) all creation operators (negative frequency parts of the field operators) stand to the left of all annihilation operators (positive frequency parts). This is partly motivated by the fact that the products like $\varphi^4(x)$ of operators out of which the interactions similar to (9.2) are built here are easily seen to be ill defined because many of their matrix elements between the state-vectors $|\alpha_0\rangle$ are infinite. Normal ordering does not spoil all the necessary (for relativistic covariance of the S-matrix) properties of the

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[3]The restriction to a finite number of types of particles is important: string theory is the best example of a relativistic theory that is not a quantum field theory. Quantum theory of strings gives rise to an infinite number of particle-like excitations with increasing masses and spins. Still, low energy interactions of the lowest (i.e. zero mass) string states can effectively cast in the form of an appropriate ordinary relativistic quantum field theory.

[4]The reader should be a warned that normal ordering of products of the (interaction picture) free operators defined in Section 5.9 should be carefully distinguished from the "normal ordering" of composite operators built out of the Heisenberg picture operators, which are used in some more advanced considerations. "Normally ordered" products of the latter type (unfortunately denoted frequently also by double colon) require in each case precise definition in terms of some their matrix elements.

interactions since any normal ordered product can be written as a linear combination of the ordinary products of operators with appropriate (formally infinite) coefficients (Section 5.9). However, even normally ordered interactions lead to ill defined expressions for S-matrix elements in higher orders of the perturbative expansion and it is therefore better to take care of all kinds of infinities by the uniform renormalization procedure discussed in Chapter 14. In fact, it seems that the only reason for building interaction Hamiltonians out of products of the interaction picture operators ordered normally with respect to the $|void\rangle$ vector is that this is necessary (see Section 5.3) for the strict equivalence of Hamiltonians of nonrelativistic N particle systems constructed within the second quantization formalism with the corresponding treatement of these systems based on N particle Schrödinger equations. Since the equivalence with the (many-body) Schrödinger equation approach is not a requirement for a relativistic theory of particles, we will use interaction Hamiltonians which will not be normally ordered.

Thirdly, to (9.2) the term $\propto \varphi^2(x)$ as well as the term $\propto \partial_\mu \varphi(x)\partial^\mu \varphi(x)$ could be added.[5] In fact, it will turn out in Section 9.7 and in Chapter 14) that precisely such terms (with appropriately adjusted coefficients) must be included in the interaction Hamiltonian densities $\mathcal{H}_{\text{int}}(x)$ in order to ensure that the Hamiltonian H has the same spectrum as H_0. However, these terms are not necessary for the calculations of S-matrix elements in the lowest order of the expansion of the formula (7.63) and for this reason will be ignored in this section.

Finally, terms with operators of dimension[6] higher than $[M]^4$, assuming that operators like $\varphi(x)$ of bosons have the dimension $[M]^1$ and those of fermions, like $\psi_\alpha(x)$ have dimension $[M]^{3/2}$ - this will acquire a justification in the approach based on field quantization but it follows also from the way these operators have been constructed in Chapter 8 - and that each derivative also counts as $[M]^1$, are known as nonrenormalizable (in four space-time dimensions) interactions and were in the past considered as not allowed (inclusion of one such term in the interaction density $\mathcal{H}_{\text{int}}(x)$ of a theory enforces inclusion of infinitely many of such terms with coupling constants which have to be determined from the data; this (superficially) seemed to lead to the complete lost of predictive power of such theories). At present renormalizability (discussed in detail in Chapter 14) is not viewed as a necessary feature of quantum field theories of low energy (low compared to, say, the electroweak or the Planck scales) particle interactions. In fact, nonrenormalizable interactions appear in the Hamiltonian of the effective (phenomenological) theory of a large class of important physical phenomena, like weak decays of leptons and hadrons, rare decays of hadrons, etc. Nonrenormalizable interactions are also necessary to account

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[5]With the terms built out of even numbers of the φ field operators only the theory has the $\mathbb{Z}_2$ symmetry $\varphi \to -\varphi$ (the Hamiltonian commutes with the parity operator $\mathcal{P}$ even if the spin zero particles are assigned negative intrinsic parity $\eta = -1$). This guarantees that no terms odd in the φ field operators will be necessary to remove infinities. Inclusion of the term $\propto \varphi^3$ would require including also at least the term $\propto \varphi$.

[6]Since we work with $\hbar = c = 1$, all dimensions of physical quantities can be expressed in terms of mass units $[M]$.

for neutrino masses in the Standard Model of particle interactions. The considerations of this section apply equally well to renormalizable and nonrenormalizable interactions.

If the spin 0 particles of a mass M carry a conserved quantum number Q and are therefore associated with the nonhermitian field operators $\phi(x)$ and $\phi^\dagger(x)$, the interaction Hamiltonian density can take the form[7]

$$\mathcal{H}_{\text{int}}(x) = \frac{\lambda}{2!2!} \left[\phi^\dagger(x)\phi(x)\right]^2 + \frac{h}{3!3!} \left[\phi^\dagger(x)\phi(x)\right]^3 + \dots, \tag{9.6}$$

where λ, h are two real coupling constants (the reasons for dividing these couplings by the factors 2!2! and 3!3! will be explained below). Again, the interaction (9.6) is locally causal and the interaction operator $V_{\text{int}}^I(t) = \int d^3\mathbf{x}\,\mathcal{H}_{\text{int}}(x)$ has the necessary property (9.5). In this case

$$H_0^{\text{bosons}} = \int d\Gamma_{\mathbf{p}} E(\mathbf{p}, M) \left[a^\dagger(\mathbf{p})a(\mathbf{p}) + a^{c\dagger}(\mathbf{p})a^c(\mathbf{p})\right]. \tag{9.7}$$

Obviously, it is also possible to construct a theory of interacting neutral spin 0 particles of mass m and charged particles of mass M. The interaction Hamiltonian density can take e.g. the form

$$\mathcal{H}_{\text{int}}(x) = \frac{g}{3!}\varphi^3(x) + h\varphi(x)\phi^\dagger(x)\phi(x) + \frac{\lambda}{4!}\varphi^4(x)$$
$$+ \frac{\kappa}{2}\varphi^2(x)\phi^\dagger(x)\phi(x) + \frac{\eta}{2!2!}\left[\phi^\dagger(x)\phi(x)\right]^2 + \dots \tag{9.8}$$

where for simplicity we have restricted $\mathcal{H}_{\text{int}}(x)$ to renormalizable (of dimension up to $[\text{M}]^4$) interactions only. The free Hamiltonian H_0 is then given by the sum of terms (9.3) and (9.7) and again the interaction $V_{\text{int}}^I = \int d^3\mathbf{x}\,\mathcal{H}_{\text{int}}(x)$ has the necessary property (9.5).

Out of the two field operators $\psi_\alpha(x)$ and $\bar{\psi}_\alpha(x)$ associated with charged spin $\frac{1}{2}$ particles and their antiparticles carrying some conserved charges Q and $-Q$, respectively (for simplicity think of Q as of the electric charge) one can form two different Lorentz scalars

$$\bar{\psi}(x)\psi(x) \equiv \sum_{\alpha=1}^{4} \bar{\psi}_\alpha(x)\psi_\alpha(x) \qquad \text{scalar}$$

$$\bar{\psi}(x)\gamma^5\psi(x) \equiv \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4} \bar{\psi}_\alpha(x)\gamma^5_{\alpha\beta}\psi_\beta(x) \qquad \text{pseudoscalar}$$

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[7]The more general interaction

$$\mathcal{H}_{\text{int}}(x) = \frac{g_3}{3!}\phi^3(x) + \frac{g_2}{2}\phi^2(x)\phi^\dagger(x) + \frac{g_2^*}{2}\phi^{\dagger 2}(x)\phi(x) + \frac{g_3^*}{3!}\phi^{\dagger 3}(x) + \frac{\lambda_4}{4!}\phi^4(x)$$
$$+ \frac{\lambda_3}{3!}\phi^3(x)\phi^\dagger(x) + \frac{\lambda_2}{2!}\phi^2(x)\phi^{\dagger 2}(x) + \frac{\lambda_3^*}{3!}\phi^{\dagger 3}(x)\phi(x) + \frac{\lambda_4^*}{4!}\phi^{\dagger 4}(x) + \dots$$

although Hermitian, does not commute with the charge operator $\hat{Q} = Q\int d\Gamma(a^\dagger a - a^{c\dagger}a^c)$ and would in higher orders split the masses of the two states, which in the zeroth order approximation seem to represent a particle and its antiparticle (thus breaking the assumptions which form the basis of our approach).

which can be used to construct interactions of these particles with neutral spin 0 particles. The simplest such interactions are:

$$\mathcal{H}_{\text{int}}(x) = h\,\varphi(x)\bar{\psi}(x)\psi(x), \quad \text{or} \quad \mathcal{H}_{\text{int}}(x) = ih\,\varphi(x)\bar{\psi}(x)\gamma^5\psi(x), \tag{9.9}$$

(i in the second type of interactions is necessary for its Hermiticity). Interactions of this general form are called *Yukawa interactions*. Both terms written above are Lorentz scalars and (after integration over $d^3\mathbf{x}$) commute with the parity operator $\mathcal{P}$ provided the field $\varphi(x)$ transforms under the space reflection as a scalar and a pseudoscalar, respectively. Coupling of a scalar field operator φ to the combination $\bar{\psi}(a + ib\gamma^5)\psi$ inevitably breaks parity conservation.[8] Since the sum of charges of each of field operators in both interactions (9.9) is zero ($Q - Q + 0 = 0$) they commute with the charge operator $\hat{Q}$. As previously, the interaction operator $V_{\text{int}}^I(t) = \int d^3\mathbf{x}\,\mathcal{H}_{\text{int}}(x)$ satisfies (9.5) if the free Hamiltonian is the sum of (9.3) and of

$$H_0^{\text{fermions}} = \int d\Gamma_{\mathbf{p}} E(\mathbf{p}, m_f) \sum_\sigma \left[b^\dagger(\mathbf{p}, \sigma)b(\mathbf{p}, \sigma) + d^\dagger(\mathbf{p}, \sigma)d(\mathbf{p}, \sigma) \right]. \tag{9.10}$$

The interactions (9.9) are locally causal and have therefore properties discussed in Section 7.5 necessary to produce Lorentz covariant S-matrices.

If the spin zero particles carry the same kind of charge as do the fermions, the simplest possible interaction conserving this charge and preserving parity is the nonrenormalizable (of dimension $[M]^5$) interaction of the form

$$\mathcal{H}_{\text{int}}(x) = h \left[\phi^\dagger(x)\phi(x) \right] \bar{\psi}(x)\psi(x). \tag{9.11}$$

Renormalizable (of dimension $[M]^4$) couplings of fermions to charged spin 0 particles (and their antiparticles) of charge Q ($-Q$) are possible only if there are two kinds of fermions (and antifermions) a and b with charges Q_a ($-Q_a$) and Q_b ($-Q_b$), such that $Q + Q_a - Q_b = 0$:

$$\mathcal{H}_{\text{int}}(x) = \phi(x)\,\bar{\psi}_b(x)\Gamma\psi_a(x) + \phi^\dagger(x)\,\bar{\psi}_a(x)\bar{\Gamma}\psi_b(x), \tag{9.12}$$

where $\Gamma = a + ib\gamma^5$ (we allow for parity non-conservation here). Hermiticity of $\mathcal{H}_{\text{int}}(x)$ requires that

$$\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0. \tag{9.13}$$

Indeed,

$$\left(\bar{\psi}_b\Gamma\psi_a \right)^\dagger = \left(\psi_b^\dagger\gamma^0\Gamma\psi_a \right)^\dagger = \psi_a^\dagger\Gamma^\dagger\gamma^0\psi_b = \psi_a^\dagger\gamma^0\gamma^0\Gamma^\dagger\gamma^0\psi_b = \bar{\psi}_a\bar{\Gamma}\psi_b,$$

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[8]Parity is indeed broken by weak interactions.

where we have used $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^0)^2 = 1$. Another important (nonrenormalizable, of dimension $[M]^6$) interaction of spin $\frac{1}{2}$ particles is the so-called *four-Fermi interaction* of the general form[9]

$$\mathcal{H}_{\text{int}} = \sum_A \left(\bar{\psi}_a \Gamma^A \psi_b\right)\left(\bar{\psi}_c \Gamma_A \psi_d\right) + \text{H.c.}, \tag{9.14}$$

in which Γ^A are some matrices in the spinor space (with real coefficients playing the role of the coupling constants) and H.c. stands for the Hermitian conjugation of the term written down explicitly. For example, if parity is to be conserved one can take $\Gamma \propto I$, $\Gamma \propto \gamma^5$, $\Gamma \propto \gamma^\mu$, $\Gamma \propto \gamma^\mu\gamma^5$, $\Gamma \propto \gamma^{[\mu}\gamma^{\nu]}$ (in other words, the whole basis (8.59) of the gamma matrix algebra can be used here). Interaction of the form (9.14), with $\gamma^\lambda(1 - \gamma^5)$ (the combination $1 - \gamma^5$ accounts for parity non-conservation) playing the role of Γ^A's accounts for a large part of low energy weak interactions of leptons.[10] The interaction Hamiltonian density takes then the "current-current" form[11]

$$\mathcal{H}_{\text{weak}}^{\text{lept}} = \frac{G_F}{\sqrt{2}} J_{\text{lept}}^\lambda (J_{\text{lept}}^\dagger)_\lambda\,, \tag{9.15}$$

with the weak leptonic "current" operator given by

$$J_{\text{lept}}^\lambda = \bar{\psi}_{(e)}\gamma^\lambda(1 - \gamma^5)\psi_{(\nu_e)} + \bar{\psi}_{(\mu)}\gamma^\lambda(1 - \gamma^5)\psi_{(\nu_\mu)} + \dots \tag{9.16}$$

(the ellipses stand for the third term with the τ lepton and its neutrino field operators) and G_F the real coupling constant (called Fermi constant) of dimension $[M]^{-2}$. This interaction gives rise, among other processes, to the decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu\,. \tag{9.17}$$

9.2 An example: the muon decay amplitude

We will now use the interaction (9.15) to calculate in the lowest order the S-matrix element corresponding to the muon decay (9.17) directly from the general formula

$$S_{\beta\alpha} = \langle\beta_-|\alpha_+\rangle = \langle\beta_0|\text{T}\exp\left(-i\int d^4x\, \mathcal{H}_{\text{int}}(x)\right)|\alpha_0\rangle$$

$$= \sum_{N=0}^\infty \frac{(-i)^N}{N!}\int d^4x_1 \dots \int d^4x_N \,\langle\Omega_0|b\dots d\dots a\dots \tag{9.18}$$

$$\text{T}\{\mathcal{H}_{\text{int}}(x_1)\dots\mathcal{H}_{\text{int}}(x_N)\}\, a^\dagger \dots d^\dagger \dots b^\dagger \dots |\Omega_0\rangle\,.$$

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[9]This form is the most general one if parity conservation is imposed. The discovery of parity violation enforced admitting a priori also other terms in the realistic effective Hamiltonian of weak interactions; the history of developement of the weak interactions theory is largely the history of elimination of most of the initially assumed interaction terms by using the experimental data.

[10]The development which led to this form of $\mathcal{H}_{\text{weak}}^{\text{lept}}$ is recalled in Chapter 12.

[11]The factor $1/\sqrt{2}$ is traditional.

This example will clearly show how the "wave function" factors - the functions $u_l(x, \mathbf{p}, \sigma)$ and $v_l(x, \mathbf{p}, \sigma)$ introduced in constructing free field operators (c.f. the formulae (8.2)) - enter the amplitudes of processes with fermions and antifermions in the initial and/or final states.

The state-vectors $|\alpha_0\rangle$ and $\langle\beta_0|$ which we have to consider in (9.18) are[12]

$$|\alpha_0\rangle = b_{(\mu)}^\dagger(\mathbf{q}, \sigma_q)|\Omega_0\rangle\,,$$
$$\langle\beta_0| = \langle\Omega_0|d_{(\nu_e)}(\mathbf{k}_1, \sigma_1)\, b_{(e)}(\mathbf{p}, \sigma_p)\, b_{(\nu_\mu)}(\mathbf{k}_2, \sigma_2)\,, \tag{9.19}$$

(the notation should be obvious). We will apply the formula (9.18) with the exponent expanded only to the first order

$$S_{\beta\alpha} = \langle\beta_0| - i\int d^4x\, \mathcal{H}_I^{\text{weak}}(x)|\alpha_0\rangle + \dots\,, \tag{9.20}$$

because a nontrivial contribution to the S-matrix element corresponding to the process (9.17) arises already in this order being generated by the term

$$\frac{G_F}{\sqrt{2}}\left(\bar\psi_{(e)}\gamma^\lambda(1 - \gamma^5)\psi_{(\nu_e)}\right)\left(\bar\psi_{\nu_\mu}\gamma_\lambda(1 - \gamma^5)\psi_{(\mu)}\right)\,, \tag{9.21}$$

present in the interction Hamiltonian density (9.15); we have also taken into account that $\langle\beta_0|\alpha_0\rangle = 0$ (the initial and final states (9.19) are different). Due to the structure of the initial and final states (9.19) the Hamiltonian entering (9.20) can be restricted to

$$\mathcal{H}_I^{\text{weak}}(x) \supset \frac{G_F}{\sqrt{2}}\left(\bar\psi_{(e)}^{(-)}\gamma^\lambda(1 - \gamma^5)\psi_{(\nu_e)}^{(-)}\right)\left(\bar\psi_{\nu_\mu}^{(-)}\gamma_\lambda(1 - \gamma^5)\psi_{(\mu)}^{(+)}\right)\,, \tag{9.22}$$

as it is only this part which can contribute in the first order (i.e. the only one which annihilates the muon and creates the electron, its antineutrino and the muonic neutrino). Using now the general structure (8.105) of the fermionic field operators one gets from (9.20):

$$\begin{aligned}
S_{\beta\alpha} \approx -i\frac{G_F}{\sqrt{2}}&\int d^4x\int d\Gamma_{\mathbf{q}'}\int d\Gamma_{\mathbf{p}'}\int d\Gamma_{\mathbf{k}_1'}\int d\Gamma_{\mathbf{k}_2'}\sum_{\sigma_q'}\sum_{\sigma_p'}\sum_{\sigma_1'}\sum_{\sigma_2'}e^{-iq'\cdot x}e^{ip'\cdot x}e^{ik_1'\cdot x}e^{ik_2'\cdot x}\\
&\times\left[\bar u_{(e)}(\mathbf{p}', \sigma_p')\gamma^\lambda(1 - \gamma^5)v_{(\nu_e)}(\mathbf{k}_1', \sigma_1')\right]\left[\bar u_{(\nu_\mu)}(\mathbf{k}_2', \sigma_2')\gamma_\lambda(1 - \gamma^5)u_{(\mu)}(\mathbf{q}', \sigma_q')\right]\\
&\times\langle\Omega_0|d_{(\nu_e)}(\mathbf{k}_1, \sigma_1)\, b_{(e)}(\mathbf{p}, \sigma_p)\, b_{(\nu_\mu)}(\mathbf{k}_2, \sigma_2)\, b_{(e)}^\dagger(\mathbf{p}', \sigma_p')\\
&\quad d_{(\nu_e)}^\dagger(\mathbf{k}_1', \sigma_1')\, b_{(\nu_\mu)}^\dagger(\mathbf{k}_2', \sigma_2')\, b_{(\mu)}(\mathbf{q}', \sigma_q')\, b_{(\mu)}^\dagger(\mathbf{q}, \sigma_q)|\Omega_0\rangle\,.
\end{aligned} \tag{9.23}$$

The integrals over $d\Gamma$'s, the summations over the primed indices as well as the exponential factors $e^{-iq'\cdot x}$ etc. come directly from the field operators present in (9.22). Notice, that only b's and d's (and $b^\dagger$'s and $d^\dagger$'s) are operators; the rest are c-number factors - owing

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[12] In our notation the operator $d_{(\nu_e)}$ annihilates the antiparticle of ν_e, that is $\bar\nu_e$.

to this they could be pulled outside the brackets. Moreover, the spinor indices of the functions u and v are completely contracted with the spinor indices of the gamma matrices just as they were in the original Hamiltonian (9.22); they are contracted within the square brackets, so that these brackets are just complex Lorentz vectors (they carry Lorentz indices λ). It should be also noted that the absolute sign of $S_{\beta\alpha}$ is unphysical because the state $\langle\beta_0|$ in (9.19) could be defined with a different ordering of the annihilation operators. However, once the initial and final states are specified, e.g. as in (9.19), this definition should be kept in computing other contributions to $S_{\beta\alpha}$ arising from higher order terms of the expansion of the formula (9.18) or from other terms of the interation Hamiltonian density contributing in the same order (absent in the considered example).

We now have to evaluate the matrix element in the expression (9.23). This can be done straightforwardly but in order to illustrate the general procedure which will be disscussed later (which is essentially the application of the Wick theorem of Section 5.9), we will do it in two steps. First we rearrange the operators, so that each creation operator stands to the right of the annihilation operator corresponding to the same type of (anti)particle. In the case at hand there is only one such ordering:

$$\left[b_{(e)}(\mathbf{p})b_{(e)}^\dagger(\mathbf{p}')\right]\left[d_{(\nu_e)}(\mathbf{k}_1)d_{(\nu_e)}^\dagger(\mathbf{k}_1')\right]\left[b_{(\nu_\mu)}(\mathbf{k}_2)b_{(\nu_\mu)}^\dagger(\mathbf{k}_2')\right]\left[b_{(\mu)}(\mathbf{q})b_{(\mu)}^\dagger(\mathbf{q}')\right] \qquad (9.24)$$

(for simplicity we have suppressed the spin labels). If there were for example two $b_{(\mu)}$'s (carrying of course independent momentum and spin labels) and two $b_{(\mu)}^\dagger$'s (also with different momentum and spin labels) two different orderings would be possible and one would have to sum expressions corresponding to both possibilities. To arrive at the ordering (9.24) from the one in (9.23) one had to pull $d_{(\nu_e)}^\dagger$ to the left past three operators and then to pull $b_{(e)}^\dagger$ to the left past one operator. Since all the operators are fermionic, these operations produce four minus signs.[13] In the second step we replace the pairs of operators enclosed in square brackets by the delta functions appearing in their anticommutation rules

$$\{b_{(e)}(\mathbf{p}',\sigma'),\ b_{(e)}^\dagger(\mathbf{p},\sigma)\} \rightarrow (2\pi)^3 2E(\mathbf{p},m)\,\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,\delta_{\sigma,\sigma'}\,, \qquad (9.25)$$

etc. The four integrals over $d\Gamma$'s, as well as four sums over spin labels, can be then performed. The last integral over d^4x produces then the delta function expressing the

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[13]In principle in this particular theory and, more generally, in the entire formalism of second quantization as presented in Chapter 5, there is nothing which would force one to adopt the convention that the creation and annihilation operators corresponding to *different* types of fermions anticommute. One could also adopt the prescription that only the annihilation and creation operators of the *same* particle and its antiparticle obey the *anticommutation* rules (because only this is required for the local causality to hold) and the ones corresponding to different particles commute. However, it sometimes turns out that particles which appeared to be unrelated to one another, are, from the point of view of a more fundamental theory, two internal states of the same particle (this is for example the case of e and ν_e, which in the Standard Model become the internal states of a doublet of the $SU(2)$ symmetry; the same is also true for μ and ν_μ and τ and ν_τ), so it is better to assume that fermionic operators always obey anticommutation rules and bosonic ones always the commutation rules.
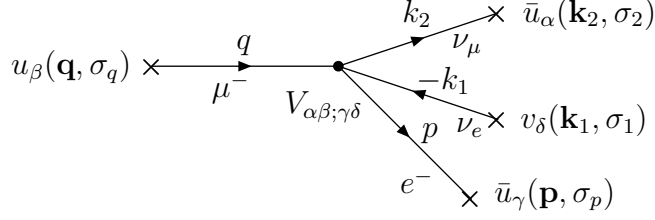
Figure 9.1: The Feynman diagram corresponding to the lowest order contribution to the amplitude of the muon decay. The lines closed (on one side) with $\times$ represent initial or final state particles. The spinor indices of the vertex factor $V_{\alpha\beta;\gamma\delta} = -i(G_F/\sqrt{2})[\gamma^\lambda(1 - \gamma^5)]_{\alpha\beta}[\gamma_\lambda(1 - \gamma^5)]_{\gamma\delta}$ are contracted with the indices of the "wave function" factors u_β, $\bar{u}_\alpha$, v_γ and u_δ.

conservation of the four-momentum and one gets:

$$S_{\beta\alpha} = (2\pi)^4 \delta^{(4)}(q - p - k_1 - k_2)\left(-i\mathcal{A}_{\beta\alpha}\right), \tag{9.26}$$

with

$$-i\mathcal{A} = -i\frac{G_F}{\sqrt{2}}\left[\bar{u}_{(e)}(\mathbf{p}, \sigma_p)\gamma^\lambda(1 - \gamma^5)v_{(\nu_e)}(\mathbf{k}_1, \sigma_1)\right]$$
$$\times \left[\bar{u}_{(\nu_\mu)}(\mathbf{k}_2, \sigma_2)\gamma_\lambda(1 - \gamma^5)u_{(\mu)}(\mathbf{q}, \sigma_q)\right]. \tag{9.27}$$

The presence of the overall delta function in the S-matrix element (9.26) is expected on general grounds discussed in Chapter 7. The amplitude $-i\mathcal{A}$ can be directly read off from the Feynman diagram shown in Figure 9.1. It is easy to see which element of the diagram corresponds to which factor of the expression (9.27). We stress only that the structure of the vertex reflects the structure of the interaction term (9.21): The two operators ψ which in (9.21) can annihilate μ^- and ν_e are in figure 9.1 identified with the lines entering the vertex (arrows pointing towards the vertex). Similarly, the two operators $\bar\psi$ which can create e^- and ν_μ are represented by the lines leaving the vertex. Note that one refers here to particles only, not to antiparticles! For this reason the lines entering and leaving the vertex are marked by the symbols of the *particles* which can be annihilated and created by the operators represented by these lines, respectively. Obviously, the same field operator $\psi_{(\nu_e)}$ which can annihilate ν_e can also create $\bar\nu_e$ and it is this second part of $\psi_{(\nu_e)}$ which contributed to the matrix element (9.22). Nevertheless, by convention the arrows in diagrams always show the possible "flow" of particles. This is why the arrow on the line representing in Figure 9.1 the electron antineutrino $\bar\nu_e$ produced in the muon decay is marked by ν_e (and not by $\bar\nu_e$) and the four-momentum of this line $(-k_1)$ is opposite to the physical four-momentum carried away by $\bar\nu_e$.

9.3   Wave functions, propagators and vertices

We now consider evaluation of the expression (9.18) with arbitrary initial and final states $|\alpha_0\rangle$ and $\langle\beta_0|$ and interactions $\mathcal{H}_{\text{int}}$ which are sums of arbitrary interaction terms like the ones discussed earlier: $\mathcal{H}_{\text{int}} = \sum_i \mathcal{H}_{(i)}$, with each $\mathcal{H}_{(i)}$'s being a local product of field operators. (Recall that derivatives of field operators, like those in (9.2), are, at this stage, treated as another kinds of operators). All terms in the perturbative expansion of the formula (9.18) can be organized according to powers of the different interaction terms they contain:

$$\mathrm{T}\exp\left(-i\int d^4x\,\mathcal{H}_{\text{int}}(x)\right) = \sum_{\{n_1,n_2,\dots\}} \tag{9.28}$$
$$\frac{(-i)^{n_1}}{n_1!}\frac{(-i)^{n_2}}{n_2!}\dots\int d^4x_{i_1}\dots\int d^4x_{i_{n_1}}\int d^4x_{j_1}\dots\int d^4x_{j_{n_2}}\dots$$
$$\mathrm{T}\big\{\mathcal{H}_{(1)}(x_{i_1})\dots\mathcal{H}_{(1)}(x_{i_{n_1}})\mathcal{H}_{(2)}(x_{j_1})\dots\mathcal{H}_{(2)}(x_{j_{n_2}})\dots\big\}\,.$$

In writing (9.28) we have used the fact that all interaction terms $\mathcal{H}_{(i)}(x)$, being necessarily bosonic operators - there is no way to couple an odd number of fermionic field operators to form a Lorentz scalar - commute under the symbol T of the time ordering operation.

   We now concentrate on evaluation of one particular term in the expansion (9.28) labeled by $\{n_1, n_2, \dots\}$ (sandwiched as in (9.18) between some initial $|\alpha_0\rangle$ and final $\langle\beta_0|$ states). This could be done "mechanically" by applying to it the Wick theorem disscussed in Section 5.9 but in order to better understand the role played by causality built in into the the formalism we will do it here step by step. Therefore we pick up first one particular ordering of the time variables $x_{i_1}^0, \dots, x_{i_{n_1}}^0, x_{j_1}^0, \dots, x_{j_{n_2}}^0, \dots$, so that the Hamiltonians $\mathcal{H}_{(i)}$ in (9.28) stand in a well defined fixed order. We split all the field operators present in $\mathcal{H}_{(i)}$'s into their positive (containing an annihilation operator) and negative (containing a creation operator) frequency parts. (As a result we get a sum of several matrix elements which must be computed.) We then pick up one, say the positive frequency part, of an operator (or an annihilation operator building the final state), and push it to the right with the aim of acting with it on the vacuum state $|\Omega_0\rangle$. This is done by successively (anti)commuting the selected operator with all other operators standing originally between it and $|\Omega_0\rangle$. Each (anti)commutation gives two terms: one in which the selected operator is already closer to $|\Omega_0\rangle$ and the second one in which there are two operators less and the remaining string of operators is multiplied by the value (which may be zero) of the (anti)commutator; in this second term we have finished with this particular operator and we repeat the same with another one. In the first term the selected positive frequency operator can either already act on $|\Omega_0\rangle$ and gives zero, or we again (anti)commute it with the next operator, get two terms and so on. After a finite number of steps we get rid of the selected operator and repeat the whole procedure with another positive frequency operator. We proceed similarly with the negative frequency parts of all operators (and with the creation operators building the initial state), except that we push them to the

left so that they eventually act on $\langle\Omega_0|$ and give zero. After a finite number of steps we get rid of all operators and, after using $\langle\Omega_0|\Omega_0\rangle = 1$, we are left with the expression which is a sum of products of c-number values of (anti)commutators and of other c-number factors originating from the Hamiltonians $\mathcal{H}_{(i)}$. Each term in the sum corresponds to one possible way of *pairing* annihilation operators (originating from the final state $\langle\beta_0|$) and positive frequency parts of field operators with creation operators (from the initial state $|\alpha_0\rangle$) and/or negative frequency parts of field operators, where "pairing" means replacing as in (9.25) the product of the annihilation (or positive frequency field) operator and the corresponding creation (or negative frequency field) operator of the same type by the c-number value of their (anti)commutator[14] (in the computation of the matrix element in (9.23) there was only one possible way (9.24) of pairing the operators). This sum has to be then multiplied by the appropriate theta functions enforcing the selected ordering of the time variables $x_{i_1}^0, \ldots, x_{i_{n_1}}^0, x_{j_1}^0, \ldots, x_{j_{n_2}}^0, \ldots$ The same has to be repeated with all other possible orderings of these variables and the results have to be added and finally integrated over all d^4x's.

As we are now going to explain, the contribution of the considered term $\{n_1, n_2, \ldots\}$ of the expansion (9.28) to $S_{\beta\alpha}$ can be obtained *without splitting* the operators into their positive and negative frequency parts as a sum of *unrestricted* integrals (i.e. without explicit theta functions introduced by the T-operation in (9.28) to divide the integration domains into regions with different fixed orderings of interaction Hamiltonians $\mathcal{H}_{(i)}$) over all d^4x's the integrands of which have the form of a product of factors which are of three basic types.

Factors of the first type are the *initial and final state wave function factors* (called shortly wave functions). They arise when a selected positive (or negative) frequency part of a field operator on its way to the right (left) gets paired with a creation (annihilation) operator building the initial (final) state. One then gets one of the following (anti)commutators

$$\left[\varphi^{(+)}(x),\ a^\dagger(\mathbf{p})\right] = e^{-ip\cdot x},$$
$$\left[a(\mathbf{p}'),\ \varphi^{(-)}(x)\right] = e^{+ip'\cdot x},$$
$$\left[\phi^{(+)}(x),\ a^\dagger(\mathbf{p})\right] = e^{-ip\cdot x},$$
$$\left[\phi^{\dagger(+)}(x),\ a^{c\dagger}(\mathbf{p})\right] = e^{-ip\cdot x},$$
$$\left[a(\mathbf{p}'),\ \phi^{(-)}(x)\right] = e^{+ip'\cdot x},$$
$$\left[a^c(\mathbf{p}'),\ \phi^{\dagger(-)}(x)\right] = e^{+ip'\cdot x}, \tag{9.29}$$
$$\left\{\psi_\alpha^{(+)}(x),\ b^\dagger(\mathbf{p},\sigma)\right\} = u_\alpha(\mathbf{p},\sigma)\,e^{-ip\cdot x},$$

[14]If in a string of operators standing between $\langle\Omega_0|$ and $|\Omega_0\rangle$ the number of creation operators of a given type (counting also those contained in the negative frequency fields) is different than the number of the corresponding annihilation operators (again counting those contained in the positive frequency fields), then not all operators can get paired and such a matrix element vanishes. For this reason some of the terms in the expansion (9.28) may trivially give zero.

$$\left\{ \bar{\psi}_\alpha^{(+)}(x), \ d^\dagger(\mathbf{p},\sigma) \right\} = \bar{v}_\alpha(\mathbf{p},\sigma)\, e^{-ip\cdot x}\,,$$
$$\left\{ b(\mathbf{p}',\sigma'), \ \bar{\psi}_\alpha^{(-)}(x) \right\} = \bar{u}_\alpha(\mathbf{p}',\sigma')\, e^{+ip'\cdot x}\,,$$
$$\left\{ d(\mathbf{p}',\sigma'), \ \psi_\alpha^{(-)}(x) \right\} = v_\alpha(\mathbf{p}',\sigma')\, e^{+ip'\cdot x}\,,$$
$$\left[V_\mu^{(+)}(x), \ a^\dagger(\mathbf{p},\sigma) \right] = \epsilon_\mu(\mathbf{p},\sigma)\, e^{-ip\cdot x}\,,$$
$$\left[a(\mathbf{p}',\sigma), \ V_\mu^{(-)}(x) \right] = \epsilon_\mu^*(\mathbf{p}',\sigma')\, e^{+ip'\cdot x}\,,$$

where $\varphi(x)$ is the field operator of neutral spin 0 particles, $\phi(x)$ and $\phi^\dagger(x)$ are the field operators of charged spin 0 particles and their antiparticles, $\psi_\alpha(x)$ and $\bar{\psi}_\beta(x)$ are the operators of spin $\frac{1}{2}$ particles and their antiparticles and finally $V_\mu(x)$ is the field operator of a neutral (massive or massless) particle of spin 1. These are the factors corresponding to the initial and final states. Four of them have already been encountered in writing down the amplitude (9.23) of the muon decay.

The second type of factors are the *vertex factors*. These are *c*-number factors (i.e. not Hilbert space operators) like the factor $V_{\alpha\beta;\gamma\delta}$ corresponding to the vertex in the diagram of Figure 9.1, present in the interaction terms $\mathcal{H}_{(i)}$. They are composed of coupling constants, matrices in the spinor space etc.[15] that are left over after stripping off all field operators present in $\mathcal{H}_{(i)}$.

The third type of factors are the *propagators*. To understand their origin we note that a selected positive frequency operator originatiging from the interaction term $\mathcal{H}_{(i)}(x)$ can on its way to the right get paired with a negative frequency operator of the same type originating from another interaction term $\mathcal{H}_{(j)}(y)$ (or even from the same interaction term, if the normal ordering is not assumed) . This can only happen when $x^0 > y^0$. For example, the operators $\varphi^{(+)}(x)$, $\phi^{(+)}(x)$, $\phi^{\dagger(+)}(x)$, $\psi_\alpha^{(+)}(x)$, $\bar{\psi}_\alpha^{(+)}(x)$ and $\partial_\mu \phi^{(+)}(x)$ can get paired with $\varphi^{(-)}(y)$, $\phi^{\dagger(-)}(y)$, $\phi^{(-)}(y)$, $\bar{\psi}_\beta^{(-)}(y)$, $\psi_\beta^{(-)}(y)$ and $\partial_\nu \phi^{\dagger(-)}(y)$, respectively. Such pairings give e.g.[16]

$$\theta(x^0 - y^0)\left[\varphi^{(+)}(x), \ \varphi^{(-)}(y) \right],$$
$$\theta(x^0 - y^0)\left[\phi^{(+)}(x), \ \phi^{\dagger(-)}(y) \right],$$
$$\theta(x^0 - y^0)\left[\phi^{\dagger(+)}(x), \ \phi^{(-)}(y) \right], \tag{9.30}$$
$$\theta(x^0 - y^0)\left\{ \psi_\alpha^{(+)}(x), \ \bar{\psi}_\beta^{(-)}(y) \right\},$$
$$\theta(x^0 - y^0)\left\{ \bar{\psi}_\alpha^{(+)}(x), \ \psi_\beta^{(-)}(y) \right\},$$
$$\theta(x^0 - y^0)\left[\partial_\mu \phi^{(+)}(x), \ \partial_\nu \phi^{\dagger(-)}(y) \right],$$

etc. The factors $\theta(x^0 - y^0)$ come from the strings of the theta functions enforcing the considered ordering of the time variables. However, among all the orderings of the time

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[15]Note that at this stage derivatives acting on field operators in $\mathcal{H}_{(i)}$'s are treated as parts of these operators and are accounted for in pairings. Later the Feynman rules will be reformulated so that the derivatives will be treated as parts of the vertex factors.

[16]Of course pairings of the operators built out of the annihilation and creation operators of different particles vanish.

variables $x^0_{i_1}, \ldots, x^0_{i_{n_1}}, x^0_{j_1}, \ldots, x^0_{j_{n_2}}, \ldots$, which must be considered in connection with the presence of the T-operation in the formula (9.28), there are also ones in which $y^0 > x^0$ and all the other time variables are in the previous order. In some of the terms arising in evaluation of the matrix elements corresponding to these orderings the operators $\varphi^{(+)}(y)$, $\phi^{\dagger(+)}(y)$, $\phi^{(+)}(y)$, $\bar{\psi}^{(+)}_\beta(y)$, $\psi^{(+)}_\beta(y)$ and $\partial_\nu \phi^{\dagger(+)}(y)$ will get paired with $\varphi^{(-)}(x)$, $\phi^{(-)}(x)$, $\phi^{\dagger(-)}(x)$, $\psi^{(-)}_\alpha(x)$, $\bar{\psi}^{(-)}_\alpha(x)$ and $\partial_\mu \phi^{(-)}(x)$ yielding

$$\theta(y^0 - x^0)\left[\varphi^{(+)}(y),\ \varphi^{(-)}(x)\right],$$
$$\theta(y^0 - x^0)\left[\phi^{\dagger(+)}(y),\ \phi^{(-)}(x)\right],$$
$$\theta(y^0 - x^0)\left[\phi^{(+)}(y),\ \phi^{\dagger(-)}(x)\right], \tag{9.31}$$
$$\theta(y^0 - x^0)\{\bar{\psi}^{(+)}_\beta(y),\ \psi^{(-)}_\alpha(x)\},$$
$$\theta(y^0 - x^0)\{\psi^{(+)}_\beta(y),\ \bar{\psi}^{(-)}_\alpha(x)\},$$
$$\theta(y^0 - x^0)\left[\partial_\nu\phi^{\dagger(+)}(y),\ \partial_\mu\phi^{(-)}(x)\right],$$

etc. The "miracle" which happens when contributions arising from different time orderings of the variables $x^0_{i_1}, \ldots, x^0_{i_{n_1}}, x^0_{j_1}, \ldots, x^0_{j_{n_2}}, \ldots$ are added is that the (anti)commutators arising for $x^0 > y^0$ and for $y^0 > x^0$ always combine together into the *Feynman propagators* like

$$\begin{aligned}
i\Delta^{\mathrm{F}}(x - y) &\equiv \theta(x^0 - y^0)\left[\phi^{(+)}(x),\ \phi^{\dagger(-)}(y)\right] \\
&+ \theta(y^0 - x^0)\left[\phi^{(+)}(y),\ \phi^{\dagger(-)}(x)\right] \\
&= \theta(x^0 - y^0)\,\Delta_+(x - y) + \theta(y^0 - x^0)\,\Delta_+(y - x) \\
&\equiv \langle\Omega_0|\mathrm{T}(\phi(x)\phi^\dagger(y))|\Omega_0\rangle,
\end{aligned} \tag{9.32}$$

corresponding to scalar field operators associated with spin 0 particles and[17]

$$\begin{aligned}
iS^{\mathrm{F}}_{\alpha\beta}(x - y) &\equiv \theta(x^0 - y^0)\{\psi^{(+)}_\alpha(x),\ \bar{\psi}^{(-)}_\beta(y)\} \\
&- \theta(y^0 - x^0)\{\bar{\psi}^{(+)}_\beta(y),\ \psi^{(-)}_\alpha(x)\} \\
&\equiv \langle\Omega_0|\mathrm{T}(\psi_\alpha(x)\bar{\psi}_\beta(y))|\Omega_0\rangle,
\end{aligned} \tag{9.33}$$

corresponding to the four-component spinorial field operators associated with fermions. (Propagators corresponding to vector fields will be discussed in Sections 9.5 and 9.6). The minus sign between the two terms in the last formula arises because fermion field operators have to be anticommuted to get the required parings. The "miraculous" combination of the (anti)commutators can be expected on the basis of our discussion at the end of Section 8.2 explaining how antiparticles save causality. The pairings (9.30) represent the probability amplitudes that a particle (antiparticle) born at y in the interaction $\mathcal{H}_{(i)}(y)$

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[17]The function $\Delta_+(x - y)$ has been defined in (8.21); $S^{\mathrm{F}}_{\alpha\beta}$ denotes here the propagator and its spinor indices; it should not be confused with the S-matrix element $S_{\alpha\beta}$! The representations of the propagators $i\Delta^F(x - y)$ and $S^{\mathrm{F}}_{\alpha\beta}(x - y)$ as the H_0 ground state expectations values of the chronological products of the free field operators follow also directly from the Wick theorem discussed in Section 5.9.

propagates to x, where it is annihilated by the interaction $\mathcal{H}_{(j)}(x)$. This can happen only for $x^0 > y^0$. But obviously, if $y^0 > x^0$ its antiparticle (particle) can be created at x by $\mathcal{H}_{(j)}(x)$ and can propagate to y to be destroyed by $\mathcal{H}_{(i)}(y)$. For $(x - y)^2 < 0$ this is required by causality; but the the structure of field operators is such that if it is required for $(x - y)^2 < 0$ it must hold also for any x and y.

This directly leads to the technique of Feynman diagrams in the position space, which represent some sort of a space-time picture of the physical process in which the initial state $|\alpha_0\rangle$ evolves into the final state $|\beta_0\rangle$. According to this picture, the total probability amplitude is given by the sum of probability amplitudes corresponding to all particular ways the system can go from $|\alpha_0\rangle$ to $|\beta_0\rangle$. In this sense it is directly rooted in the basic principles of quantum mechanics stating that amplitudes corresponding to all physically indistinguishable alternatives should be added.

To show how the theta functions introduced by the T-operation in (9.28) can be encoded in the propagators we now derive compact representations for these objects. Consider the spin zero particle propagator (9.32) first. Using the integral representations of the theta function

$$\theta(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega \, \frac{e^{-i\omega t}}{\omega + i0} = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega \, \frac{e^{+i\omega t}}{\omega - i0} \,, \tag{9.34}$$

$i\Delta^{\mathrm{F}}(x - y)$ defined in (9.32) can be rewritten in the form

$$i\Delta^{\mathrm{F}}(x - y) = i \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i(\omega + E_{\mathbf{p}})(x^0 - y^0)} e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\omega + i0}$$

$$- i \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i(\omega - E_{\mathbf{p}})(x^0 - y^0)} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\omega - i0} \,.$$

Substituting next $\omega = p^0 - E_{\mathbf{p}}$ in the first integral and $\omega = p^0 + E_{\mathbf{p}}$ in the second one (and changing in the latter $\mathbf{p} \to -\mathbf{p}$) we obtain the covariant form of the propagator

$$i\Delta^{\mathrm{F}}(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i0} e^{-ip\cdot(x-y)} \,. \tag{9.35}$$

The propagator (9.33) corresponding to pairings of the operators $\psi_\alpha(x)$ and $\bar{\psi}_\beta(y)$ is obtained similarly (but some manipulations with the gamma matrices are necessary) and reads

$$iS^{\mathrm{F}}_{\alpha\beta}(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\slashed{p} + m)_{\alpha\beta}}{p^2 - m^2 + i0} e^{-ip\cdot(x-y)}$$

$$\equiv \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{\slashed{p} - m + i0} \right)_{\alpha\beta} e^{-ip\cdot(x-y)} \,, \tag{9.36}$$

The spinor indices $\alpha\beta$ are explicitly shown here.[18]

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[18]It is perhaps worth saying that the functions (9.35) and (9.36) satisfy the equations

$$(\partial_\mu \partial^\mu + M^2) \, i\Delta^{\mathrm{F}}(x - y) = -i\delta^{(4)}(x - y) \,,$$

381

Before summarizing the evaluation of the formula (9.28) between arbitrary initial and final states let us illustrate using a simple example how the propagators arise. Let us consider a theory of neutral spin 0 bosons and spin $\frac{1}{2}$ fermions and their antifermions the interaction of which is given by

$$\mathcal{H}_{\text{int}}(x) = \varphi(x)\,\bar{\psi}_\alpha(x)\,\Gamma_{\alpha\beta}\psi_\beta(x)\,, \tag{9.37}$$

with $\Gamma = hI$ or $ih\gamma^5$ (h is a real coupling constant), so that $\mathcal{H}_{\text{int}}^\dagger = \mathcal{H}_{\text{int}}$, depending on whether the spin zero particle is a scalar or a pseudoscalar, and let us compute the amplitude of the elastic scattering of two fermions. The first nontrivial contribution to the relevant S-matrix element comes from the second term of the expansion (9.28):

$$\frac{(-i)^2}{2!} \int d^4x \int d^4y \,\langle\Omega_0|b_{1'}b_{2'}\,\mathrm{T}\,\{\mathcal{H}_{\text{int}}(x)\mathcal{H}_{\text{int}}(y)\}\,b_2^\dagger b_1^\dagger|\Omega_0\rangle\,, \tag{9.38}$$

where we have simplified the notation: $b(\mathbf{p}_1,\sigma_1) \equiv b_1$ etc. The first order term in the expansion (9.28) gives zero, because with a single $\mathcal{H}_{\text{int}}(x)$ not all operators can be paired. By definition the zeroth order term

$$\langle\Omega_0|b_{1'}b_{2'}b_2^\dagger b_1^\dagger|\Omega_0\rangle = \delta_\Gamma(1-1')\delta_\Gamma(2-2') - \delta_\Gamma(1-2')\delta_\Gamma(2-1')\,, \tag{9.39}$$

does not contribute to the amplitude $\mathcal{A}$. Taking into account the composition of the initial and final states, we can restrict the field operators in (9.38) to the following two terms

$$\frac{(-i)^2}{2!} \int d^4x \int d^4y \,\{\theta(x^0 - y^0)\,[\bar\psi\Gamma\psi^{(+)}\varphi^{(+)}(x)]\,[\bar\psi^{(-)}\Gamma\psi\varphi^{(-)}(y)]$$
$$+\theta(y^0 - x^0)\,[\bar\psi\Gamma\psi^{(+)}\varphi^{(+)}(y)]\,[\bar\psi^{(-)}\Gamma\psi\varphi^{(-)}(x)]\}\,.$$

The terms we have dropped can act directly on $|\Omega_0\rangle$ or $\langle\Omega_0|$ and give zero.[19] In the $x^0 > y^0$ term $\varphi^{(+)}(x)\varphi^{(-)}(y) = \varphi^{(-)}(y)\varphi^{(+)}(x) + \Delta_+(x-y)$ but then $\varphi^{(+)}(x)$ can act on $|\Omega_0\rangle$

$$(i\,\slashed\partial - m)\,iS^{\mathrm{F}}_{\alpha\beta}(x - y) = i\delta_{\alpha\beta}\delta^{(4)}(x - y)\,,$$

that is, they are the (Feynmanian) Green's functions of the Klein-Gordon and Dirac equations respectively. In many textbooks (see eg. the ones of Bjorken and Drell or of Itzykson and Zuber) propagators appear in the prescription for computing S-matrix elements as Green's functions of the wave equations (taking the Feynmanian Green's functions - which propagate positive frequency waves forward and negative frequency waves backward in time - and not e.g. the advanced ones which might seem more appropriate on the basis of causality - is then decided by appealing to the messy idea of Feynman that while particles propagate in time forward, antiparticles propagate backward) making the false impression that quantum field theory is based on wave equations. This is not the point of view adopted here.

[19] It is tacitly assumed here that that the interaction $\mathcal{H}_{\text{int}}(x)$ is normally ordered, $\mathcal{H}_{\text{int}}(x) =: \mathcal{H}_{\text{int}}(x):$ so that $:\bar\psi_\alpha^{(+)}(x)\psi_\beta^{(-)}(x): = -\psi_\beta^{(-)}(x)\bar\psi_\alpha^{(+)}(x)$ and therefore there is no extra anticommutator term when e.g. $\psi_\beta^{(-)}(x)$ in the first term is moved to the left to act on $\langle\Omega_0|$. Alternatively, without normal ordering, we would get some extra (divergent) contribution to the S-matrix element, which would be proportional to the zeroth order term (9.39) and would have to be renormalized away - see the discussion at the end of Section 9.4.

giving zero, so only $\Delta_+(x-y)$ remains. Likewise, in the $y^0 > x^0$ term we are left with the $\Delta_+(y-x)$ term. Next we move $\psi^{(+)}(x)$ in the $x^0 > y^0$ term to the right past $\bar\psi^{(-)}(y)$. This gives two terms: $-\bar\psi^{(-)}(y)\psi^{(+)}(x)$ and the anticommutator $\{\psi^{(+)}(x),\ \bar\psi^{(-)}(y)\}$. Similarly, in the $y^0 > x^0$ term we get $-\bar\psi^{(-)}(x)\psi^{(+)}(y)$ and the anticommutator $\{\psi^{(+)}(y),\ \bar\psi^{(-)}(x)\}$. Let us consider first the terms with $-\bar\psi^{(-)}(y)\psi^{(+)}(x)$ and $-\bar\psi^{(-)}(x)\psi^{(+)}(y)$ in the $x^0 > y^0$ and $y^0 > x^0$ pieces, respectively (the other terms will be briefly considered below). Let us pick up the negative and positive frequency parts of the operators $\bar\psi$ and ψ standing on the extreme left and right, respectively in both these pieces. The terms arising from taking their negative and positive frequency parts, respectively, give

$$-\frac{(-i)^2}{2!}\,\Gamma_{\alpha\gamma}\Gamma_{\beta\delta}\int d^4x \int d^4y\,\theta(x^0-y^0)\,\Delta_+(x-y)$$
$$\times\langle\Omega_0|b_{1'}b_{2'}\bar\psi_\alpha^{(-)}(x)\bar\psi_\beta^{(-)}(y)\psi_\gamma^{(+)}(x)\psi_\delta^{(+)}(y)b_2^\dagger b_1^\dagger|\Omega_0\rangle$$
$$-\frac{(-i)^2}{2!}\,\Gamma_{\alpha\gamma}\Gamma_{\beta\delta}\int d^4x \int d^4y\,\theta(y^0-x^0)\,\Delta_+(y-x)$$
$$\times\langle\Omega_0|b_{1'}b_{2'}\bar\psi_\alpha^{(-)}(y)\bar\psi_\beta^{(-)}(x)\psi_\gamma^{(+)}(y)\psi_\delta^{(+)}(x)b_2^\dagger b_1^\dagger|\Omega_0\rangle\,.$$

In the second term we simultaneously interchange the order within the two pairs of operators: in $\bar\psi_\alpha^{(-)}(y)\bar\psi_\beta^{(-)}(x)$ and in $\psi_\gamma^{(+)}(y)\psi_\delta^{(+)}(x)$ so that the two resulting minuses cancel each other. In addition we interchange the names of the spinor indices $\alpha\leftrightarrow\beta$ and $\gamma\leftrightarrow\delta$. After these operations the string of operators in the $y^0 > x^0$ piece is the same as the one in the $x^0 > y^0$ piece. Hence, using the definition (9.32), the considered contribution to the S-matrix element takes the form

$$-\frac{(-i)^2}{2!}\,\Gamma_{\alpha\gamma}\Gamma_{\beta\delta}\int d^4x \int d^4y\,i\Delta^F(x-y)$$
$$\times\langle\Omega_0|b_{1'}b_{2'}\bar\psi_\alpha^{(-)}(x)\bar\psi_\beta^{(-)}(y)\psi_\gamma^{(+)}(x)\psi_\delta^{(+)}(y)b_2^\dagger b_1^\dagger|\Omega_0\rangle\,. \tag{9.40}$$

There are now four possible pairings of the fermion field operators with the creation and annihilation operators building the initial and final states. They give pairwise equal contributions, so that the factor $1/2!$ in (9.40) arising from the expansion of the exponent in (9.28) is canceled out and the result is

$$(-i)^2\,[\bar u(1')\Gamma u(1)]\,[\bar u(2')\Gamma u(2)]$$
$$\times\int d^4x \int d^4y\,e^{ip_1'\cdot x}e^{-ip_1\cdot x}\,i\Delta^F(x-y)\,e^{ip_2'\cdot y}e^{-ip_2\cdot y}$$
$$-(-i)^2\,[\bar u(2')\Gamma u(1)]\,[\bar u(1')\Gamma u(2)] \tag{9.41}$$
$$\times\int d^4x \int d^4y\,e^{ip_2'\cdot x}e^{-ip_1\cdot x}\,i\Delta^F(x-y)\,e^{ip_1'\cdot y}e^{-ip_2\cdot y}\,.$$

Taking now the integrals over x and y and using (9.35) one obtains the following contribution to the S-matrix element:

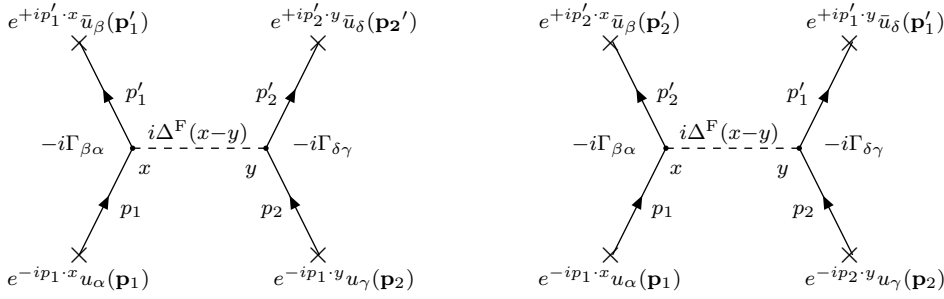$$(S-1)_{\beta\alpha} = (2\pi)^4\delta^{(4)}(p_2'+p_1'-p_2-p_1)$$

Figure 9.2: Graphical interpretation of the two contributions (9.41) to the fermion-fermion scattering amplitude. The amplitude is obtained by integrating the expressions corresponding to these diagrams over $d^4x \, d^4y$ - the space-time positions of the two interaction points.

$$\times (-i)^2 \left\{ [\bar{u}(1')\Gamma u(1)] \frac{i}{(p_1' - p_1)^2 - M^2 + i0} [\bar{u}(2')\Gamma u(2)] \right. \tag{9.42}$$
$$\left. - [\bar{u}(2')\Gamma u(1)] \frac{i}{(p_2' - p_1)^2 - M^2 + i0} [\bar{u}(1')\Gamma u(2)] \right\}.$$

The two terms enter in (9.41) with the relative minus sign (which comes out automatically from the the parings) because they correspond, loosely speaking, to interchanging two fermions in the final (or in the initial) state. These two terms are represented graphically by the two Feynman diagrams in Figure 9.2.

Having identified three basic types of factors we can reformulate the procedure of evaluating the $\{n_1, n_2, \ldots\}$ term of the formula (9.28) as follows. We first pull all the c-number vertex factors and integrals over d^4x's out of the Dirac brackets. This leaves between the vacuum vectors $\langle \Omega_0 |$ and $| \Omega_0 \rangle$ only a string of creation and annihilation operators building the initial and final states and field operators (under the sign of time ordering) arising from $\mathcal{H}_{(i)}$'s. To evaluate this vacuum matrix element we write down all possible groupings of these operators into pairs in which the operators ψ_α, ϕ, φ and the annihilation operators (building the final states) stand to the left of the operators $\bar{\psi}_\beta$, $\phi^\dagger$, φ and the creation operators (building the initial state). That is, only the orderings consisting of pairs (note that now we do not split field operators into their positive and negative frequency parts)

$$[b\bar{\psi}], \quad [d\psi], \quad [\psi b^\dagger], \quad [\bar{\psi}d^\dagger], \quad [\psi\bar{\psi}], \quad [bb^\dagger], \quad [dd^\dagger],$$

and

$$[a\phi^\dagger], \quad [a^c\phi], \quad [\phi a^\dagger], \quad [\phi^\dagger a^{c\dagger}], \quad [\phi\phi^\dagger], \quad [a\varphi], \quad [\varphi a^\dagger], \quad [\varphi\varphi],$$

etc. are considered. The sign of each ordering is $+$ or $-$ depending on how many interchanges of fermionic operators are needed to get a given ordering[20] from the original one.

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[20]Note that pairs of operators, that is the entire brackets $[\ldots]$, in a given sequence of pairs can be

For example, the matrix element of the chronological product of operators in (9.38) yields the following strings of pairs (not assuming normal ordering of $\mathcal{H}_{\text{int}}$)

$$\langle\Omega_0|b_{1'}b_{2'}\mathrm{T}\big\{\bar{\psi}_\alpha(x)\psi_\gamma(x)\varphi(x)\bar{\psi}_\beta(y)\psi_\delta(y)\varphi(y)\big\}b_2^\dagger b_1^\dagger|\Omega_0\rangle$$

$$\begin{aligned}
\rightarrow\; &+[b_{1'}\bar{\psi}_\alpha(x)][b_{2'}\bar{\psi}_\beta(y)][\varphi(x)\varphi(y)][\psi_\delta(y)b_2^\dagger][\psi_\gamma(x)b_1^\dagger]\\
&-[b_{1'}\bar{\psi}_\beta(y)][b_{2'}\bar{\psi}_\alpha(x)][\varphi(x)\varphi(y)][\psi_\delta(y)b_2^\dagger][\psi_\gamma(x)b_1^\dagger]\\
&-[b_{1'}\bar{\psi}_\alpha(x)][b_{2'}\bar{\psi}_\beta(y)][\varphi(x)\varphi(y)][\psi_\gamma(x)b_2^\dagger][\psi_\delta(y)b_1^\dagger]\\
&+[b_{1'}\bar{\psi}_\beta(y)][b_{2'}\bar{\psi}_\alpha(x)][\varphi(x)\varphi(y)][\psi_\gamma(x)b_2^\dagger][\psi_\delta(y)b_1^\dagger]\\
&-[b_{1'}b_1^\dagger][b_{2'}b_2^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\beta(y)][\psi_\delta(y)\bar{\psi}_\alpha(x)]\\
&+[b_{1'}b_2^\dagger][b_{2'}b_1^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\beta(y)][\psi_\delta(y)\bar{\psi}_\alpha(x)]\\
&+[b_{1'}b_1^\dagger][b_{2'}b_2^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\alpha(x)][\psi_\delta(y)\bar{\psi}_\beta(y)]\\
&-[b_{1'}b_2^\dagger][b_{2'}b_1^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\alpha(x)][\psi_\delta(y)\bar{\psi}_\beta(y)]\\
&-[b_{1'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_1^\dagger][\varphi(x)\varphi(y)][\psi_\delta(y)\bar{\psi}_\beta(y)][b_{2'}b_2^\dagger]\\
&-[b_{1'}\bar{\psi}_\beta(y)][\psi_\delta(y)b_1^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\alpha(x)][b_{2'}b_2^\dagger]\\
&+[b_{1'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_2^\dagger][\varphi(x)\varphi(y)][\psi_\delta(y)\bar{\psi}_\beta(y)][b_{2'}b_1^\dagger]\\
&+[b_{1'}\bar{\psi}_\beta(y)][\psi_\delta(y)b_2^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\alpha(x)][b_{2'}b_1^\dagger]\qquad(9.43)\\
&-[b_{2'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_2^\dagger][\varphi(x)\varphi(y)][\psi_\delta(y)\bar{\psi}_\beta(y)][b_{1'}b_1^\dagger]\\
&-[b_{2'}\bar{\psi}_\beta(y)][\psi_\delta(y)b_2^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\alpha(x)][b_{1'}b_1^\dagger]\\
&+[b_{2'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_1^\dagger][\varphi(x)\varphi(y)][\psi_\delta(y)\bar{\psi}_\beta(y)][b_{1'}b_2^\dagger]\\
&+[b_{2'}\bar{\psi}_\beta(y)][\psi_\delta(y)b_1^\dagger][\varphi(x)\varphi(y)][\psi_\gamma(x)\bar{\psi}_\alpha(x)][b_{1'}b_2^\dagger]\\
&+[b_{1'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)\bar{\psi}_\beta(y)][\psi_\delta(y)b_1^\dagger][\varphi(x)\varphi(y)][b_{2'}b_2^\dagger]\\
&+[b_{1'}\bar{\psi}_\beta(y)][\psi_\delta(y)\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_1^\dagger][\varphi(x)\varphi(y)][b_{2'}b_2^\dagger]\\
&-[b_{1'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)\bar{\psi}_\beta(y)][\psi_\delta(y)b_2^\dagger][\varphi(x)\varphi(y)][b_{2'}b_1^\dagger]\\
&-[b_{1'}\bar{\psi}_\beta(y)][\psi_\delta(y)\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_2^\dagger][\varphi(x)\varphi(y)][b_{2'}b_1^\dagger]\\
&+[b_{2'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)\bar{\psi}_\beta(y)][\psi_\delta(y)b_2^\dagger][\varphi(x)\varphi(y)][b_{1'}b_1^\dagger]\\
&+[b_{2'}\bar{\psi}_\beta(y)][\psi_\delta(y)\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_2^\dagger][\varphi(x)\varphi(y)][b_{1'}b_1^\dagger]\\
&-[b_{2'}\bar{\psi}_\alpha(x)][\psi_\gamma(x)\bar{\psi}_\beta(y)][\psi_\delta(y)b_1^\dagger][\varphi(x)\varphi(y)][b_{1'}b_2^\dagger]\\
&-[b_{2'}\bar{\psi}_\beta(y)][\psi_\delta(y)\bar{\psi}_\alpha(x)][\psi_\gamma(x)b_1^\dagger][\varphi(x)\varphi(y)][b_{1'}b_2^\dagger]\;.
\end{aligned}$$

To obtain the amplitude (9.41) only the first four terms of (9.43) have been used. The remaining ones will be discussed briefly in the language of Feynman diagrams at the end of the next section.

In the next step, after all possible sequences of pairs of operators are written down, the operators in the brackets are paired, i.e. the brackets are replaced by the corresponding

interchanged without changing the sign; permutations of entire brackets are not counted as different orderings. It is the composition of pairs that matters, not their relative place in the sequence.

c-number propagators:

$$[\psi_\alpha(x)\bar\psi_\beta(y)] \to iS^{\mathrm{F}}_{\alpha\beta}(x-y)\,,$$
$$\left[\phi(x)\phi^\dagger(y)\right] \to i\Delta^{\mathrm{F}}(x-y)\,, \tag{9.44}$$

etc., the factors (9.29) for the brackets like $[\psi_\gamma(x)b_1^\dagger]$, and deltas $\delta_\Gamma^{(3)}(1-1')$ for brackets like $[b_{1'}b_1^\dagger]$, etc. The sum of the terms obtained in this way multiplied by the c-number vertex factors is (after contracting appropriately all the spinor indices) the value of the matrix element.

9.4   Feynman diagrams and rules

From the discussion of the preceding section it follows, that (9.28) evaluated between the initial $|\alpha_0\rangle$ and final $\langle\beta_0|$ states takes schematically the form

$$S_{\beta\alpha} = \sum_{n_1,n_2,\ldots} \int d^4x_{i_1} \ldots \int d^4x_{i_{n_1}} \int d^4x_{j_1} \ldots \int d^4x_{j_{n_2}} \ldots \tag{9.45}$$

$$\sum_{\text{all possible pairings}} \Big(\text{product of wave functions, vertex factors and propagators}\Big).$$

(The factors $(-i)^{n_1}\ldots$, which were present in (9.28), have been included now into the vertex factors; the omission of factors $1/(n_1!\ldots)$ will be explained below.) Each of Feynman diagrams, which we are going to introduce now, represents graphically the content of one bracket in (9.45) corresponding to one of the possible pairings (and fixed $n_1, n_2, \ldots$) and the Feynman rules allow to quickly write down this content. Comparison of the two terms of the amplitude (9.42) with their graphical representation in Figure 9.2 should facilitate understanding of the general procedure described below.

   Applying the Feynman diagrams technique we begin by drawing lines representing particles and antiparticles in the initial $|\alpha_0\rangle$ and final $\langle\beta_0|$ states. Let us adopt the convention that the time flows from the bottom to the top of the diagram. Each fermion (particle) in the initial (final) state is represented by a solid line with the arrow starting with a cross[21] (ending) at the bottom (top) of the diagram. The lines are marked with the values $p_1,\ldots,p_r$ $(p'_1,\ldots,p'_s)$ of the four-momenta carried by these particles. In contrast, antifermions (antiparticle) in the initial (final) state are represented by solid lines with the arrows ending (beginning) at the bottom (top) of the diagram. Initial (final) state antiparticle lines are marked $-p_{r+1},\ldots,-p_N$ $(-p'_{s+1},\ldots,-p'_M)$ where $+p_{r+1},\ldots,+p_N$ $(+p'_{s+1},\ldots,+p'_M)$ are the values of the physical four-momenta of these antiparticles. The

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[21]Marking lines representing (anti)particles in the initial/final state with a cross is not a common convention. We do it in order to distinguish such lines (to which no propagator is associated) from the external lines of off-shell Green's functions (to be introduced in Section 13.1) to which in the analytical expressions appropriate propagators do correspond.

$$-ih\delta_{\beta\alpha} \qquad -ih^*\delta_{\beta\alpha} \qquad -i\lambda \qquad -i\sum_A (\Gamma^A)_{\beta\alpha}(\bar{\Gamma}^A)_{\gamma\delta}$$
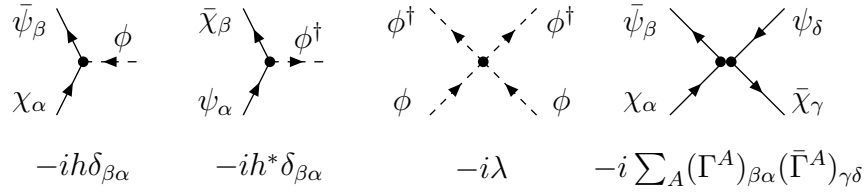
Figure 9.3: Interaction vertices arising from the interaction Hamiltonian density (9.46). The factor written below each vertex is the corresponding vertex Feynman rule.

rule for antiparticles follow from the fact that the four-momentum we ascribe to a line refers always to the direction of the arrow carried by this line, which in turn always (by convention) shows the direction of the possible flow of particles (opposite to that of antiparticles). Similar rules apply also to spin 0 charged particles except that by convention instead of solid lines dashed lines are used to represent them (spin 1 particles - to be discussed in Sections 9.5 and 9.6 - are represented usually by wavy lines). Lines representing spinless, spin $\frac{1}{2}$ or spin 1 particles which are their own antiparticles do not carry arrows related to the "flow" of particles. Four-momentum variables marking lines representing such particles in the initial or final state can be, therefore, assigned naturally: $+p$ and $+p'$, respectively. Thus, a neutral spin 0 particle in the initial (final) state is represented simply by a dashed line starting (ending) with a cross at the bottom (top) of the diagram and marked by the value of the physical four-momentum of these particle.

Interaction terms $\mathcal{H}_{(i)}$ will be represented in Feynman diagrams by dots called *vertices*. Each operator ψ ($\bar{\psi}$) in $\mathcal{H}_{(i)}$ is represented by a solid line with an arrow entering (leaving) the vertex. Similarly, each ϕ ($\phi^\dagger$) in $\mathcal{H}_{(i)}$ is represented by a dashed line with an arrow entering (leaving) the vertex. Finally, to each Hermitian field φ in $\mathcal{H}_{(i)}$ there corresponds a dashed line without an arrow attached to the vertex. For example, all vertices corresponding to the theory in which interactions of two different kinds of spin $\frac{1}{2}$ particles (and their antiparticles) with spin 0 particles and their antiparticles are built using the field operators ψ, χ and ϕ, $\phi^\dagger$ are given by

$$\mathcal{H}_{\mathrm{int}} = h\,\phi\,\bar{\psi}_\alpha\chi_\alpha + h^*\phi^\dagger\bar{\chi}_\alpha\psi_\alpha + \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$$
$$+ \sum_A \left[\bar{\psi}_\beta(\Gamma^A)_{\beta\alpha}\chi_\alpha\right]\left[\bar{\chi}_\gamma(\bar{\Gamma}^A)_{\gamma\delta}\psi_\delta\right], \tag{9.46}$$

(h, λ are coupling constants and Γ^A are some matrices in the spinor space with $\bar{\Gamma}^A = \gamma^0\Gamma^{A\dagger}\gamma^0$ and summations over repeated spinor indices is understood) are shown in figure 9.3.

In order to write down the products of factors corresponding to a fixed set of numbers $\{n_1, n_2, \ldots\}$ in the schematic formula (9.45), one draws first lines representing particles (and antiparticles) in the initial and final states according to the rules given above. One

draws also n_1 vertices labeled by their space-time positions $x_{i_1}, \ldots, x_{i_{n_1}}$ corresponding to the interaction $\mathcal{H}_{(1)}$, n_2 vertices labeled by their space-time positions $x_{j_1}, \ldots, x_{j_{n_2}}$ corresponding to the interaction $\mathcal{H}_{(2)}$, etc. all together with the attached lines corresponding to field operators in these interaction terms. To obtain Feynman diagrams one connects lines in the diagram in all possible ways, but respecting their orientations (if the lines carry arrows distinguishing the direction of the flow of particles), so that each line representing an initial state particle gets connected to some line (of the same type) in one of the vertices or to a line representing a particle of the same type in the final state. Similarly, each line representing a final state particle is connected to a vertex or to a line representing an initial state particle. Finally, all lines from vertices must be connected either to lines of similar type in some vertices[22] or to lines representing initial or final state particles. Usually there is more than one way[23] of connecting the lines (the number of possibilities grows quickly with the number $N = n_1 + n_2 + \ldots$ of interaction vertices) and one obtains many different Feynman diagrams (for a given set $\{n_1, n_2, \ldots\}$). Each of the diagrams obtained in this way corresponds to one particular product of factors in the formula (9.45) and is therefore in a one-to-one correspondence with some term arising from direct evaluation (described above) of the $\{n_1, n_2, \ldots\}$ term of the expansion (9.28) inserted between states $\langle \beta_0 |$ and $| \alpha_0 \rangle$. Moreover, the analytic form of the corresponding product in (9.45) can readily be written down just by looking at the diagram: to each initial or final state particle line connected to a vertex there corresponds one of the factors listed in (9.29); to a line going through the whole diagram without being connected to any vertex (this happens when an initial state particle line is directly connected to a line representing a final state particle) corresponds a factor $(2\pi)^3 2E_\mathbf{p} \delta^{(3)}(\mathbf{p}' - \mathbf{p}) \delta_{\sigma'\sigma}$ (where $\mathbf{p}$, σ and $\mathbf{p}'$, σ' are the momentum and spin labels of these particles); to each dashed (spin 0 particle) line connecting two vertices having space time labels x and y there corresponds a propagator factor $i\Delta^\mathrm{F}(x - y)$ while to each solid line (spin $\frac{1}{2}$ fermion) corresponds a propagator factor $iS^\mathrm{F}_{\alpha\beta}(x - y)$; similar propagator factors correspond also to other cases (to be discussed in Sections 9.5, 9.6); if a line carries an arrow, x in the propagator factor is the space-time label of the vertex into which the line enters. To each vertex of the type (i) corresponds $-i$ times the appropriate vertex factor which can be obtained directly from $\mathcal{H}_{(i)}$ by stripping off the field operators (modulo some combinatoric factors to be explained below). The vertex factors corresponding to the interaction Hamiltonian (9.46) are given in Figure 9.3 (notice that the coupling λ in the rule for the third vertex in Figure 9.3 does not have $1/4$ present in the corresponding term $\mathcal{H}_{(i)}$ in (9.46) - this is related to combinatorics which will be explained below). All these factors have to be assembled together in a unique way dictated by the structure of the diagram (since the amplitude is

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[22]If the interaction terms are normal ordered $:\mathcal{H}_{(i)}: = \mathcal{H}_{(i)}$, a line originating from a given vertex cannot be connected to another line in the same vertex. This restriction does not apply if the interaction terms are not normal ordered.

[23]If one or more lines of some type cannot be connected anywhere (because e.g. the number of lines of this type associated with the initial and the final state and vertices is odd, or the orientation of lines do not allow for that) then a diagram cannot be drawn. This simply means that the corresponding term in (9.45) gives vanishing contribution to the considered S-matrix element.

a pure complex number, all spinor indices have to be summed, etc.) and integrated over the space-time positions of the vertices. The sum of analytic expressions corresponding to all topologically distinct diagrams is just the sum over all pairings of products in the formula (9.45) and thus gives the complete contribution of the $\{n_1, n_2, \ldots\}$ term of the expansion (9.28) to the S-matrix element $S_{\beta\alpha}$.

Assembling analytical factors corresponding to a given Feynman diagram is easy. We describe it here only for diagrams corresponding to interactions $\mathcal{H}_{(i)}$ built out of the ψ, $\bar{\psi}$, ϕ, $\phi^\dagger$ and φ operators. Generalization to other cases is straightforward. We select a fermion line (passing throughout at least one vertex) and begin with its end to which the arrows point. If this end of the line is at the top (bottom) of the diagram we write the factor[24] $\bar{u}_\alpha(\mathbf{p}', \sigma')e^{ip'\cdot x}$ ($\bar{v}_\alpha(\mathbf{p}, \sigma)e^{-ip\cdot x}$). As we follow this line back, we encounter a vertex labeled x, whose vertex factor necessarily has a spinor index (which in $\mathcal{H}_{(i)}$ was contracted with the spinor index α of $\bar{\psi}_\alpha$) with which the spinor index of $\bar{u}_\alpha$ (or $\bar{v}_\alpha$) has to be contracted. This same vertex factor must necessarily have at least one more spinor index, say γ, (contracted in $\mathcal{H}_{(i)}$ with the spinor index of ψ_γ) which now has to be contracted either with with the spinor index of $u_\gamma(\mathbf{p}, \sigma)e^{-ip\cdot x}$ or $v_\gamma(\mathbf{p}', \sigma')e^{+ip'\cdot x}$ (if the fermion line we are considering after leaving the vertex at x goes directly to the bottom or the top of the diagram, respectively), or with the left spinor index of the fermion propagator $iS_{\gamma\delta}^{\mathrm{F}}(x - y)$, if the line passes through another vertex labeled y. In the latter case the spinor index δ of $iS_{\gamma\delta}^{\mathrm{F}}(x - y)$ has to be contracted with the appropriate spinor index δ of the vertex factor and another spinor index of this vertex factor, call it β, is contracted either with $u_\beta(\mathbf{p}, \sigma)e^{-ip\cdot y}$ or $v_\beta(\mathbf{p}', \sigma')e^{+ip'\cdot y}$ or with the spinor index of a successive propagator $iS_{\beta\rho}^{\mathrm{F}}(y - z)$ etc. Having finished with one fermion line we proceed to the next one and do the same. To write down the analytical expression corresponding to a sequence of fermion line which together form a closed loop we begin with any vertex through which such a line passes and write down its vertex factor. Next, we follow the line in the direction opposite to its arrow multiplying for each line segment connecting successive vertices by the propagator iS^{F} and for each vertex by the corresponding vertex factor. The spinor indices of the propagators should be contracted appropriately with the spinor indices in the consecutive vertex factors. Since the line returns to the vertex from which it has started, the summations over spinor indices can be concisely written as the trace in the spinor space. In addition we multiply the resulting expression by (-1). As a result, to a closed fermion loop going through vertices labeled $x_1, x_2, \ldots, x_K$ there corresponds the expression

$$(-1)\mathrm{tr}\left[\Gamma_1\, iS^{\mathrm{F}}(x_1 - x_2)\Gamma_2\, iS^{\mathrm{F}}(x_2 - x_3)\Gamma_3 \ldots \Gamma_K\, iS^{\mathrm{F}}(x_K - x_1)\right] \qquad (9.47)$$

in which Γ_i's denote vertex factors of the consecutive vertices. Spin 0 particle lines are dealt with in a similar manner. They are easier as no spinor indices are involved and final and initial state factors are simply $e^{+ip'\cdot x}$ and $e^{-ip\cdot x}$, respectively. To lines connecting vertices there correspond propagators $i\Delta^{\mathrm{F}}$. No extra minuses are associated

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[24]In this description we do not pay attention to the labeling of the particle's momenta and spins; we only use primes to denote momenta and spins of particles in the final state.

with closed loops formed by lines of spin 0 particles. As follows form the formula (9.45), the contribution of a Feynman diagram to the S-matrix element $S_{\beta\alpha}$ is given by integrating this expression over space-time positions of all vertices.

At this point one has to comment on the combinatoric factors. We first note that drawing all Feynman diagrams corresponding to the $\{n_1, n_2, \ldots\}$ term of the expansion (9.28) we get $n_1! \cdot n_2! \cdot \ldots$ identical diagrams in which n_i vertices representing n_i interaction terms $\mathcal{H}_{(i)}$ differ only by their space-time labels x_i. After integration over these variables one gets $n_1! \cdot n_2! \cdot \ldots$ identical contributions. Hence, it is sufficient to draw only one of these diagrams dropping at the same time the factor $1/(n_1! \cdot n_2! \cdot \ldots)$ in the formula (9.28). This is why we have omitted $1/(n_1! \cdot n_2! \cdot \ldots)$ in (9.45). Secondly, if in an interaction term $\mathcal{H}_{(i)}$ there are m identical field operators (operator and its Hermitian conjugate are not considered identical), the corresponding vertex has m identical legs to which a line of the appropriate type can be connected. One gets therefore $m!$ identical Feynman diagrams, all of which give the same analytical expression. Hence, a factor $1/m!$ is usually inserted in the interaction vertex which in most cases compensates for $m!$ identical diagrams. The Feynman rule is then written without the $1/m!$ factor and one draws only one Feynman diagram instead of $m!$ identical ones. (This is why there is no $1/4 = 1/(2!2!)$ in the third vertex Feynman rule in Figure 9.3). These $1/m!$ factors are not always completely canceled, though. Consider for example the self-interaction of neutral scalar particles of the form $\mathcal{H}_{\text{int}}(x) = (\lambda/4!)\varphi^4(x)$ and the diagram shown in Figure 9.4. The two lines which are connected to the same vertex can be attached to any of its four legs; there are $4 \cdot 3$ ways of doing this at each vertex. But having connected the four external lines pairwise to the two vertices we are left with two free legs in each vertex. These lines have to be sewn together and there are clearly only two ways of doing this. Hence, one has only $2 \cdot (3 \cdot 4)^2$ indistinguishable possibilities to build the diagram shown in figure 9.4 and the two factors $1/4!$ arising with the two vertices are not completely canceled; the factor $1/2$ is left and it has to be included in the analytical expression corresponding to the diagram of figure 9.4. Note, that the counting would be different for the same diagram if the lines carried arrows! Similar *symmetry factors* arise in many cases and with a little bit of practice can be easily calculated by counting the number of ways in which the lines and vertices can be connected to each other to yield a given diagram.

In some special situations even the $1/N!$ factor in the formula (9.18) or one of $1/n_i!$ in the formula (9.28) can be canceled only incompletely. Consider the interaction term

$$\mathcal{H}_{\text{int}} = \bar{\psi}(x)\Gamma[\varphi(x)]\psi(x), \tag{9.48}$$

in which $\Gamma[\varphi(x)]$ is some matrix function of a classical external field $\varphi(x)$ (which is, therefore, *not* a field operator). With this interaction term we can consider calculation of the following matrix element[25]

$$\langle\Omega_0|\text{T}\exp\left(-i\int d^4x\,\mathcal{H}_{\text{int}}(x)\right)|\Omega_0\rangle \tag{9.49}$$

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[25] At this stage this matrix element does not have clear physical interpretation.
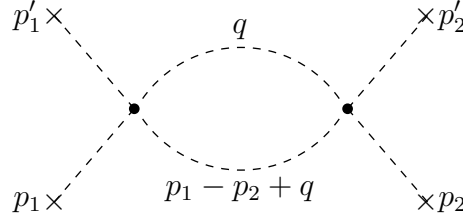
Figure 9.4: One of the three second order contributions to the amplitude of the elastic scattering of two spin 0 neutral particles. In the other two diagrams (not drawn) the lines marked by p_1 and p'_2 and p_1 and p_2 join in the same vertex.

$$= \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \int d^4x_1 \ldots \int d^4x_N \, \langle \Omega_0 | \mathrm{T}\big\{ [\bar{\psi}\Gamma\psi(x_1)] \cdots [\bar{\psi}\Gamma\psi(x_N)] \big\} \, |\Omega_0\rangle \,.$$

Consider now computing the N-th term in this sum. Following the rules formulated above, we draw N vertices each having one incoming and one outgoing fermion line. All these lines should be connected with other lines entering/leaving vertices because there are no lines corresponding to initial/final states (we consider the vacuum to vacuum transition). Let us concentrate on the connected diagram, i.e. the one which consists of a single closed loop formed from all these vertices. It is clear that the outgoing line from the first vertex marked x_1 can be connected to the incoming line of any of $N-1$ other vertices. The outgoing line of the second vertex can be connected to the incoming line of any of $N-2$ remaining vertices, and so on. Thus, there are only $(N-1) \cdot (N-2) \cdot \ldots \cdot 2 \cdot 1 = (N-1)!$ possible connected diagrams which, after integrating over $d^4x_1 \ldots d^4x_N$ give identical contributions. Hence, there remains a factor $(N-1)!/N! = 1/N$.

The signs of various terms contributing to the amplitude of a given process are unambiguously determined by the procedure described at the end of Section 9.3. For example, as has been said, each closed fermion loop in the diagram gives rise to a minus sign, so that the amplitude corresponding to a diagram with L_f closed loops of fermion lines acquires an extra factor $(-1)^{L_f}$. The origin of these minuses is easy to understand. They correspond to the following sequences of pairs of fermion field operators

$$\ldots [\psi_{\alpha_N}(x_N)\bar{\psi}_{\beta_1}(x_1)][\psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_2}(x_2)] \ldots [\psi_{\alpha_{N-1}}(x_{N-1})\bar{\psi}_{\beta_N}(x_N)] \ldots$$

To obtain such strings of pairs out of the string

$$\ldots \bar{\psi}_{\beta_1}(x_1)\psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_2}(x_2)\psi_{\alpha_2}(x_2) \ldots \bar{\psi}_{\beta_n}(x_N)\psi_{\alpha_N}(x_N) \ldots$$

arising under the sign of the T-operation from several $\mathcal{H}_{(i)}$'s bilinear in fermion field operators it is necessary to make an odd number of interchanges of fermionic field operators.

The relative signs of various contributions to the amplitude can frequently be read off directly from the relevant Feynman diagrams. One example is the amplitude (9.42):
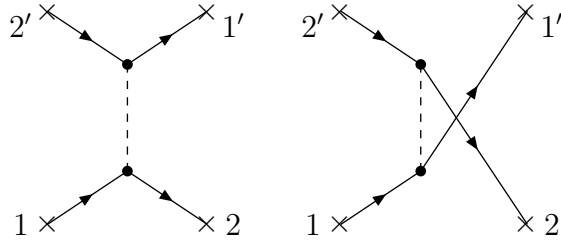
391

Figure 9.5: Lowest order diagrams corresponding in the theory (9.37) to the $f\bar{f} \to f\bar{f}$ scattering. Time flows from the bottom upwards. The expressions corresponding to these two diagrams have a relative minus sign.

the two terms have a relative minus sign because their corresponding Feynman diagrams shown in Figure 9.2 can be obtained one from another by interchanging the labels of the fermions e.g. in the final state: $(\mathbf{p}'_1, \sigma'_1) \leftrightarrow (\mathbf{p}'_2, \sigma'_2)$. Less obvious can be the relative minus sign of the two contributions to the $f\bar{f} \to f\bar{f}$ scattering amplitude shown in Figure 9.5. It follows from the general procedure in the same way as does the relative minus sign in (9.42). It can also be justified on the ground that, as will be explained in Section 13.1, both amplitudes: the $ff \to ff$ scattering amplitude and the $f\bar{f} \to f\bar{f}$ one, can be obtained from one and the same off-shell Green's function by continuing its four-momentum arguments analytically to two different kinematical domains. Hence, at the level of the Feynman diagram it matters only that the two diagrams of Figure 9.5 are obtained one from another by interchanging fermion particle labels, e.g. $(\mathbf{p}_2, \sigma_2) \leftrightarrow (\mathbf{p}'_1, \sigma'_1)$ (or, more precisely, the labels p_2 and p'_1 ascribed to these lines). In cases in which the relative signs cannot be unambiguously determined from Feynman diagrams (this is so if e.g. Majorana fermions are involved) one can always return to the general procedure described at the end of Section 9.3.

In the Feynman rules formulaed as above, interaction vertices are marked by their space-time positions over which one has eventually to integrate. In this form they make clear the quantum-mechanical foundation of the method: one sums amplitudes of all indistinguishable alternative ways the system can go from the initial to the final state: there can be arbitrary number of each type of interactions (summation over numbers $\{n_1, n_2, \ldots\}$) which can occur at arbitrary space-time points (integrations over d^4x's).

As we have seen in considering the example of the fermion-fermion elastic scattering, expressions for amplitudes obtained by using Feynman rules formulated above can be further simplified by inserting the propagators in the form of the Fourier transforms (9.35) and (9.36). Integrals over all d^4x's can be then explicitly taken and are replaced by integrations over independent four-momenta of all *internal* lines (i.e. lines connecting vertices) of the diagram. Hence, when a Feynman diagram is interpreted in the momentum space, each of its internal lines should be marked with a four-momentum q flowing through this line (in the direction of the arrow, if the line carries one). Since integrations over space-time positions of the vertices give four-dimensional delta functions (times $(2\pi)^4$
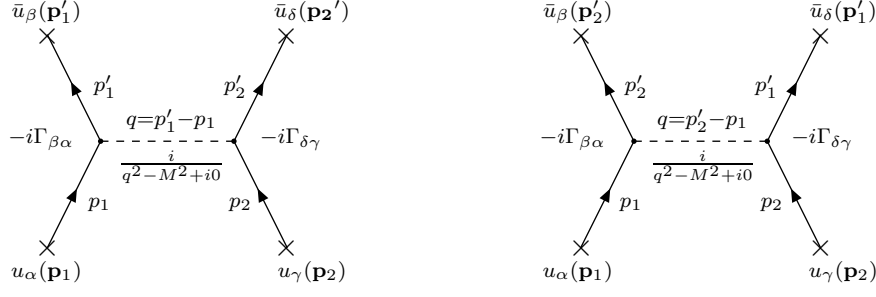
392

Figure 9.6: Lowest order momentum space Feynman diagrams corresponding to fermion-fermion scattering mediated by a neutral spin 0 particle. The four-momenta p_1, p_2, p_1', p_2' are "on-shell", i.e. such that $p_1^2 = m^2$ etc., whereas $q^2 \neq M^2$.

each) which express the four-momentum conservation in each of the vertices, some of the of integrals over four-momenta of internal lines associated with propagators can immediately be performed. This reduces to eliminating those four-momenta of internal lines which can unambiguously be expressed by the combinations of the four-momenta of other lines by using the four-momentum conservation in each vertex. It is not difficult to understand that the number of independent integrals over the four-momenta which are left equals the number of independent closed loops formed by internal lines in the Feynman graph. For example, integration over d^4x and d^4y in both terms of (9.41) gave (in each term) two delta functions which allowed us to integrate over d^4q obtaining (again in each term) a single delta function $(2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)$ expressing the overall four-momentum conservation, and the momentum space propagator with q replaced by the linear combination of the four-momenta of incoming and outgoing fermions. The result (9.42) could have been obtained by drawing from the beginning the two Feynman diagrams as in Figure 9.6 and ascribing to their internal lines the propagators

$$i\Delta^{\mathrm{F}}(q) = \frac{i}{q^2 - M^2 + i0} , \tag{9.50}$$

with $q = p_1' - p_1$ in the first one and $q = p_2' - p_1$ in the second.

The diagrams shown in Figure 9.6 give directly the scattering amplitude $-i\mathcal{A}$ appearing in the formula (7.87). The spinless particle mediating interaction of the two fermions in figure 9.6 is said to be *virtual* because its four-momentum is *off shell*, that is, it does not satisfy the relation $q^2 = M^2$ (where M is the mass of the physical particle). This should be contrasted with the old-fashioned perturbation calculus based on the formula (7.62) in which intermediate states consists of particles which are *on shell*, but instead energy is not conserved (only three-momentum is).

Thus, the Feynman rules can be formulated directly in the momentum space as follows: *i)* Draw lines corresponding to all initial and final state particles at the bottom and top of the diagram respectively. Mark them with the particle four-momenta according to the rules already explained. *ii)* To evaluate the contribution of the $\{n_1, n_2, \ldots\}$ term to the

393

formula (9.28) draw n_1 vertices corresponding to $\mathcal{H}_{(1)}$, n_2 vertices corresponding to $\mathcal{H}_{(2)}$, etc. together with the lines entering/leaving these vertices as already explained. Connect the lines representing the initial and final state (anti)particles with the lines entering and leaving vertices in all possible ways (respecting the directions of arrows and the types of lines). *iii)* Ascribe to the l-th internal (i.e. connecting two vertices) line an independent four-momentum q_l flowing in the direction of the arrow (if the line carries one) or in the arbitrary direction (if the line does not carry any arrow). *iv)* Write down the product of external line wave function factors, vertex factors and propagators following the same procedure previously. The difference is that the external line wave function factors do not have now the $e^{\pm i p_k^{(\prime)} x_l}$ factors and the propagators ascribed to internal lines of spin 0 particles are now given by

$$\frac{i}{q_l^2 - M^2 + i0} \tag{9.51}$$

and, those ascribed to internal lines of spin $\frac{1}{2}$ particles, by

$$\left[\frac{i(\slashed{q}_l + m)}{q_l^2 - m^2 + i0} \right]_{\alpha\beta} \tag{9.52}$$

The vertex factors are unchanged.[26]   *v)* Supply per each interaction vertex the delta function

$$(2\pi)^4 \delta^{(4)}\left(\sum p_{\text{in}} - \sum p_{\text{out}} \right),$$

in which $\sum p_{\text{in}}$ ($\sum p_{\text{out}}$) is the sum of all four-momenta flowing into (flowing into out of) the vertex, which expresses the conservation of the four-momenta of lines entering and leaving the vertex. The combinatoric factors (and relative minus signs) are as previously: identical vertices are indistinguishable (their permutations give identical contributions - so the factors $1/(n_1! \cdot n_2! \cdot \dots)$ are canceled), lines entering/leaving a vertex corresponding to identical field operators in $\mathcal{H}_{(i)}$'s are also indistinguishable. *vi)* Integrate over $d^4 q_l/(2\pi)^4$ per each internal line l.

Going further in automatizing the procedure of computing amplitudes, instead of writing delta functions corresponding to vertices and integrals over $d^4 q_l/(2\pi)^4$ for each internal line, one can from the beginning ascribe to each internal line the four-momentum following from the four-momentum conservation in the diagram's vertices.[27] If a diagram contains L independent closed loops, there are L four-momenta of internal lines which

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[26]In formulating in the momentum space the rules for propagators arising from pairings of field operators involving explicit derivatives it will be possible (after ensuring proper cancellations of noncovariant terms - see Section 9.5) to replace the corresponding propagator factors by the propagators obtained from pairings of operators without derivatives and to include the derivatives in the vertex factors as the appropriate four-momentum factors ascribed to lines on which the derivatives acted.

[27]Different ways of introducing four-momenta over which one integrates may, if the integrals are linearly divergent, lead to different results; this is related to anomalies discussed in Chapter 23.
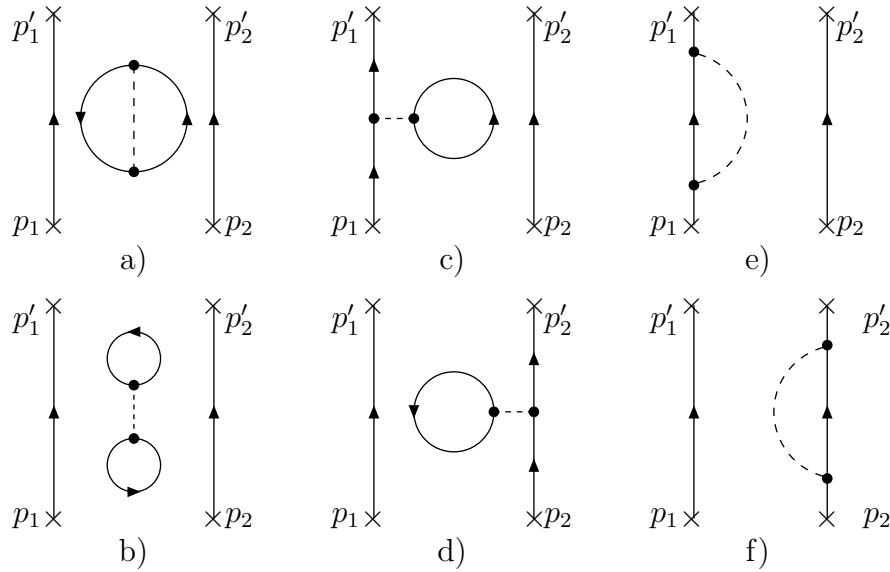
Figure 9.7: Additional diagrams arising in the calculation of the matrix element (9.38). To each of these diagrams there is a companion diagram with momenta p'_1 and p'_2 interchanged.

are not determined by the four-momentum conservation (see for example the diagram of Figure 9.4, in which q is an independent - i.e. not determined by the conservation law - four-momentum) and the contribution of such a diagram has to be integrated over $d^4q/(2\pi)^4$ only for each independent four-momentum. In this approach the overall delta function in the S-matrix element (more precisely, in the matrix element of the T_0 operator) is restored by just appealing to the four-momentum conservation. More precisely, if a diagram can be divided into C disconnected pieces without cutting any of its lines, then, as a result of eliminating delta functions arising in vertices, one obtains C four-dimensional delta functions expressing conservation of the four-momenta within disjoint groups of the initial and final state particles. However, computing amplitudes of physical processes one is interested only in the connected part of the S-matrix. The reason is that non-connected parts of the S-matrix combine to the transition amplitudes of all possible far-away processes that can occur along with the particular process we are interested in (characteristics of which are measured by our detector). When the sum over all possible final states which can be realized in these far-away processes is performed, unitarity plus cluster decomposition property of the S-matrix (see Section 7.8) ensure that they sum up to 1 (multiplying the S-matrix element we are interested in).

Using the Feynman diagrams language it is easy to complete the analysis of the matrix element (9.38), that is to give an interpretation to each of the remaining terms in (9.43) (only the first four terms of (9.43) contributed to the amplitude (9.42)). In the considered order we have two interaction vertices $\mathcal{H}_{\text{int}}$, two incoming and two outgoing fermion lines. Hence, in addition to the two diagrams shown in Figure 9.2 one can draw the diagrams

shown in Figure 9.7. They correspond to the remaining terms in the list (9.43) of all possible strings of pairings of the operators.[28] Diagrams 9.7b - 9.7d arise only if the interaction $\mathcal{H}_{\mathrm{int}}$ is not normal ordered. If present, they can be treated in a similar fashion as the others shown in this figure. All expressions corresponding to the diagrams shown in figure 9.7 involve divergent integrals.

The entire contribution of the diagrams 9.7c - 9.7f can be removed by adding to the original interaction Hamiltonian density $\mathcal{H}_{\mathrm{int}}$ extra terms of the form

$$\Delta \mathcal{H}_{\mathrm{int}} = C\,\bar{\psi}\psi + D\,\bar{\psi}\,i\slashed{\partial}\,\psi\,, \tag{9.53}$$

where C and D are called *renormalization constants*. With $\Delta \mathcal{H}_{\mathrm{int}}$ assumed to start at order g^2 there are additional two order g^2 contributions to the considered S-matrix element. The constants C and D can be then adjusted in such a way to cancel the entire contribution of the diagrams 9.7c - 9.7f, or at least its divergent part.

Canceling unwanted contributions by adding extra terms to the interaction is at the heart of the renormalization procedure discussed in detail in Chapters 13 and 14. Here let us only remark, that if all divergences arising in perturbative expansions of all possible S-matrix elements can be removed by introducing only *a finite number* of operator structures like the ones in (9.53), multiplied by the renormalization constants (like C and D) which get adjusted order by order in the perturbative expansion, the theory is said to be *renormalizable*. Other theories, which require introducing new operator structures at each order of the perturbative expansion (so, in effect, and infinite number) are called *nonrenormalizable*. The renormalization procedure does not affect computations of the S-matrix elements from diagrams which do not contain closed loops, i.e. from *tree* diagrams.

From the perspective of the general renormalization procedure as it is applied to Green's functions (to be defined in Chapter 13), canceling the entire contribution of the diagrams 9.7c - 9.7f is a special choice. However, as will be argued in Section 9.7, it is enforced by the consistency of the whole scheme of calculating S-matrix elements developed here: it is necessary to ensure that the full Hamiltonian H and H_0 have the same spectrum (one of the basic assumptions made in Chapter 7). Other choices of the renormalization scheme are possible but they require going beyong the scheme developed here. This will be discussed only in Chapters 13 and 14.

Finally, we consider diagrams 9.7a and 9.7b which are also divergent. By carefully analyzing the combinatorics of Feynman diagrams it can be shown that the contribution of all such *vacuum diagrams* factorizes and exponentiates, that is any matrix element of the operator S_0 between the free Hamiltonian eigenstates can be written in the form

$$\langle\beta_0|\mathrm{T}\exp\left(-i\int d^4x\,\mathcal{H}_{\mathrm{int}}(x)\right)|\alpha_0\rangle \tag{9.54}$$

---

[28]In Figure 9.7 there are less diagrams than terms in the expression (9.43) because some of the terms of (9.43) give identical contributions and simply cancel the appropriate numerical factors (of the type $1/n_i!$) arising from the expansion.
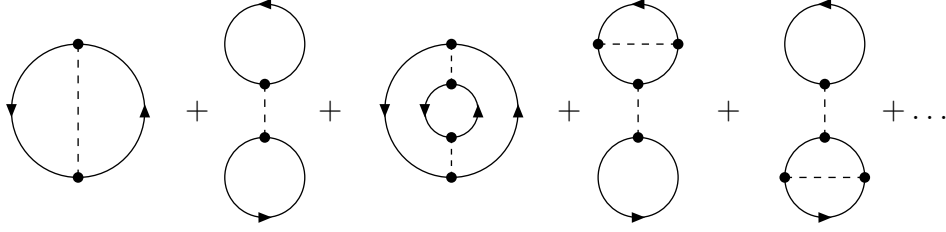
Figure 9.8: Connected vacuum diagrams. Their sum equals $-i\Xi$.

$$= e^{-i\Xi} \times \langle \beta_0 | \mathrm{T} \exp\left(-i\int d^4x\, \mathcal{H}_{\mathrm{int}}(x)\right) |\alpha_0\rangle_{\mathrm{without\ vacuum\ graphs}},$$

in which the purely imaginary factor $-i\Xi$ is the sum of all *connected* vacuum diagrams, i.e. it is given by the matrix element

$$-i\Xi = \langle \Omega_0 | \mathrm{T} \exp\left(-i\int d^4x\, \mathcal{H}_{\mathrm{int}}(x)\right) |\Omega_0\rangle_{\mathrm{connected\ graphs}}. \tag{9.55}$$

Diagrams contributing to $-i\Xi$ if the interaction has the form (9.37) are shown graphically in Figure 9.8. The phase factor $e^{-i\Xi}$ is just the scalar product of the *in* and *out* vacuum state-vectors:

$$\langle \Omega_- | \Omega_+ \rangle = e^{-i\Xi}. \tag{9.56}$$

Indeed, since $H_0 |\Omega_0\rangle = 0$, from the formula (7.39) it formally follows that

$$|\Omega_-\rangle = \lim_{\tau_2 \to \infty} \lim_{\tau_1 \to -\infty} e^{i(\tau_2-\tau_1)H} |\Omega_+\rangle. \tag{9.57}$$

so that Ξ is just[29]

$$\Xi = \lim_{\tau_2 \to \infty} \lim_{\tau_1 \to -\infty} (\tau_2 - \tau_1) E_\Omega = \int d^4x\, \mathcal{E}_\Omega. \tag{9.58}$$

(In the actual calculation of Ξ using Feynman diagrams the space-time volume $\int d^4x$ arises as the factor $(2\pi)^4 \delta^{(4)}(0)$; obtaining a finite value of the vacuum energy density $\mathcal{E}_\Omega$ requires expressing it in terms of properly defined renormalized (finite) parameters of the theory like particle masses and coupling constants). Since the energy of the ground (vacuum) state of H_0 is $E_{\Omega_0} = 0$, it follows that, to satisfy the assumptions made in Chapter 7, one should add to $\Delta\mathcal{H}_{\mathrm{int}}$ a constant term (another renormalization constant) allowing to remove also the contribution of all vacuum diagrams, i.e. to "renormalize the vacuum energy" (or, in the context of coupling to gravity, the cosmological constant). In

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[29]This agrees with the formula (1.38) (for $E_{\Omega_0} = 0$) but cannot, of course, be taken for a rigorous proof; a better justification of the statement that the sum of all connected "vacuum diagrams" gives the difference of $\mathcal{E}_\Omega - \mathcal{E}_{\Omega_0}$ is provided by the appropriate analytic continuation of the formula for the partition function of a given system of interacting particles.

practice, instead of doing ths one simply drops all the vacuum diagrams. The "vacuum - to - vacuum" amplitude (9.56) becomes important in cases of systems interacting with external agents (i.e. with external classical fields). It cannot be then simply discarded (renormalized to zero).

9.5 Propagators of vector-like field operators

As we have explained in Section 8.4, with spin 0 particles also field operators of the form $\partial_\mu \phi(x)$ transforming as Lorentz vectors can be associated. Suppose an interaction (one of $\mathcal{H}_{(i)}$'s) of spin 0 particle is of the form

$$\mathcal{H}_{(i)} = \partial_\mu \phi(x) J^\mu(x), \tag{9.59}$$

where $J^\mu(x)$ is a Lorentz vector constructed out of field operators of other (or the same) particles and consider a pairing of two operators $\partial \phi$ from two such interaction vertices in $\mathrm{T}[...\partial_\mu \phi(x) J^\mu(x) ... \partial_\nu \phi(y) J^\nu(y) ...]$. According to the general procedure, the propagator ascribed to the line corresponding to this pairing arises as the ombination

$$
\begin{aligned}
i\Delta_{\mu\nu}^{\mathrm{F}}(x-y) &\equiv \theta(x^0 - y^0) \left[\partial_\mu \phi^{(+)}(x),\ \partial_\nu \phi^{(-)}(y) \right] \\
&\quad + \theta(y^0 - x^0) \left[\partial_\nu \phi^{(+)}(y),\ \partial_\mu \phi^{(-)}(x) \right] \\
&= \theta(x^0 - y^0)\, \partial_\mu^{(x)} \partial_\nu^{(y)} \int d\Gamma_{\mathbf{p}}\, e^{-ip\cdot(x-y)} \\
&\quad + \theta(y^0 - x^0)\, \partial_\mu^{(x)} \partial_\nu^{(y)} \int d\Gamma_{\mathbf{p}}\, e^{+ip\cdot(x-y)} \\
&= -\theta(x^0 - y^0)\, \partial_\mu^{(x)} \partial_\nu^{(x)} \int d\Gamma_{\mathbf{p}}\, e^{-ip\cdot(x-y)} \\
&\quad - \theta(y^0 - x^0)\, \partial_\mu^{(x)} \partial_\nu^{(x)} \int d\Gamma_{\mathbf{p}}\, e^{+ip\cdot(x-y)} \\
&= \langle \Omega_0 | \mathrm{T}(\partial_\mu \phi(x) \partial_\nu \phi^\dagger(y)) | \Omega_0 \rangle.
\end{aligned} \tag{9.60}
$$

This differs from $-\partial_\mu^{(x)} \partial_\nu^{(x)} i\Delta^{\mathrm{F}}(x-y)$ because in the latter expression derivatives with respect to x^0 act also on the theta functions present in (9.32). Since the integrals are over the three-momentum $\mathbf{p}$ and the time-like component p^0 in the exponents is just $\sqrt{\mathbf{p}^2 + M^2}$ the factor $(p^0)^2$ under the integral can be replaced by $\mathbf{p}^2 + M^2$. This means, that $\partial_0^{(x)} \partial_0^{(x)}$ can be replaced by $\boldsymbol{\Delta} - M^2$ (as it produces the same factor under the integral). In (9.60) we have therefore three possibilities:

$$
\begin{aligned}
i\Delta_{ij}^{\mathrm{F}}(x-y) &= -\partial_i^{(x)} \partial_j^{(x)} i\Delta^{\mathrm{F}}(x-y), \\
i\Delta_{i0}^{\mathrm{F}}(x-y) &= -\partial_i^{(x)} \partial_0^{(x)} i\Delta^{\mathrm{F}}(x-y) \\
&\quad + \partial_i^{(x)} \delta(x^0 - y^0) \int d\Gamma_{\mathbf{p}}\, e^{-ip\cdot(x-y)}
\end{aligned} \tag{9.61}
$$

$$- \partial_i^{(x)} \delta(y^0 - x^0) \int d\Gamma_{\mathbf{p}} \, e^{+ip\cdot(x-y)}$$

$$= -\partial_i^{(x)} \partial_0^{(x)} i\Delta^{\mathrm{F}}(x-y) \,,$$

$$i\Delta_{00}^{\mathrm{F}}(x-y) = -\left(\mathbf{\Delta} - M^2\right) i\Delta^{\mathrm{F}}(x-y) \,.$$

In the case of mixed time-spatial derivatives the two terms cancel each other because the function $\Delta_+(x-y)$ defined in (8.21) is symmetric and for $(x-y)^2 < 0$ (enforced by the delta functions) the two integrals give the same result - see the discussion around the equation (8.22). Thus the pairing (9.60) can concisely be written as[30]

$$i\Delta_{\mu\nu}^{\mathrm{F}}(x-y) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{i\,e^{-ip\cdot(x-y)}}{p^2 - M^2 + i0} \left[p_\mu p_\nu - \delta_\mu^0 \delta_\nu^0 \left(p^0 p^0 - \mathbf{p}^2 - M^2\right)\right]$$

$$= \partial_\mu^{(x)} \partial_\nu^{(y)} i\Delta^{\mathrm{F}}(x-y) - i\delta_\mu^0 \delta_\nu^0 \, \delta^{(4)}(x-y) \,, \tag{9.62}$$

where we have used the explicit form (9.35) of $i\Delta^{\mathrm{F}}(x-y)$.

Thus, pairings of the vector field operators built out of the creation and annihilation operators of spin 0 particles acquire a noncovariant term. One may wonder, what is the reason for the appearence of this term: unlike the case of massless spin 1 particles which will be discussed in the next Section, the operators $\partial_\mu \phi$ transform as true vectors and, moreover,

$$[\partial_\mu \phi(x), \, \partial_\nu \phi(y)] = 0 \qquad \text{for} \qquad (x-y)^2 < 0 \,, \tag{9.63}$$

so that all conditions for Lorentz covariance of the S-matrix formulated in Section 7.5 are satisfied. The answer is that while the condition (9.63) is indeed satisfied when $x \neq y$, vanishing of the (anti)commutators at $x = y$ is not guaranteed by the construction of Chapter 8 of free field operators (which are in fact *operator valued distributions*): their products taken at the same space-time point are singular objects. For this reason, matrix elements of the commutator (9.63) may not vanish at $x = y$ and this manifests itself in the noncovariant term (nonvanishing only for $x = y$) in the propagator (9.62).

The noncovariant term in (9.62) can be compensated for by adding to the interaction Hamiltonian density (9.59) a noncovariant term

$$\Delta \mathcal{H}_{\mathrm{int}} = \frac{1}{2} J^0(x) J^0(x) \,, \tag{9.64}$$

which in Feynman diagrams gives rise to a local contact interactions shown in Figure 9.9. To each Feynman diagram in which somewhere two currents J_μ and J_ν are connected by the propagator $i\Delta_{\mu\nu}^{\mathrm{F}}$ (9.62) there is a companion diagram in which all other elements are the same but the propagator $i\Delta_{\mu\nu}^{\mathrm{F}}$ is replaced by the contact interaction vertex and the

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[30]From the above analysis it readily follows that contraction of $\partial_\mu \phi(x)$ with $\phi(y)$ (such operators may e.g. originate from two different terms of the interaction Hamiltonian delsity $\mathcal{H}_{\mathrm{int}}(x)$) does not lead to any noncovariant terms; it simply gives $\partial_\mu i\Delta^F(x-y)$.

$$(-i)^2 \int d^4x \int d^4y \, J^\mu(x) i\Delta_{\mu\nu}^{\mathrm{F}}(x-y) J^\nu(y)$$
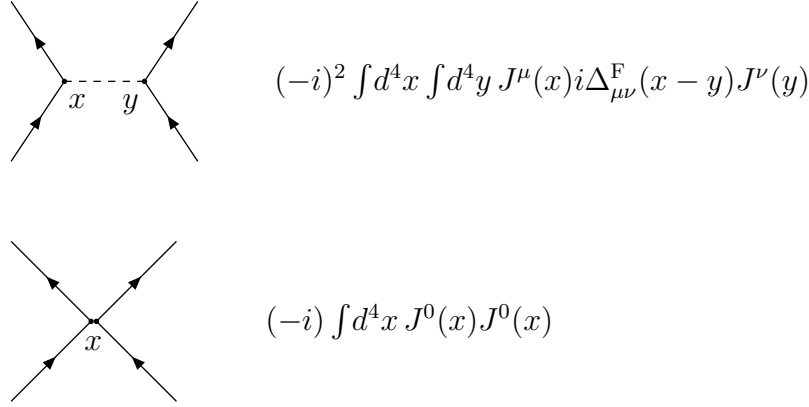
$$(-i) \int d^4x \, J^0(x) J^0(x)$$

Figure 9.9: Cancellation of the noncovariant term in the pairing (9.62) by the local contact interaction (9.64).

two noncovariant contributions to the amplitude cancel each other in the sum of these two diagrams. As we will see in Section 11.9, precisely the contact interaction (9.64) appears in the interaction Hamiltonian density if $\mathcal{H}_{\mathrm{int}}$ if a derivative interaction of spin 0 particles is obtained by canonically quantizing a classical field theory.

Since the only role of the additional interaction term (9.64) is to cancel the noncovariant term in the propagator, in practical calculations one can replace everywhere (9.62) by $\partial_\mu^{(x)} \partial_\nu^{(y)} i\Delta^{\mathrm{F}}(x-y)$ forgetting at the same time about the interaction (9.64). Derivatives acting on ϕ or $\phi^\dagger$ can be then treated as part of the vertex factors rather than of the propagator. In the momentum space each derivative ∂_μ acting in the vertex on an operator ψ, ϕ or φ ($\bar\psi$, $\phi^\dagger$, φ) translates then into the factor $-iq_\mu$ ($+iq_\mu$) where q is the four-momentum of the line entering (leaving) the vertex "through" this operator.

A noncovariant term similar to the one in (9.62) arises (for the same reasons) in the propagator of massive spin 1 particles represented in the interaction Hamiltonian density by the vector field operator (8.143):

$$
\begin{aligned}
i\Delta_{\mu\nu}^{\mathrm{F}}(x-y) &\equiv \theta(x^0-y^0) \left[V_\mu^{(+)}(x), \, V_\nu^{(-)}(y) \right] \\
&\quad + \theta(y^0-x^0) \left[V_\nu^{(+)}(y), \, V_\mu^{(-)}(x) \right] \\
&= \theta(x^0-y^0) \int d\Gamma_{\mathbf{p}} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right) e^{-ip\cdot(x-y)} \\
&\quad + \theta(y^0-x^0) \int d\Gamma_{\mathbf{p}} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right) e^{+ip\cdot(x-y)},
\end{aligned}
\tag{9.65}
$$

where the sum (8.137) over polarizations has been used. Repeating the arguments one gets therefore

$$
i\Delta_{\mu\nu}^{\mathrm{F}}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i0} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right) e^{-ip\cdot(x-y)}
$$

$$-\frac{i}{M^2}\,\delta_\mu^0\delta_\nu^0\,\delta^{(4)}(x-y)\,. \tag{9.66}$$

As in the previous case the noncovariant terms in the propagator of spin 1 massive particle can be canceled by adding to the interaction Hamiltonian density a noncovariant local contact interaction term

$$\Delta\mathcal{H}_{\mathrm{int}} = \frac{1}{2M^2}\,J^0(x)J^0(x)\,, \tag{9.67}$$

which arises automatically if such interactions of massive spin 1 particles appear as a result of the canonical quantization of a classical vector field coupled to other fields through a current J_μ. Again, in practical calculations one can simply forget about the interaction term (9.67) dropping at the same time the noncovariant piece of (9.66). The rest of the Feynman rules for massive spin 1 particles can be formulated as in the case of spin 0 and spin $\frac{1}{2}$ particles.

9.6   Feynman rules for massless spin 1 particles

In Section 8.5 we have learned that constructing a field operator transforming as a true Lorentz vector out of the creation and annihilation operators of a massless spin 1 particle[31] is impossible. The operator (8.180) transforms inhomogeneously (see the formula (8.175)) and moreover it does not satisfy the local causality requirement, i.e. the commutator $[A_\mu(x),\ A_\nu(y)]$ does not vanish for $(x-y)^2 < 0$ (it vanishes only if $x^0 = y^0$). These deficiencies of $A_\mu(x)$ result in noncovariant terms in the photon propagator and must be appropriately compensated by a very special form of interactions.

The propagator of a massless, spin 1 particle (the photon) arises as a combination of two commutators

$$
\begin{aligned}
iD^{\mathrm{F}}_{\mu\nu}(x-y) &\equiv \theta(x^0 - y^0)\left[A^{(+)}_\mu(x),\ A^{(-)}_\nu(y)\right]\\
&\quad +\theta(y^0 - x^0)\left[A^{(+)}_\nu(y),\ A^{(-)}_\mu(x)\right]\\
&= \theta(x^0 - y^0)\int d\Gamma_{\mathbf{k}}\, e^{-ik\cdot(x-y)}\sum_{\lambda=\pm1}\epsilon_\mu(\mathbf{k},\lambda)\epsilon_\nu^*(\mathbf{k},\lambda)\\
&\quad +\theta(y^0 - x^0)\int d\Gamma_{\mathbf{k}}\, e^{+ik\cdot(x-y)}\sum_{\lambda=\pm1}\epsilon_\nu(\mathbf{k},\lambda)\epsilon_\mu^*(\mathbf{k},\lambda)\\
&= \int d\Gamma_{\mathbf{k}}\, P_{\mu\nu}(\mathbf{k})\left[\theta(x^0 - y^0)\,e^{-ik\cdot(x-y)} + \theta(y^0 - x^0)\,e^{+ik\cdot(x-y)}\right],
\end{aligned}
\tag{9.68}
$$

in which

$$P_{00}(\mathbf{k}) = P_{i0}(\mathbf{k}) = P_{0j}(\mathbf{k}) = 0\,, \qquad P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\,. \tag{9.69}$$

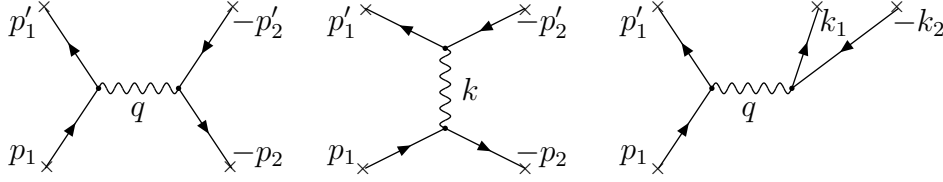
[31] For definiteness we will speak about photons.

Figure 9.10: Examples of the Feynman diagrams with a single spin 1 massless particle exchange (the two left ones contribute to the invariant amplitude of the fermion-antifermion scattering; the right one is strictly speaking unphysical for the process $f \to ff\bar{f}$ is forbidden by the four-momentum conservation).

Using the same trick as the one used to derive the propagator of spin 0 particles one obtains

$$iD_{\mu\nu}^{\mathrm{F}}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{iP_{\mu\nu}(\mathbf{k})}{k^2 + i0} e^{-ik\cdot(x-y)} . \tag{9.70}$$

With the help of the time-like vector $t^\mu = (1, 0, 0, 0) = \delta_0^\mu$ the "tensor" $P_{\mu\nu}$ can be rewritten in the form

$$P_{\mu\nu}(\mathbf{k}) = -g_{\mu\nu} + \frac{k_0(k_\mu t_\nu + k_\nu t_\mu) - k_\mu k_\nu - k^2 t_\mu t_\nu}{|\mathbf{k}|^2} . \tag{9.71}$$

Let us now see if the noncovariant terms in the photon propagator arising from the second term in (9.71) can be dealt with. Suppose A_μ couples in the interaction Hamiltonian density to the current $J^\mu = Q\, \bar{\psi}\gamma^\mu\psi$ constructed out of field operators of spin $\frac{1}{2}$ particles carrying the charge Q (in units of $e > 0$):

$$\mathcal{H}_{\mathrm{int}}(x) = e\, J^\mu(x)A_\mu(x) = e\, Q\, \bar{\psi}(x)\gamma^\mu\psi(x)\, A_\mu(x) . \tag{9.72}$$
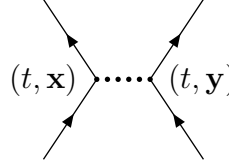
This Hamiltonian density is not a Lorentz scalar, of course, because while $J^\mu(x)$ does transform as a four-vector (we assume it is constructed to transform in this way), $A_\mu(x)$ does not: $A_0(x) \equiv 0$ in any Lorentz frame.

The arguments will be now inductive. We consider first the lowest order diagrams shown in Figure 9.10 and check that in these diagrams the terms in the propagator proportional to k_μ and/or k_ν can be dropped. Indeed, in the momentum space using the vertex factor from the interaction (9.72) (as well as the rules formulated in the preceding section) the first diagram of Figure 9.10 leads to the expression (spin labels σ are suppressed)

$$(-ie\, Q)^2\, \bar{u}(\mathbf{p}_1')\gamma^\mu u(\mathbf{p}_1) \frac{iP_{\mu\nu}(\mathbf{q})}{q^2 + i0} \bar{v}(\mathbf{p}_2)\gamma^\nu v(\mathbf{p}_2') , \tag{9.73}$$

in which $q = p_1 - p_1' = p_2' - p_2$. It is then easy to check that owing the properties (8.100) of the spinors u and v

$$\bar{u}(\mathbf{p}_1')\gamma^\mu u(\mathbf{p}_1)\, q_\mu = \bar{v}(\mathbf{p}_2)\gamma^\nu v(\mathbf{p}_2')\, q_\nu = 0 . \tag{9.74}$$

402

$(t, \mathbf{x}) \cdots\!\!\!< (t, \mathbf{y}) \qquad -ie^2 \int dt \int d^3\mathbf{x} \int d^3\mathbf{y} \, \frac{J^0(t,\mathbf{x})J^0(t,\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}$

Figure 9.11: Noncovariant nonlocal current-current contact interaction and the corresponding Feynman rule.

The same is true of the other two diagrams of Figure 9.10.

The vanishing of the terms proprtional to k_μ and/or k_ν can be in this case attributed to the fact that in (9.72) the current $J^\mu(x)$ as the interaction picture operator (built out of the free field operators satisfying the "Dirac equations" (8.106)) is conserved,[32] $\partial_\mu J^\mu(x) = 0$, but vanishing of the four-divergence of the interaction picture operator is by itself not enough (as will be seen below on the example of spin 0 particles coupled to photons) to ensure the covariance of all S-matrix elements.

The only noncovariant piece in the amplitude (9.73) (and in the amplitudes corresponding to the other two diagrams of Figure 9.10) arises therefore only from the $k^2 t_\mu t_\nu$ term in (9.71). In the position space this part of $P_{\mu\nu}(\mathbf{k})$ gives in the propagator the term

$$iD^{\mathrm{F}}_{\mu\nu}(x - y) \supset \int \frac{d^4k}{(2\pi)^4} \, \delta^0_\mu \delta^0_\nu \, \frac{-i}{|\mathbf{k}|^2} \, e^{-ik^0(x^0-y^0)} e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{9.75}$$

$$= -i\delta(x^0 - y^0) \, \delta^0_\mu \delta^0_\nu \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, \frac{e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{|\mathbf{k}|^2} \tag{9.76}$$

$$= -i\delta(x^0 - y^0) \, \frac{\delta^0_\mu \delta^0_\nu}{4\pi|\mathbf{x} - \mathbf{y}|} \, .$$

Its contributions to amplitudes can however be canceled if, in addition to the term (9.72), the operator $V^I_{\mathrm{int}}(t) = \int d^3\mathbf{x} \, \mathcal{H}_{\mathrm{int}}(t, \mathbf{x})$ coupling charged particles to the massless spin 1 particle (the photon) is completed with the *spatially nonlocal* term

$$V^I_{\mathrm{nonlocal}}(t) = \frac{e^2}{2} \int d^3\mathbf{x} \int d^3\mathbf{y} \, \frac{J^0(t, \mathbf{x})J^0(t, \mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|} \, , \tag{9.77}$$

in which $J^0(x) = Q \, \bar{\psi}(x)\gamma^0\psi(x)$. Through the formula (9.28) such a term of $V^I_{\mathrm{int}}(t)$ gives rise to a new interaction vertex having the form of the contact interactions (it differs from the standard ones in that one integrates afterwards not over its space-time position but over its variables $t, \mathbf{x}, \mathbf{y}$) of the particles the creation and annihilation operators of which enter the current J_μ in (9.72). The Feynman rule this new vertex gives rise to is shown in Figure 9.11. The factor $1/2$ present in (9.77), is absent in the rule in Figure 9.11 due to

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[32]This should not be confused (as in Weiberg's textbook) with vanishing of the divergence of the Heisenberg picture current operator $J^\mu_H(x)$.
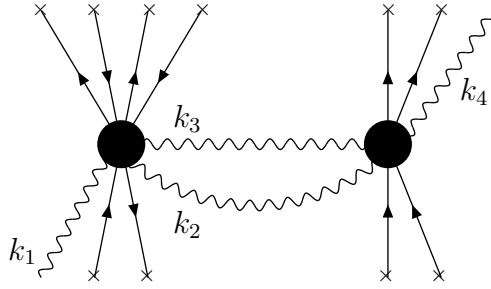
Figure 9.12: Example of a diagram in quantum electrodynamics of spin $\frac{1}{2}$ particles. Black blobs represent all possible diagrams (at a given order in e) that can be drawn with the given set of lines entering them.

combinatorics. It is clear that if somewhere in a Feynman diagram a photon line connects with the propagator (9.70) two vertices through which fermion lines are passing (i.e. a photon is exchanged between two currents J_μ), a similar diagram with this photon line replaced by the contact interaction of Figure 9.11 can be drawn. It follows, that when the contributions of both these diagrams are added, the contact interaction will precisely cancel the contribution of the noncovariant term in the photon propagator. As will be shown in Section 11.7, the interaction term (9.77) is obtained automatically if the classical electromagnetic field coupled to a curent J^μ is quantized in a consistent manner.

By analyzing amplitudes obtained from Feynman diagrams which include insertion(s) of the vertex(ices) arising from the term (9.77) added to the interaction Hamiltonian, their following important property, usually called *gauge invariance* (of the quantum theory), can be proved[33] (as the second step of the induction). Consider the complete set of Feynman diagrams contributing (at a given order in the coupling constant e) to one of the black blobs (say, the left one) shown in Figure 9.12. Some of the photon lines entering or leaving this blob can be connected (as in the figure) to some other interaction vertices forming a more complicated Feynman diagram and some may represent photons in the initial and/or final state. Provided all lines of *charged* particles passing through the left blob in Figure 9.12 *are on-shell* (i.e. provided all such lines represent particles in the initial and final states which means that their four-momenta satisfy the respective mass-shell conditions, $p^2 = m^2$, and are "closed up" with the appropriate "wave functions" $u(\mathbf{p}, \sigma)$, $v(\mathbf{p}, \sigma)$ or $\bar{u}(\mathbf{p}', \sigma')$, $\bar{v}(\mathbf{p}', \sigma')$), the analytical expression corresponding to this

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[33]In the approach based on quantization of the classical electromagnetic field coupled to particles of to other fields this property comes out formally as the so-called Ward-Takahashi identity related to the gauge symmetry of the classical action. In actual calculations based on the operator approach (as opposed to the one exploiting path integrals) it is satisfied due to a delicate cancellation of the Schwinger terms in the equal time commutator of two electromagnetic current Heisenberg picture operators against the so called sea-gull contributions covariantizing the chronological product of the current operators. The cancellation holds automatically in special regularization prescriptions which mafestly preserve gauge invariance.

blob $\mathcal{N}_{\mu_1,\mu_2,\ldots,\mu_n}(k_1, k_2, \ldots, k_n, \mathbf{p}_1, \ldots)$ satisfies the gauge invariance condition[34]

$$k_i^{\mu_i} \mathcal{N}_{\mu_1,\mu_2,\ldots,\mu_n}(k_1, k_2, \ldots, k_n, \mathbf{p}_1, \ldots) = 0. \tag{9.78}$$

With this property holding[35] for any such blob up to order e^n in the coupling constant e, in diagrams (which as wholes are of higher order in e) in which photon lines connect like in Figure 9.12 blobs of order e^k, where $k \leq n$, the noncovariant terms in the propagators connecting blobs can be dropped: the k_μ and/or k_ν terms will not contribute and the terms originating from the $t_\mu t_\nu$ parts of the propagators $i D_{\mu\nu}^{\mathrm{F}}(x - y)$ are canceled by the contact interaction as in Figure 9.11. The argument can be inductively extended to arbitrary orders in the coupling constant e. Thus, the S-matrix of a theory involving massless spin 1 particles coupled to spin $\frac{1}{2}$ particles will be Lorentz-covariant provided the interaction (9.72) is supplemented with the contact interaction (9.77).

Things are a little bit more complicated if the photon couples to spin 0 particles. Consider first the interaction of the form

$$\mathcal{H}_{(1)} = e\, J^\mu(x) A_\mu(x) = ie\, Q\left(\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi\right) A_\mu, \tag{9.79}$$

(i is inserted for Hermiticity). The current operator $J^\mu(x)$ in (9.79) is conserved, $\partial_\mu J^\mu(x) = 0$, because the interaction picture field operators $\phi(x)$ and $\phi^\dagger(x)$ out of which it is formed satisfy the "Klein-Gordon equation" (8.27). It is easy to check that the terms proportional to k_μ and k_ν arising in the photon propagator in the first diagram of Figure 9.13 (representing the lowest order contribution to the S-matrix element corresponding to elastic scattering of a spin 0 antiparticle on its particle) again do not contribute if the four-momenta of all spin 0 particles are on-shell (the u and v functions are both trivial - equal unity - in this case). Indeed, in the momentum space to the diagram 9.13a corresponds the expression

$$-i\mathcal{A} = (-ie\, Q)^2 (-p_1 - p_1')^\mu \frac{iP_{\mu\nu}(\mathbf{k})}{k^2 + i0} (p_2 + p_2')^\nu, \tag{9.80}$$

and since[36] $k_\mu = p_1 - p_1'$, the k_μ term in the photon propagator gives $(-p_1 - p_1') \cdot (p_1 - p_1') = p_1'^2 - p_1^2 = M^2 - M^2 = 0$. The term with k_ν gives zero in the same way. The remaining term in the photon propagator can be canceled by adding to the interaction (9.79) the contact interaction (9.77). However if to the blob in the diagram 9.13b (representing the lowest order contribution to the S-matrix element corresponding to the Compton scattering of

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[34]In the position space k_μ is replaced by the derivative ∂_μ; as we will argue, (9.78) generalizes the conservation of the current formed out of free field operators to the conservation of the current formed out of interacting Heisenberg picture field operators which will be introduced in due course.

[35]In higher orders in e one has to ensure that divergences of integrals corresponding to closed loops inside diagrams contributing to such blobs do not spoil this property. This can be achieved by judiciously choosing the regularization procedure (see Chapter 19).

[36]Lines corresponding to neutral particles like the photon do not carry arrows. Yet, one has to choose a direction of the flow of the four-momentum through such a line. Here we choose it from the left to the right.

photons off charged spin 0 particles) contributed (in the order e^2) only the diagrams 9.13c and 9.13d, the corresponding analytical expression would not have the property (9.78). The analytical expression the sum of these two diagrams gives rise to has the form

$$-i\mathcal{A} = \mathcal{N}^{\mu\nu}(k_1, k_2, \mathbf{p}_1, \mathbf{p}_2)\, \epsilon_\mu(\mathbf{k}_1)\, \epsilon_\nu^*(\mathbf{k}_2)\,, \tag{9.81}$$

with

$$\mathcal{N}^{\mu\nu} = (-ie\,Q)^2 i\left[\frac{(2p_2 + k_2)^\nu(2p_1 + k_1)^\mu}{(p_1 + k_1)^2 - M^2 + i0} + \frac{(2p_2 - k_1)^\mu(2p_1 - k_2)^\nu}{(p_2 - k_1)^2 - M^2 + i0}\right].$$

Contracting $\mathcal{N}^{\mu\nu}$ with e.g. k_1^μ keeping lines representing initial and final state spin 0 particles on-shell (i.e. setting $p_1^2 = p_2^2 = M^2$, so that the denominators of the two propagators become $2k_1 \cdot p_1 + k_1^2$ and $-2k_1 \cdot p_2 + k_1^2$, respectively) we get

$$k_1^\mu \mathcal{N}_{\mu\nu} = -ie^2 Q^2\left[(2p_2 + k_2)_\nu - (2p_1 - k_2)_\nu\right] = -2ie^2 Q^2 k_1^\mu g_{\mu\nu}\,. \tag{9.82}$$

We have used here the equality $p_2 - p_1 + k_2 = k_1$ (note that the conditions $k_1^2 = k_2^2 = 0$ have not been imposed!). Satisfying in order e^2 the identity (9.78) requires an additional contribution (despite the fact that the current $J^\mu(x)$ is conserved as the interaction picture operator). Such a contribution is provided by the diagram 9.13e which arises if the interaction (9.79) is supplemented by the yet another term

$$\mathcal{H}_{(2)} = -e^2 Q^2 \phi^\dagger \phi A_\mu A^\mu\,, \tag{9.83}$$

which gives rise to the diagram 9.13e and contributes to $\mathcal{N}^{\mu\nu}$ in (9.81) a term[37]

$$\Delta\mathcal{N}^{\mu\nu} = +2ie^2 Q^2 g^{\mu\nu}\,, \tag{9.84}$$

(the factor of 2 comes from two possible ways to attach two photon lines to the two lines in the vertex arising from (9.83)).

With the interaction $\mathcal{H}_{\text{int}}$ consisting of the sum of $\mathcal{H}_{(1)}$ (9.79), $\mathcal{H}_{(2)}$ (9.83) and of $V_{\text{nonlocal}}^I(t)$ (9.77) the property (9.78) can be proved[38] to hold to an arbitrary order in the coupling constant e also in the theory of massless spin 1 particles (photons) interacting with charged spin 0 particles and their antiparticles.

It will become clear in Sections 11.7 and 11.9 that the interactions (9.72) and (9.79) together with (9.83) which ensure the gauge invariance property (9.78) of amplitudes are precisely those which arise[39] from the canonical quantization of the classical theory of a

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[37]From the point of view of the Ward-Takahashi identity this term is just the so-called "seagull diagram" contribution which covariantizes the chronological product of the electromagnetic currents which ensures cancellation of the (regularization independent part) of the Schwinger term in the equal-time commutator of the Heisenberg picture electromagnetic current operators in such a theory.

[38]We do not attempt to prove it here; firstly, because we have first to consider renormalization and, secondly, the property (9.78) will be proved in a different way in Chapter 19.

[39]Since the interaction (9.79) involve derivatives of the field operators, in higher orders in e extra noncovariant terms in the interaction Hamiltonian will be needed to covariantize the contractions of operators involving space-time derivatives; also such terms come out automatically from the canonical operator quantization procedure.
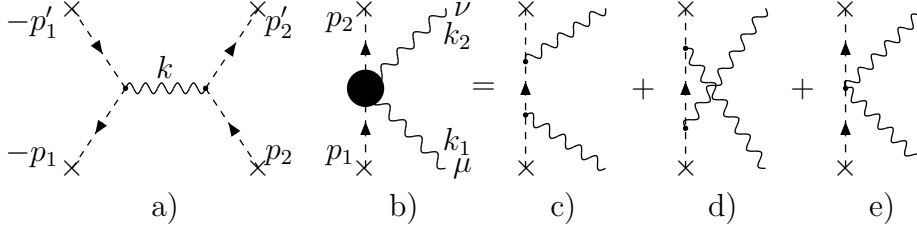
Figure 9.13: Examples of the Feynman diagrams in electrodynamics of spin 0 particles. The leftmost one contributes to the particle-antiparticle scattering; the remaining ones give the order e^2 contribution to the amplitude of the Compton process (photon scattering off a charged particle).

vector field coupled to a spinor or a complex scalar field in a gauge invariant manner, i.e. in such a way that the classical theory is invariant under the gauge transformations

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \Theta(x),$$
$$\psi(x) \to \psi'(x) = e^{-iQ\Theta(x)} \psi(x),$$
$$\phi(x) \to \phi'(x) = e^{-iQ\Theta(x)} \phi(x).$$
(9.85)

In particular, we will see in Section 11.7 that when the electromagnetic field interacting with matter is quantized in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the interaction $V_{\text{int}}^I(t)$ expressed in terms of the interaction picture operators takes the form

$$V_{\text{int}}^I(t) = -e \int d^3\mathbf{x}\, \mathbf{J}(t,\mathbf{x}) \cdot \mathbf{A}(t,\mathbf{x}) + \frac{e^2}{2} \int d^3\mathbf{x} \int d^3\mathbf{y}\, \frac{J^0(t,\mathbf{x}) J^0(t,\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}.$$
(9.86)

Because in any Lorentz frame the time-like component of the photon field operator (8.180), (8.162) vanishes, the first term above can be written in the formally covariant form (9.72). The second term is precisely the nonlocal interaction required to cancel the contributions of the noncovariant term in the propagator. Summarizing, provided the theory is regularized and renormalized in the appropriate manner so that the property (9.78) is maintained order by order in the expansion, one can simply drop the nonlocal term (9.77) in the interaction and use everywhere the covariant photon propagator

$$iD_{\mu\nu}^{\text{F}}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2+i0} e^{-ik\cdot(x-y)}.$$
(9.87)

There remains however an intriguing question what happens then to the Coulomb interaction energy represented by the second term in (9.86)? It is of course hidden in the covariant propagator (9.87). To see this, let us consider the photon propagator connecting two conserved currents $J_\mu(k)$ (in the momentum space) and take without loss of generality a Lorentz frame in which $k^\mu = (\omega, 0, 0, |\mathbf{k}|)$. Since the photon is off-shell (lines connecting vertices represent virtual particles) $\omega \neq |\mathbf{k}|$. In the chosen frame the current conservation condition $k_\mu J^\mu(k) = 0$ implies

$$J^3(k) = \frac{\omega}{|\mathbf{k}|} J^0(k).$$
(9.88)

407

The momentum space analytical expression which in the amplitude corresponds to the photon propagator connecting two currents can be then rewritten with the help of the relation (9.88) as

$$J^\mu \frac{g_{\mu\nu}}{k^2 + i0} J^\nu = \frac{J^0 J^0}{k^2 + i0} - \frac{\mathbf{J}_T \cdot \mathbf{J}_T}{k^2 + i0} - \frac{J^3 J^3}{k^2 + i0} \tag{9.89}$$

$$= \frac{J^0 J^0}{k^2 + i0} \left(1 - \frac{\omega^2}{\mathbf{k}^2} \right) - \frac{\mathbf{J}_T \cdot \mathbf{J}_T}{k^2 + i0} = -\frac{J^0 J^0}{\mathbf{k}^2} - \frac{\mathbf{J}_T \cdot \mathbf{J}_T}{k^2 + i0}. \tag{9.90}$$

where $\mathbf{J}_T$ is the current component orthogonal to the photon three-momentum. The first term is just the Coulomb interaction in the momentum space.

The property (9.78) of amplitudes (their gauge invariance) can be exploited to simplify the sums over polarizations of the incoming and/or outgoing photons. Consider an amplitude corresponding to an S-matrix element between initial and final states with one or more massless spin 1 particles. Concentrate on one of these particles carrying the (on-shell) four-momentum k and write the S-matrix element in the form

$$-i\mathcal{A} = \epsilon_\mu^{(*)}(\mathbf{k}, \lambda) \mathcal{N}^\mu(\mathbf{k}, \ldots), \tag{9.91}$$

factorizing explicitly the polarization vector $\epsilon_\mu^{(*)}(\mathbf{k}, \lambda)$ of the considered incoming (outgoing) photon. Gauge invariance (9.78) means that

$$k_\mu \mathcal{N}^\mu(\mathbf{k}, \ldots) = 0. \tag{9.92}$$

Owing to this relation, when the modulus of the amplitude squared is summed over spin projections of the photon present in the initial or final state, the rule

$$\sum_{\lambda=\pm 1} \epsilon_\mu(\mathbf{k}, \lambda) \epsilon_\nu^*(\mathbf{k}, \lambda) = P_{\mu\nu}(\mathbf{k}), \tag{9.93}$$

can be *effectively* replaced by the simpler (covariant) one[40]

$$\sum_{\lambda=\pm 1} \epsilon_\mu(\mathbf{k}, \lambda) \epsilon_\nu^*(\mathbf{k}, \lambda) \to -g_{\mu\nu}. \tag{9.94}$$

Indeed, let $J_{\mu\nu}(\mathbf{k}, \ldots) \equiv \mathcal{N}_\mu(\mathbf{k}, \ldots) \mathcal{N}_\nu^*(\mathbf{k}, \ldots)$. Summing over polarizations one should in principle write

$$\sum_{\lambda=\pm 1} \epsilon_\mu(\mathbf{k}, \lambda) \epsilon_\nu^*(\mathbf{k}, \lambda) J_{\mu\nu}(\mathbf{k}, \ldots) = P_{\mu\nu}(\mathbf{k}) J^{\mu\nu}(\mathbf{k}, \ldots). \tag{9.95}$$

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[40]A warning: in general this rule does not apply in non-Abelian gauge theories! The relation (9.78) holds provided external lines of *all charged* particles are on shell (that is also "closed up" with the appropriate functions u, u^*, v or v^*); since in non-Abelian theories massless spin 1 particles analogous to photons *are charged*, their external lines must be "closed up" with the true polarization vectors $\epsilon_\mu(\mathbf{k}, \lambda)$ or $\epsilon_\mu^*(\mathbf{k}, \lambda)$. It follows that if the amplitude has several external lines of such massless spin 1 particles the rule (9.94) can be applied only to one of these line if the remaining ones are on shell.

However - taking without loss of generality the (on-shell) photon four-momentum k^μ in the form $(|\mathbf{k}|, |\mathbf{k}|, 0, 0)$, and using the explicit form (9.69) of $P^{\mu\nu}(\mathbf{k})$ - one can write:

$$\begin{aligned} P_{\mu\nu}(\mathbf{k}) J^{\mu\nu}(\mathbf{k}, \ldots) &= J^{22}(\mathbf{k}, \ldots) + J^{33}(\mathbf{k}, \ldots) \\ &= J^{00}(\mathbf{k}, \ldots) - J^{11}(\mathbf{k}, \ldots) - J^{\mu}{}_{\mu}(\mathbf{k}, \ldots) . \end{aligned} \tag{9.96}$$

Since $k_\mu J^{\mu\nu}(\mathbf{k}, \ldots) = k_\nu J^{\mu\nu}(\mathbf{k}, \ldots) = 0$, in the chosen frame in which $k_\mu = (|\mathbf{k}|, |\mathbf{k}|, 0, 0)$ one has $J^{0\nu} = J^{1\nu}$ and $J^{\mu 0} = J^{\mu 1}$. Hence, setting $\nu = 0$ and $\mu = 1$, one finds that $J^{00} = J^{11}$ and the first two terms in the second line of (9.96) cancel each other. This proves the rule (9.94).

9.7 Corrections to external lines

In the preceding sections we have formulated the general method for computing S-matrix elements by perturbatively evaluating the right hand side of the formula

$$S_{\beta\alpha} = \langle \beta_0 | \mathrm{T} \exp\left(-i \int_{-\infty}^{\infty} dt \, V_{\mathrm{int}}^{I}(t) \right) | \alpha_0 \rangle . \tag{9.97}$$

The resulting expansion has been conveniently encoded in the Feynman diagrams and the Feynman rules. The method allows, in principle, to compute the desired S-matrix element in a given theory to arbitrarily high orders of the perturbative expansion.

However, in computing contributions to matrix elements arising from diagrams containing closed loops one encounters ill defined expressions which require introducing additional rules to correctly handle them. Within the approach to computing S-matrix elements formulated in Chapters 7, 8 and this one, these new rules require modifying the original interaction operator V_{int} one started with by adding to it in each order of the expansion (in powers of the coupling constant) new terms which can be used to cure the encountered problems. The most important class of ill defined expressions is due to integrating over arbitrarily large four-momenta of (virtual) particles in loops and is, therefore, related to ultraviolet properties of the considered theories. Their systematic treatement will be outlined in Chapter 14. Another class of ill defined expressions is associated with corrections to external lines of Feynman diagrams.[41] Although their treatement by modifying the interaction is intimately related to the treatement of the divergent integrals, their origin is conceptually different and because they seem (at first sight) to render the rules of calculating S-matrix elements invalid they must be therefore discussed already here. It should be said that this type of ill defined contributions to the S-matrix elements does not appear in nonrelativistic theories (formulated in the language of second quantization of Chapter 5) - indeed no such problem has been encountered in

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[41] Yet another class of ill defined contributions is encountered in theories of interacting massless particles. Their treatement is different (also different is the physical reason for their appearance) and they will be discussed in Section 19.6.

the computation of the elastic scattering amplitude within a nonrelativistic effective theory in Section 7.4. In such theories the interaction Hamiltonians of which have the form (5.75) or (7.70) by construction - see Section 5.3 - normally ordered with respect to the vector $|\text{void}\rangle$ and preserving the number of particles of each kind separately (processes of creation or annihilation of particle-antiparticle pairs are impossible), scattering amplitudes can be computed using directly the formula (9.97) essentially (apart for the need of renormalization of the couplings) along the lines outlined in this chapter.

The problem is best illustrated by considering the computation of the elastic scattering amplitude within the theory of spin 0 particles of mass M with the original interaction of the form $\mathcal{H}_{\text{int}} = (\lambda/4!)\varphi^4$. It should be clear (this will be shown rigorously in Chapter 17) that all Feynman diagrams contributing to this amplitude can be organized as shown in Figure 9.14, that is into (one-particle irreducible) diagrams contributing to the central white blob in the upper left skeleton diagram and the ones contributing to one of the four white blobs on the external lines. In turn, diagrams contributing to the blobs on the external lines can be organized into black blobs (consisting of one-particle irreducible diagrams), called *self energies* and denoted $-i\Sigma$, connected with one another by the ordinary propagators. As a result, the considered scattering amplitude, written down according to the Feynman rules discussed in this chapter, takes the schematic form

$$-i\mathcal{A} = -i\left(\lambda + \ldots\right)\prod_{j=1}^{4}\left\{1 + \frac{i}{p_j^2 - M^2}\left(-i\Sigma(p_j^2)\right)\right.$$
$$\left. + \frac{i}{p_j^2 - M^2}\left(-i\Sigma(p_j^2)\right)\frac{i}{p_j^2 - M^2}\left(-i\Sigma(p_j^2)\right) + \ldots\right\}, \quad (9.98)$$

With the interaction consisting of the single operator $\mathcal{H}_{\text{int}} = (\lambda/4!)\varphi^4$ the first $(-i\Sigma$ is by definition at least of the one-loop order) contribution to the self-energy[42] Σ is given by the diagram shown in the upper right panel of figure 9.14.

$$\Sigma(p^2) = \frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - M^2 + i0} + \ldots \quad (9.99)$$

Since to obtain from (9.98) the S-matrix element each of the four-momenta p_j must be taken on-shell, i.e. one has to set $p_j^2 = M^2$, the scattering amplitude (9.98) is plainly infinite due to the presence in the curly brackets of the propagators $i/(p_j^2 - M^2)$ and this singular behavior is independent of the question of finiteness of the self-energy $\Sigma(p^2)$ itself!

The resolution of the problem lies in recalling the postulates underlying the approach to calculating S-matrix elements developed in Chapter 7. This approach is based on the assumption that the Hamiltonians H and H_0 have the same spectrum, and that there

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[42]It is the peculiarity of the φ^4 interaction that the one-loop contribution to $\Sigma(p^2)$ is independent of the four-momentum p. For the sake of illustration we can however ignore this pathology because in higher orders $\Sigma(p^2)$ does depend on p.
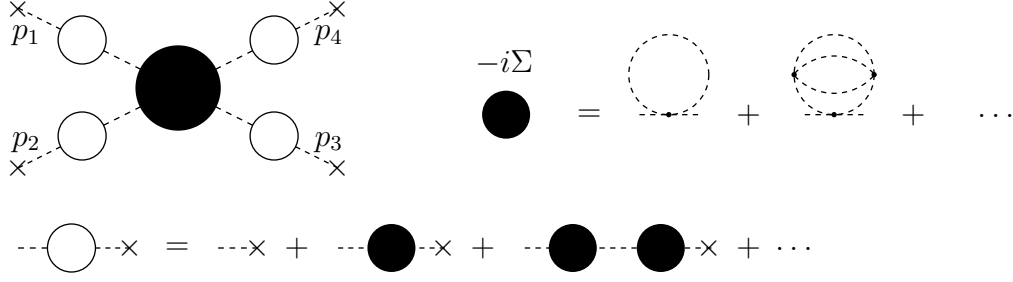
Figure 9.14: The "skeleton" representation of the amplitude $-i\mathcal{A}$ of the elastic scattering of spin 0 particles the interaction of which is given by $\mathcal{H}_{\text{int}} = (\lambda/4!)\varphi^4$. The representation makes the factorization of corrections to external lines explicit. The central white blob represents the sum of all one-particle irreducible Feynman diagrams. The sum of diagrams contributing to the white blobs on external lines of the upper left "skeleton" diagram is organized as in the lower part of the figure into sums of self energy insertions $-i\Sigma(p^2)$ (to which only one-particle irreducible diagrams coontribute as shown in the upper right part of the figure) connected by the ordinary propagators. Crosses on lines mean that these lines are on shell. The white and black blobs do not include the propagators corresponding to the lines entering into them.

is a well defined, specified by the relation (7.39), one-to-one correspondence between the full Hamiltonian *in* and *out* eigenvectors and the eigenvectors of H_0. These postulates imply, in the first place, that the ground states of H_0 and H must have the same energies (as we have already discussed at the end of Section 9.4, this can trivially be satisfied by adding to $\mathcal{H}_{\text{int}}$ a constant term allowing to cancel the relative phase between the *in* and *out* vacuum vectors) and, secondly, that the mass parameter(s) M in H_0 (and hence also in free propagator(s)) is (are) exactly equal to the mass(es) M_{ph} of the physical particle(s) represented by the *in* and *out* states. Moreover, the postulated relation (7.39) implies that the matrix element of the Heisenberg picture field operator $\tilde{\varphi}_H(x)$ constructed out of the free field operator $\varphi(x)$ (obtained in Chapter 8) according to the prescription[43]

$$\tilde{\varphi}_H(x) = \tilde{\varphi}_H(t, \mathbf{x}) \equiv e^{iHt}\,\varphi(0, \mathbf{x})\,e^{-iHt}\,, \tag{9.100}$$

between the *in* vacuum $|\Omega_+\rangle$ (the *out* vacuum $|\Omega_-\rangle$) state-vector and the *in* one-particle vector $|(\mathbf{p})_+\rangle$ (the *out* vector $|(\mathbf{p})_-\rangle$) is exactly equal to the matrix element of the free (interaction picture) field operator $\varphi(x)$ between the corresponding free Hamiltonian eigenvectors. This follows from taking formally the limit

$$\lim_{t\to\mp\infty}\langle\Omega_0|\varphi(x)|(\mathbf{p})_0\rangle = \lim_{t\to\mp\infty}\langle\Omega_0|e^{iH_0 t}e^{-iHt}e^{iHt}\varphi(0, \mathbf{x})\,e^{-iHt}e^{iHt}e^{-iH_0 t}|(\mathbf{p})_0\rangle$$
$$= \lim_{t\to\mp\infty}\langle\Omega_\pm|\tilde{\varphi}_H(t, \mathbf{x})|(\mathbf{p})_\pm\rangle\,, \tag{9.101}$$

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[43]We denote the Heisenberg picture operator constructed in this way $\tilde{\varphi}_H(x)$ (in Section 11.10 this operator will be denoted $\varphi_{\text{ph}}(x)$), and not just $\varphi_H(x)$, because it will turn out to be related to the canonical operator $\varphi_H(x)$ obtained by quantizing the corresponding classical field theory by an appropriate rescaling (renormalization) transformation.

using the assumed relation (7.39). But since $\langle \Omega_0|\varphi(x)|(\mathbf{p})_0 \rangle = e^{-ipx}$ and, by Poincaré invariance,

$$\langle \Omega_\pm|\tilde{\varphi}_H(x)|(\mathbf{p})_\pm \rangle = \langle \Omega_\pm|e^{i\hat{P}x}\,\tilde{\varphi}_H(0)\,e^{-i\hat{P}x}|(\mathbf{p})_\pm \rangle = e^{-ipx}\,\langle \Omega_\pm|\varphi(0)|(\mathbf{p})_\pm \rangle = \text{Const.}\, e^{-ipx}\,,$$

(assuming that $\hat{P}^\mu|\Omega_\pm \rangle = 0$) the relation (9.101) implies that the equality must hold for any instance t, not only asymptotically. As will be shown in Chapter 13, these two conditions (the equality of M in the propagator to the physical mass and (9.101)) together imply that the momentum space two-point Green's function $\tilde{G}^{(2)}(p^2)$ defined as the Fourier transform

$$(2\pi)^4\delta^{(4)}(p+p')\,i\tilde{G}^{(2)}(p^2) \equiv \int d^4x \int d^4y\, e^{-ip\cdot x}e^{-ip'\cdot y}\,\langle \Omega_-|\mathrm{T}[\tilde{\varphi}_H(x)\tilde{\varphi}_H(y)]|\Omega_+ \rangle\,, \quad (9.102)$$

of the position space Green's function $G^{(2)}(x-y)$ - the relativistic theory counterpart of the nonrelativistic single-particle Green's function considered in Section 5.7 - has a simple pole at $p^2 = M^2 = M_{\mathrm{ph}}^2$ with the residue exactly equal to i.

Obviously, in the presence of the interaction term such as $\mathcal{H}_{\mathrm{int}} = (\lambda/4!)\varphi^4$ physical particles experience self-interaction (the upper right diagram in Figure 9.14 represents one of its possible effects) which cannot be switched off and which inevitably shifts the pole of the two-point Green's function $\tilde{G}^{(2)}(p^2)$ away from $p^2 = M^2$ and causes the pole residue to deviate from i. Therefore, to satisfy the postulates, the original interaction $(\lambda/4!)\varphi^4$ has to be supplemented by some additional self-interaction terms which must cancel out the effects of the particle self-interaction due to the $(\lambda/4!)\varphi^4$ term. In the case of the theory of interacting spinless particles the terms which can ensure this are[44]

$$\Delta\mathcal{H}_{\mathrm{int}} = -\frac{1}{2}\,\delta Z\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\,\Delta M^2\varphi^2 - \frac{1}{2}\,\delta Z'\partial_0\varphi\partial_0\varphi\,. \quad (9.103)$$

The coefficients δZ, $\delta Z'$ and ΔM^2 in (9.103) should be treated as power series in the coupling constant(s) (here λ) and adjusted in each order of the perturbative expansion so as to satisfy the conditions specified below. In computing amplitudes in the perturbative expansion, the terms of (9.103) with the coefficients determined in lower orders of the expansion must be then used as interaction vertices in building diagrams of higher orders. Since the first of the added terms involves derivatives acting on the field operators, in some Feynman diagrams it will give rise to noncovariant terms in the propagators as discussed in Section 9.5. For this reason one has also to add to $\Delta\mathcal{H}_{\mathrm{int}}$ the last, noncovariant term, the only role of which is to cancel the noncovariant terms in the propagators in order to produce a Lorentz covariant S-matrix.[45] Thus, in practice the last term can be dropped provided one neglects simultaneously noncovariant pieces of the propagators.

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[44]The proper notation for these "counterterms" will be fixed in Chapter 14 in which the general procedure of removing divergences arising from integrations over four-momenta of internal lines of loop diagrams with the help of suitable counterterms will be discussed.

[45]Again, it will be seen that precisely such terms with $\delta Z'$ appropriately correlated with δZ will come out automatically in the approach based on quantization of fields.

If the terms (9.103) are included in the interaction and used as specified above, the self energy $\Sigma^{(n)}(p^2)_{\text{complete}}$ which includes all contribution up to (and including) the n-th order can be written as the sum of two parts:[46]

$$-i\Sigma^{(n)}(p^2)_{\text{complete}} = -i\left[\Sigma^{(n)}(p^2) - \delta Z^{(n)}p^2 + \Delta^{(n)}M^2\right], \tag{9.104}$$

where to $-i\Sigma^{(n)}(p^2)$ contribute, in addition to the interactions generated by the original term $\mathcal{H}_{\text{int}} = (\lambda/4!)\varphi^4$, also the terms (9.103) with coefficients up to the $(n-1)$-st order and $\delta Z^{(n)}$ and $\Delta^{(n)}M^2$ denote the n-th order contributions to δZ and ΔM^2. If now $\Sigma^{(n)}(p^2)$ is expanded in the Taylor series around $p^2 = M^2$

$$\Sigma^{(n)}(p^2) = \Sigma^{(n)}(M^2) + (p^2 - M^2)\Sigma^{(n)\prime}(M^2) + \frac{1}{2}(p^2 - M^2)^2\Sigma^{(n)\prime\prime}(M^2) + \dots$$

it becomes clear that to eliminate the problem with corrections to external lines, it is sufficient to adjust (in every order of the perturbative expansion) $\delta Z^{(n)}$ and $\Delta^{(n)}M^2$ so that the first two terms in this Taylor expansion of $\Sigma^{(n)}(p^2)$ are canceled. In this way the resulting "subtracted" self-energy $\Sigma(p^2)$ (which will be called renormalized self-energy and denoted $\Sigma_R(p^2)$) will always (in every order in the coupling constant(s)) satisfy this condition. Since the remaining terms of $\Sigma_R(p^2)$ vanish faster than $p^2 - M^2$, the whole curly bracket in (9.98) will, when the limit $p_j^2 \to M^2$ is taken, reduce to unity (thus will no longer be singular). As a result, if the original interaction is completed with the terms (9.103) the coefficients of which are recursively fixed by the procedure described here, one can simply omit all contributions to the scattering amplitudes which in the language of Feynman diagrams can be interpreted as corrections to external lines (the white blobs on external lines in Figure 9.14).

Similarly, in the case of external lines corresponding to spin $\frac{1}{2}$ particles the necessary additional interactions which must be added to $\mathcal{H}_{\text{int}}$ are those given (omitting possible noncovariant terms) in (9.53). Their contributions to amplitudes allow to cancel out, order by order in the expansion, the first two terms in the Taylor series expansion of fermionic self-energy[47]

$$\Sigma^{(n)}(\not{p}) = \Sigma^{(n)}(m) + (\not{p} - m)\Sigma^{(n)\prime}(m) + \frac{1}{2}(\not{p} - m)^2\Sigma^{(n)\prime\prime}(m) + \dots, \tag{9.105}$$

and to avoid in scattering amplitudes the dangerous terms which arise as corrections to external lines.

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[46]In these manipulations one assumes that the theory is regularized and is successively being, order by order, renormalized. This will be descussed in Chapter 14.

[47]Since the fermion self-energy Σ is a matrix in the spinor indices, it depends on $\not{p}$ (it should be written as $\Sigma(\not{p})$); therefore m as the argument of Σ and its derivatives should be understood as m times the unit matrix in the spinor indices. The limit $\not{p} \to m$ must be then understood as taken on amplitudes "closed up" with the wave functions $u(\mathbf{p}, \sigma)$, $\bar{u}(\mathbf{p}, \sigma)$, etc. (because $\not{p}u(\mathbf{p}, \sigma) = mu(\mathbf{p}, \sigma)$ etc.). This is precisely how $\Sigma(\not{p})$ appears in the expression resulting from the diagrams 9.7c - 9.7f. Since $\Sigma(\not{p})$ is in this expression taken for $p^2 = m^2$, canceling the first two terms in the Taylor series in (9.105) eliminates the entire contribution of the diagrams 9.7c - 9.7f, as suggested at the end of section 9.4.

In the following chapters it will become clear that canceling the first two terms of the Taylor expansion around the value $p^2 = M^2$ in the case of bosons and around $\not{p} = m$ in the case of fermions of the self-energy insertions Σ against the additional interactions like those specified in (9.103) or (9.53) is precisely what is required to ensure that the physical masses of the particles represented by the *in* and *out* state-vectors $|\alpha_\pm\rangle$ are identical to masses of free-particles represented by the H_0 eigenvectors $|\alpha_0\rangle$ and that the relation (7.39) of these eigenvectors of the full H (which now includes the additional terms (9.103)) and of H_0 is preserved because the two-point Green's function $i\tilde{G}(p)$, which in the case of spin 0 particles takes the form $i/[p^2 - M^2 - \Sigma_R(p^2)]$ will have a simple pole at $p^2 = M^2$ with the residue i as is implied by these relations. The subtraction procedure described above will be shown in Chapter 14 to work (in a special class of theories) to all orders of the perturbative expansion. As said, in higher orders the additional terms (9.103) in the total interaction $\mathcal{H}_{\text{int}}$ will generate interaction vertices entering also those parts of Feynman diagrams that cannot be interpreted as corrections to external lines and will prove necessary to remove some of the divergences of the first class (related to the UV behaviour of amplitudes). In this way, the procedure outlined here, called the *On-Shell scheme* is part of the general renormalization procedure which will be discussed in Chapter 14. However, it is important to stress that from the conceptual point of view the On-Shell scheme is enforced by the assumptions specified in Section 7.3 and is therefore a necessary ingredient of the direct (i.e. based on the formula (9.18)) calculation of S-matrix elements.

It should, however, be said that the procedure of avoiding singularities on external lines outlined here may turn impossible to be carried out in some theories because the factors like $\delta Z^{(n)}$ and $\Delta^{(n)} M^2$ must be real (to maintain Hermiticity of the interaction operator) while the self energies evaluated on shell, like $\Sigma(M^2)$, may turn out to have a nonzero imaginary part. By the unitarity relations discussed in Section 7.6 such an imaginary part signals that the particle corresponding to the considered external line is in fact not stable (can decay) and, therefore, the corresponding *in* and *out* one-particle state-vectors do not exist (the full Hamiltonian H does not possess the eigenvectors which would represent such particles). In this way the assumed one-to-one correspondence od H_0 and H spectra breaks down and such theories cannot, strictly speaking, be developed along the lines adopted here - they should be formulated using the approach based on Green's functions. In fact, carrying out the procedure discussed here, and therefore the preservation of the assumed one-to-one correspondence, is possible only in very special theories (of which quantum electrodynamics is the most prominent and, among realistic theories, almost the unique example).

In Chapter 13 a more general approach to investigating properties of quantum field theory models will be formulated. It will be based on Green's functions and will not rely on the assumptions (impossible to satisfy in many theories) about the one-to-one correspondence between the free Hamiltonian eigenvectors $|\alpha_0\rangle$ and the *in* and *out* eigenvectors of the full Hamiltonian $H = H_0 + V_{\text{int}}$. In this more general approach S-matrix elements are accessible (if particle-like *in* and *out* states of the full Hamiltonian do exist) through the Lehman-Symanzik-Zimmermann (LSZ) prescription (described in Section 13.4) ap-

plied to appropriate Green's functions. This approach will give a considerable freedom in setting the perturbative expansion (of Green's functions) which will imply a corresponding freedom in formulating renormalization conditions (specification of how subtractions in self-energies and other important functions are made). This flexibility will be then exploited in the renormalization group methods discussed in Chapter 18. The On-Shell scheme formulated here is, from this point of view, only a one particular possibility (singled out by its relation to the old-fashioned, shaped by the historical development of quantum electrodynamics and not always applicable, way of computing S-matrix elements) out of many other possible, renormalization prescriptions.