10 Decay rates and cross sections

In this chapter we relate S-matrix elements to measurable rates of physical processes. This can be done in many different ways. In the spirit of our approach, we chose the method which has direct connection with the general quantum mechanical approach to transition rates presented in Chapter 2. To this end we generalize first the Fermi's Golden Rule to higher orders of the perturbation expansion.

10.1 Fermi's Golden Rule to higher orders

To extend the Fermi's Golden Rule to higher orders of the perturbative expansion it is convenient to use a slightly different approach than the one adopted in Chapter 2. Here from the beginning we specify the formulae to the case of particle interactions. We thus have $H = H_0 + V_{\rm int}$, where

$$V_{\rm int} = \int d^3\mathbf{x}\, \mathcal{H}_{\rm int}(0, \mathbf{x})\,, \tag{10.1}$$

is a time independent interaction constructed out of free field operators and the eigenstates of H_0 are represented by the state-vectors $|\alpha_0\rangle$ introduced in Chapter 7. In Chapter 2 we were interested in the time developement of th system prepared in the state $|\alpha_0\rangle$ at $t = 0$. Here insted the the initial conditions will be set at $t = -\infty$ and correspondingly the interaction $V_{\rm int}$ given by (10.1) will be modified to

$$V_{\rm int} \to V_{\rm int}\, e^{\varepsilon t}\,, \tag{10.2}$$

to ensure its adiabatic switching off in the far past, $t \to -\infty$. The limit $\varepsilon \to 0^+$ will be taken at the end. We assume that for $t = -\infty$ the system was in a state $|\alpha_0\rangle$ and compute the probability of it being in a state $|\beta_0\rangle$ at some later time t. From the general formulae (2.4) we obtain

$$\mathcal{A}^{(1)}_{\beta\alpha}(t) = \frac{1}{i\hbar}\int_{-\infty}^{t} dt'\, V_{\beta\alpha}\, e^{i(\omega_{\beta\alpha}-i\varepsilon)t'} = \frac{1}{i\hbar}\,\frac{e^{i(\omega_{\beta\alpha}-i\varepsilon)t}}{i(\omega_{\beta\alpha}-i\varepsilon)}\, V_{\beta\alpha}\,, \tag{10.3}$$

where $V_{\beta\alpha} = \langle\beta_0|V_{\rm int}|\alpha_0\rangle$, and

$$\begin{aligned}
\mathcal{A}^{(2)}_{\beta\alpha}(t) &= \frac{1}{i\hbar}\int_{-\infty}^{t} dt'\, V_{\beta\gamma}\, e^{i(\omega_{\beta\gamma}-i\varepsilon)t'}\, a^{(1)}_{\gamma\alpha}(t') \\
&= \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{t} dt' \int d\gamma\, V_{\beta\gamma}V_{\gamma\alpha}\, \frac{e^{i(\omega_{\beta\alpha}-2i\varepsilon)t'}}{i(\omega_{\gamma\alpha}-i\varepsilon)} \\
&= \left(\frac{1}{i\hbar}\right)^2 \int d\gamma\, \frac{e^{i(\omega_{\beta\alpha}-2i\varepsilon)t}}{i(\omega_{\beta\alpha}-2i\varepsilon)i(\omega_{\gamma\alpha}-i\varepsilon)}\, V_{\beta\gamma}V_{\gamma\alpha}\,.
\end{aligned} \tag{10.4}$$

The formulae for $\mathcal{A}^{(n)}_{\beta\alpha}(t)$ for $n > 2$ can also be readily written down.

The (Schrödinger picture) state-vector $|\Psi(t)\rangle$ of the system at time t is then given (setting $\varepsilon = 0^+$) by

$$|\Psi(t)\rangle = e^{-iE_\alpha t/\hbar}\left\{|\alpha_0\rangle + \int d\beta\, \frac{V_{\beta\alpha}}{E_\alpha - E_\beta + i0}\,|\beta_0\rangle \right. \\ \left. + \int d\beta \int d\gamma\, \frac{V_{\beta\gamma}V_{\gamma\alpha}}{(E_\alpha - E_\beta + i0)(E_\alpha - E_\gamma + i0)}\,|\beta_0\rangle + \dots\right\}. \quad (10.5)$$

The second term in the brackets resembles the ordinary first order correction to the wave function obtained in the time-independent perturbation calculus. In fact, with λ set to zero, the perturbation $V_{\rm int}$ considered here is time independent and indeed one could guess at the formula (10.5) on the basis of the Rayleigh-Schrödinger stationary perturbative expansion.[1] The advantage of the procedure used here is that it automatically gives the prescription "$+i0$" for avoiding singularities in the denominator in (10.5) occurring in the integration over the continuous part of the spectrum.

According to the discussion of Section 2.1, the modulus squared of the amplitude of finding the system at time t in the H_0 eigenstate $|\beta_0\rangle \neq |\alpha_0\rangle$ is given by $|\mathcal{A}_{\beta\alpha}(t)|^2 = \left|\mathcal{A}^{(1)}_{\beta\alpha}(t) + \mathcal{A}^{(2)}_{\beta\alpha}(t) + \dots\right|^2$. Thus, still before setting $\varepsilon \to 0$,

$$|\mathcal{A}_{\beta\alpha}(t)|^2 = \frac{e^{2\varepsilon t/\hbar}}{(E_\beta - E_\alpha)^2 + \varepsilon^2}\left|V_{\beta\alpha} + \int d\gamma\, \frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha - E_\gamma + i\varepsilon} + \dots\right|^2, \quad (10.6)$$

(where we have redefined $\hbar\varepsilon \to \varepsilon$). The rate (which, because the states $|\alpha_0\rangle$ are not normalizable, has not yet the physical dimension allowing to interpret it as the transition probability per unit time) per unit time can be now defined as[2]

$$w(\alpha \to \beta) \equiv \frac{d}{dt}\,|\mathcal{A}_{\beta\alpha}(t)|^2 = \frac{2\pi}{\hbar}\delta(E_\beta - E_\alpha)\left|V_{\beta\alpha} + \int d\gamma\, \frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha - E_\gamma + i0} + \dots\right|^2, \quad (10.7)$$

where we have taken at the end the limit $\varepsilon \to 0$. Thus we recover the result of Section 2.6 that the amplitude squared $|\mathcal{A}_{\beta\alpha}(t)|^2$ of the transition grows linearly with time and, therefore, the rate per unit time is constant.

[1] The Rayleigh-Schrödinger expansion usually gives corrections to discrete energy levels (discrete energy levels of the full Hamiltonian $H = H_0 + V_{\rm int}$ are shifted with respect to the corresponding discrete levels of H_0). In the continuous part of the spectrum one cannot distinguish "unperturbed" and "perturbed" energy levels and, consequently, there is no correction to the energy levels.

[2] We use here the formula

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)} = \delta(x)\,.$$

10.2 Decay rates and cross sections

Comparing the expression (10.7) with the formula (7.62) for $t_{\beta\alpha}(E_\alpha)$ we see that the rate per unit time is given by

$$w(\alpha \to \beta) = \frac{2\pi}{\hbar}\delta(E_\beta - E_\alpha)\,|t_{\beta\alpha}|^2\,. \tag{10.8}$$

In the case of particle interactions the final state $|\beta_0\rangle$ always belongs to a continuous spectrum of H_0. Experimentally it is almost indistinguishable from other states[3] with momenta infinitesimally different from the momenta of particles in the state $|\beta_0\rangle$. Therefore, following the arguments given in section 2.6, one is interested in the transition rate to any of the states having energy between E_β and $E_\beta + dE_\beta$ (and other characteristics like directions of the particle momenta and their spins also similar to those of the state $|\beta_0\rangle$):

$$dw(\alpha \to \beta) = \frac{2\pi}{\hbar}\delta(E_\beta - E_\alpha)\,|t_{\beta\alpha}|^2\,\rho(E_\beta)dE_\beta\,. \tag{10.9}$$

The factor $\rho(E_\beta)dE_\beta$ gives the number of states with required characteristics (spin projections, directions of the momenta) in the interval $(E_\beta,\ E_\beta + dE_\beta)$.

The density of states $\rho(E)$ can only be computed if the system is enclosed in a box of finite volume $V = L^3$ so that the H_0 spectrum becomes discrete. Recall, that the state-vectors representing free particles $|\alpha_0\rangle = |\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle$ are normalized according to the rule

$$\langle \mathbf{p}'_n\sigma'_n, \dots \mathbf{p}'_1\sigma'_1|\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle = 2E_{\mathbf{p}_1}(2\pi)^3\delta_{\sigma_1\sigma'_1}\delta^{(3)}(\mathbf{p}_1 - \mathbf{p}'_1)\dots \\ \pm \text{ permutations,} \tag{10.10}$$

where "permutations" pertains to groups of labels of identical particles. Enclosing the system in a box of volume $V = L^3$ and imposing periodic boundary conditions we select only some subset of the one-particle wave vectors - namely those of the form $\mathbf{p}_i = (2\pi/L)\mathbf{n}_i$ ($\mathbf{n}_i$ are the vectors whose components take integer values). In the box the delta functions are replaced with

$$(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}') \to \int_V d^3\mathbf{x}\, e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} = V\delta_{\mathbf{p},\mathbf{p}'}\,, \tag{10.11}$$

and the multiparticle state-vectors one works with are normalized so that

$$\langle \mathbf{p}'_n\sigma'_n, \dots \mathbf{p}'_1\sigma'_1|\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle = 2E_{\mathbf{p}_1}V\delta_{\sigma_1\sigma'_1}\delta_{\mathbf{p}_1,\mathbf{p}'_1}\dots \\ \pm \text{ permutations.} \tag{10.12}$$

[3]Such other states can also represent more particles than does the state $|\beta_0\rangle$ itself provided the energy of the additional particles is sufficiently low (i.e. they must be massless and "soft") so that they cannot be experimentally distinguished from the ones represeted by $|\beta_0\rangle$. This observation is important for avoiding infrared divergences arising due to massless virtual particle exchanges in quantum electrodynamics and chromodynamics.

In order to facilitate ascribing the usual probabilistic quantum mechanical interpretation to the Fermi's Golden Rule it is convenient to work with the state-vectors

$$|\alpha_0\rangle_V \equiv |\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle_V = \frac{1}{\sqrt{2E_{\mathbf{p}_1}V \dots 2E_{\mathbf{p}_n}V}}\,|\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle\,, \tag{10.13}$$

normalized so that

$$\begin{aligned} {}_V\langle\mathbf{p}'_n\sigma'_n, \dots \mathbf{p}'_1\sigma'_1|\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle_V = \delta_{\sigma_1\sigma'_1}\delta_{\mathbf{p}_1,\mathbf{p}'_1}\dots \\ \pm \text{ permutations.} \end{aligned} \tag{10.14}$$

In the box of volume $V = L^3$ the number of states representing n_β free particles having their momenta (or, rather, the wave-vectors) infinitesimally different from $\mathbf{p}_1 \dots \mathbf{p}_n$ is given by

$$d\beta = \prod_{i=1}^{n_\beta} \frac{V}{(2\pi)^3}\, d^3\mathbf{p}_i \equiv \rho(E_\beta)\, dE_\beta\,. \tag{10.15}$$

The matrix elements $V_{\beta\alpha}$ etc. in the formulae (10.5)-(10.7) must then consistently be computed between the properly normalized state-vectors[4] $|\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle_V$; the form of the formula (10.8) leading to the proper probabilistic interpretation therefore is

$$dw(\alpha\to\beta) = \frac{2\pi}{\hbar}\delta(E_\beta - E_\alpha)\,|V_{\beta_V\alpha_V} + \dots|^2 \prod_{j=1}^{n_\beta} \frac{V}{(2\pi)^3}\, d^3\mathbf{p}_j\,, \tag{10.16}$$

with $V_{\beta_V\alpha_V} \equiv {}_V\langle\beta_0|V_{\rm int}|\alpha_0\rangle_V$. Expressing the state-vectors $|\beta_0\rangle_V$ and $|\alpha_0\rangle_V$ in (10.16) back in terms of the vectors $|\beta_0\rangle = |\mathbf{p}_1\sigma_1, \dots \mathbf{p}_n\sigma_n\rangle$ and $|\alpha_0\rangle = |\mathbf{k}_1 s_1, \dots\rangle$, respectively, we obtain[5]

$$dw(\alpha\to\beta) = \frac{2\pi}{\hbar}\delta(E_\beta - E_\alpha)\,|V_{\beta\alpha} + \dots|^2\, \frac{1}{2E_{\mathbf{k}_1}V\dots} \prod_{j=1}^{n_\beta} \frac{d^3\mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}}\,, \tag{10.17}$$

that is,

$$dw(\alpha\to\beta) = \frac{2\pi}{\hbar}\delta(E_\beta - E_\alpha)\,|t_{\beta\alpha}|^2\, \frac{1}{2E_{\mathbf{k}_1}V\dots} \prod_{j=1}^{n_\beta} \frac{d^3\mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}}\,, \tag{10.18}$$

The factors of V corresponding to the final state $|\beta_0\rangle_V$ have canceled completely against the factors of V in the density of the final states $\rho(E_\beta)$ and the factors $2E_{\mathbf{p}_i}$ from the

[4] These state-vectors are not properly normalized only if momenta of two or more identical particles coincide (see Section 5.1), i.e. in a zero measure subset of the final states phase space.

[5] When the state-vectorss $|\gamma\rangle_V$ are normalized in the box, the integrals $\int d\gamma$ in the formula (10.7) are replaced by $\prod_{i=1}^{N}\sum_{\mathbf{q}_i}$. In the limit $V\to\infty$ each $\sum_{\mathbf{q}_i}$ goes over into $[V/(2\pi)^3]\int d^3\mathbf{q}_i$ which combines with the appropriate factor $(1/\sqrt{2E_{\mathbf{q}_i}V})^2$ arising from expressing $|\gamma\rangle_V\,{}_V\langle\gamma|$ through $|\gamma\rangle\langle\gamma|$ to give the Lorentz invariant integral $\int d\Gamma_{\mathbf{q}_i}$.

normalization of the state $|\beta_0\rangle_V$ have completed the integration measures making them Lorentz invariant. The factors $2E_{\mathbf{k}_1}V\dots$ in the denominator come from the initial state.

Next, to express $|t_{\beta\alpha}|^2$ in terms of the modulus square of the amplitude $\mathcal{A}_{\beta\alpha}$ we use the relation (7.82)

$$t_{\beta\alpha} = (2\pi)^3\delta^{(3)}(\mathbf{P}_\beta - \mathbf{P}_\alpha)\mathcal{A}_{\beta\alpha}\,,$$

which in the box (see (10.11)) takes the form[6]

$$t_{\beta\alpha} = V\,\delta_{\mathbf{P}_\beta,\mathbf{P}_\alpha}\mathcal{A}_{\beta\alpha}\,. \tag{10.19}$$

Thus

$$|t_{\beta\alpha}|^2 = V^2\,\delta_{\mathbf{P}_\beta,\mathbf{P}_\alpha}\,|\mathcal{A}_{\beta\alpha}|^2 = V\,|\mathcal{A}_{\beta\alpha}|^2\int_V d^3\mathbf{x}\,e^{i(\mathbf{P}_\beta-\mathbf{P}_\alpha)\cdot\mathbf{x}}\,, \tag{10.20}$$

where we have used $(\delta_{\mathbf{P}_\beta,\mathbf{P}_\alpha})^2 = \delta_{\mathbf{P}_\beta,\mathbf{P}_\alpha}$. Hence, the final formula reads

$$\begin{aligned} dw(\alpha\to\beta) &= \frac{V}{2E_{\mathbf{k}_1}V\dots}\,|\mathcal{A}_{\beta\alpha}|^2 \\ &\quad\times\frac{2\pi}{\hbar}\,\delta(E_\beta - E_\alpha)\int_V d^3\mathbf{x}\,e^{i(\mathbf{P}_\beta-\mathbf{P}_\alpha)\cdot\mathbf{x}}\prod_{j=1}^{n_\beta}\frac{d^3\mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}}\,. \end{aligned} \tag{10.21}$$

In the limit $V\to\infty$ the integral over $d^3\mathbf{x}$ in this formula is replaced by $(2\pi)^3\delta^{(3)}(\mathbf{P}_\beta-\mathbf{P}_\alpha)$ completing the four-dimensional Lorentz-invariant delta function $(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)$.

As explained in Section 2.6 the rate $dw(\alpha\to\beta)$ should be interpreted within the statistical ensemble of identical systems: it is the fraction of the number of systems in the ensemble which in the unit time make transitions from the initial state to the specified final state.

Dividing $dw(\alpha\to\beta)$ (10.21) by the volume factor V one obtains the quantity called the *differential reaction density*

$$\begin{aligned} \frac{dw(\alpha\to\beta)}{V} &= \frac{1}{2E_{\mathbf{k}_1}V\dots}\,|\mathcal{A}_{\beta\alpha}|^2(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)\,d\beta\,, \\ d\beta &\equiv \prod_{j=1}^{n_\beta}d\Gamma_{\mathbf{p}_j} \equiv \prod_{j=1}^{n_\beta}\frac{d^3\mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}}\,, \end{aligned} \tag{10.22}$$

which gives the number of reactions leading to the specified final state occuring per unit volume and per unit time. It is finds applications in e.g. cosmology and various kinetic theory considerations (see Appendix G).

[6]We have already expressed $|\beta_0\rangle_V$ and $|\alpha_0\rangle_V$ in terms of the states $|\beta_0\rangle$ and $|\alpha_0\rangle$ but we still work in the box and the momenta in the states $|\beta_0\rangle$ and $|\alpha_0\rangle$ are still discrete.

Decay rates

In the case of a decay of a particle the initial state is a one-particle state $|\alpha_0\rangle = |\mathbf{k}, \sigma\rangle$ (so $P_\alpha = k$). The explicit volume factors in (10.21) cancel then out and in the limit $V \to \infty$ for $dw(\alpha \to \beta) \equiv d\Gamma$ one gets (we set $\hbar = 1$) the expression

$$d\Gamma = \frac{1}{2M} \left|\mathcal{A}_{\beta\alpha}\right|^2 (2\pi)^4 \delta^{(4)}(\sum_j p_j - k) \prod_{j=1}^{n_\beta} \frac{d^3\mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}}\,, \tag{10.23}$$

in which $E_{\mathbf{k}}$ has been replaced by the mass M of the decaying particle (one usually considers the decay of a particle at rest). With the adopted normalizations the last two factors in (10.23): the squared matrix element $\left|\mathcal{A}_{\beta\alpha}\right|^2$, and the final state phase factor (wich includes the delta function) are separately Lorentz invariant. The differential rate $d\Gamma$ can be integrated over some or over all the momenta $\mathbf{p}_j$ of the produced particles and summed over their spin variables σ_j, depending on the actual experimental conditions. The integrated *partial decay width* Γ_f of the particle i into the definite set f of other particles is given by

$$\begin{aligned}\Gamma_f &= \int \prod_{j=1}^{n_f} \left(\frac{d^3\mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}} \sum_{\sigma_j} \right) \frac{dw(\alpha \to \beta)}{d\beta} \\ &= \frac{1}{2M} \int \prod_{j=1}^{n_f} \left(\frac{d^3\mathbf{p}_j}{(2\pi)^3 2E_{\mathbf{p}_j}} \sum_{\sigma_f} \right) \left|\mathcal{A}_{fi}\right|^2 (2\pi)^4 \delta^{(4)}(\sum_j p_j - k)\,. \end{aligned} \tag{10.24}$$

As the state-vector $|\alpha_0\rangle_V$ is normalized so that there is one decaying particle in the volume V, the rate $d\Gamma$ (10.23) is the (differential) transition probability per unit time. Interpreted within the statistical ensemble of identical systems it gives the fraction of the number of systems in the ensemble which make the transition to the specified group of final states (infinitesimally close to $|\beta_0\rangle$) in unit time. Thus, in a real experiment occurring during some time interval dt (which from the microscopic point of view is so large that the Fermi's Golden Rule is valid) the number of registered particles corresponding to the final states around $|\beta_0\rangle$ is proportional to the number of decaying particles and to $dw(\alpha \to \beta) \cdot dt$. Correspondingly, the initial number $N(0)$ of decaying particles decreases in dt by $dN(t) = \sum_f dN_f(t) = -N(t) \sum_f \Gamma_f dt$, where the sum is over all final states into which the initial particle can decay. This leads to the standard exponential decay law $N(t) = N(0) \exp(-\sum_f \Gamma_f t) \equiv N(0) \exp(-\Gamma t)$. The quantity $\hbar/\Gamma$ is called the particle's *lifetime*.

If the final state f is a two-particle state, the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ of the two produced particles are fixed by the delta function in (10.23) up to a common direction (the direction of $\hat{\mathbf{p}}_f \equiv \mathbf{p}_1/|\mathbf{p}_1| = -\mathbf{p}_2/|\mathbf{p}_2|$). Then, if the decaying particle is unpolarized (either because it is spinless, or the rate is averaged over all spin projections of the decaying particle - the averaging, i.e. the factor $(2s_i+1)^{-1} \sum_{\sigma_i}$, is then in (10.25) and (10.26) implicit), the sum

$\sum_{\sigma_1\sigma_2} |\mathcal{A}_{fi}|^2$ can depend only on $\mathbf{p}_1$ and $\mathbf{p}_2$ (but not on other vectors, like polarization vectors) and one gets[7]

$$\Gamma_f = \frac{|\mathbf{p}_f|}{32\pi^2 M^2} \sum_{\sigma_1\sigma_2} |\mathcal{A}_{fi}|^2 \int d\Omega_{\hat{\mathbf{p}}_f} = \frac{|\mathbf{p}_f|}{8\pi M^2} \sum_{\sigma_1\sigma_2} |\mathcal{A}_{fi}|^2 \,. \tag{10.25}$$

If there are n identical (indistinguishable) particles in the final state, the formula for the decay width obtained by integrating the right hand side of (10.23) over the entire phase space has to be divided by $n!$ Thus, if the two final particles in a two-body decay are identical (and have mass m) the formula (10.25) is replaced by

$$\Gamma_f = \frac{|\mathbf{p}_f|}{16\pi M^2} \sum_{\sigma_1\sigma_2} |\mathcal{A}_{fi}|^2 = \frac{1}{32\pi M} \sum_{\sigma_1\sigma_2} |\mathcal{A}_{fi}|^2 \sqrt{1 - \frac{4m^2}{M^2}} \,. \tag{10.26}$$

Once again it should be stressed that strictly speaking within the rigorous formalism of quantum field theory decay processes cannot be considered because the true asymptotic states of decaying particles cannot be defined (quantities like decay widths should be determined indirectly, from analyses of resonant behaviour of partial amplitudes of processes in which only stable particles take part). Nevertheless, the formulae given here can be applied to situations in which decays are induced by weak interactions - the amplitudes of decays can be then computed in the lowest order approximation without encountering any problems.

Cross sections

If the initial state is a two-particle state $|\alpha_0\rangle = |\mathbf{k}_1\sigma_1, \mathbf{k}_2\sigma_2\rangle$, the factor in the denominator in the first line of (10.21) is $V^2 2E_{\mathbf{k}_1} 2E_{\mathbf{k}_2}$ and the explicit volume factors do not cancel out in $dw(\alpha \to \beta)$: there is one factor V^{-1} left. To understand its presence one has to inquire what in this case is really measured experimentally.

Experimentally one measures the number of events corresponding to the production of a particular final state (e.g. the definite number and types of particles, with specified momenta and spins) as a result of colliding particles 1 with particles 2 either per unit time (if there is a constant flux of 1 particles falling on a fixed target consisting of particles 2) or in a definite time interval (in colliders in which bunches of particles, usually colliding "head-on", have finite sizes). Obviously, the number of registered events depends not only on the microscopic interaction but also on the experimental conditions, i.e. on the density of particles 1 in the beam and on the density of particles 2 in the target/beam.

Let us consider first two typical experimental configurations. If N_2 particles 2 forms a fixed target at rest, the number of events of a specified kind registered by the detectors

[7] The integral over the final particles phase space yields also a factor $\theta(M - m_1 - m_2)$ which we omit in (10.25) assuming implicitly that $M > m_1 + m_2$.

is proportional to N_2 and to the flux $\mathcal{F}_1$ (assumed to be uniform) of particles 1 falling on the target. The cross section $d\sigma$ (possibly still differential with respect to the event's kinematical characteristics) is then experimentally *defined* as the ratio

$$d\sigma \equiv \frac{\text{number of events of a specified kind per unit time}}{\mathcal{F}_1\, N_2}\,. \tag{10.27}$$

If the target particles 2 are localized in a volume[8] $V = A \cdot d_2$, then $\mathcal{F}_1 N_2 = n_1\, |\mathbf{v}_1|\, n_2\, V$, where n_2 and n_1 are the densities of particles 2 and 1, respectively and $|\mathbf{v}_1|$ is the velocity of particles 1, all the quantities (n_2, n_1 and $|\mathbf{v}_1|$) measured in the frame, in which the particles 2 are at rest (the laboratory frame). It will be convenient to rephrase (10.27) slightly differently: the number of reactions (collisions) leading to a specified final state which take place in the volume V in the time interval Δt is proportional to $d\sigma$ and to the factor $(n_1\, |\mathbf{v}_1|\, n_2\, V\, \Delta t)_{\rm LAB}$.

Consider now a frame $\mathcal{O}$ in which the target of particles 2 moves with velocity $\mathbf{v}_2$ and the particles 1 have velocity $\mathbf{v}_1$ antiparallel to $\mathbf{v}_2$. This corresponds to the typical collider experiment configuration. Let the target be a cylindrical bunch of length d_2 and cross section area A in which the density of particles 2 is n_2 (d_2 and n_2 are measured in the frame $\mathcal{O}$ in which the detectors are at rest). The flux of particles 1 colliding with this bunch is then[9] $\mathcal{F}_1 = n_1\, |\mathbf{v}_1 - \mathbf{v}_2|$ (again, n_1 and the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are measured in the frame $\mathcal{O}$). Suppose now that particles 1 also form a cylindrical bunch, of length d_1 and the section area A (in any case, what matters is the section area A of the overlap of the two bunches). During the time interval $\Delta t_1 = d_1/|\mathbf{v}_1 - \mathbf{v}_2|$ (measured in $\mathcal{O}$) there is then a constant flux $\mathcal{F}_1$ of particles 1 falling on the moving target consising of particles 2. Therefore, the number of events of a specified kind registered in detectors over the time Δt_1 will be

$$\begin{aligned}(\text{number of events of a specified kind in } \Delta t_1) &= d\sigma \cdot N_2 \cdot \mathcal{F}_1 \cdot \Delta t_1 \\ &= d\sigma \cdot (d_1 \cdot n_1 \cdot A \cdot n_2 \cdot d_2)\,. \end{aligned} \tag{10.28}$$

This formula *defines* the cross section $d\sigma$ measured experimentally in the frame $\mathcal{O}$ in the conditions specified above. Usually the right hand side of (10.28) is expressed in terms of the *luminosity* $\mathcal{L}$ (of dimension $[\mathrm{T}]^{-1}[\mathrm{L}]^{-2}$) defined as

$$\mathcal{L} = n_b(t) \cdot \mathcal{C}\,, \tag{10.29}$$

where $n_b(t)$ is the number of collisions of bunches occurring per second in the collider and the quantity

$$\mathcal{C} = \int d^2\mathbf{b} \int dz_1\, dz_2\, n_1(z_1, \mathbf{b})\, n_2(z_2, \mathbf{b})\,, \tag{10.30}$$

[8] Part of this definition is the implicit limit $d_2 \to 0$, necessary to neglect possible multiple scatterings of particles 1 inside the target.

[9] Note that this is as in the *nonrelativistic* kinematics. There is no conflict with relativity here, as there is nothing wrong in obtaining the value, say, $1.8c$ for the relative velocity of two particles as seen *in the same* frame. If we see two electrons moving towards one another, each with the velocity $0.9c$ *in our frame*, the time (again: measured in our frame) after which they will collide is correctly given by dividing the (initial) distance separating them by $1.8c$.

in which $\mathbf{b}$ is a two-dimensional vector in the plane perpendicular to the common axis of particle beams, generalizes the quantity $d_1 \cdot n_1 \cdot A \cdot n_2 \cdot d_2$ in (10.28) to the more realistic case in which the particles in the bunches are not uniformly distributed and the densities n_1 and n_2 are not constant throughout the bunches. The reaction cross section $d\sigma$ is then determined by counting the number of events of specified kind registered in detectors in some time interval (t_i, t_f) and dividing this number by the integrated luminosity

$$\int_{t_i}^{t_f} dt\, \mathcal{L}(t)\,. \tag{10.31}$$

In practice, the luminosity $\mathcal{L}$ (of a given experimental setup - the accelerator) is determined by computing theoretically the cross section (see below) of some well studied reaction (like the Bhabha e^+e^- elastic scattering in the case of e^+e^- colliders) and counting the corresponding events produced by the collider over some period (t_i, t_f).

Thus in both typical situations the number of events of the specified kind which result from reactions occuring in the volume V of the target and in the time interval Δt during which the target of particles 2 is exposed to the constant flux of particles 1 is proportional to the relative velocity, the number densities of particles and to the product $V\Delta t$ of the volume in which the reactions take place and to the quantity $d\sigma$ of dimension of the area.

We now want to generalize these considerations to particles colliding in arbitrary kinematical configurations, i.e. to non-parallel and nonzero velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the initial state particles. Doing this we will also show that the cross section $d\sigma$ is a Lorentz invariant quantity. Of course, if one counts "events" corresponding to momenta of the final state particles in a specific domain of the phase space (i.e. one measures a differential cross section) this domain should be transformed appropriately:

$$d\sigma(\mathbf{p}_1, \dots, \mathbf{p}_n) = d\sigma^{\Lambda}(\mathbf{p}_1^{\Lambda}, \dots, \mathbf{p}_n^{\Lambda})\,. \tag{10.32}$$

The total reaction cross sections integrated over all final state momenta are then the same in all Lorentz frames: $\sigma^{\Lambda} = \sigma$.

To this end we consider the number of events *registered in the LAB system* (in which the particles 2 are at rest) occuring in the target volume $V_{\rm LAB}$ during the time interval $(\Delta t)_{\rm LAB}$. This number is, on one hand, by definition Lorentz invariant (as is any quantity defined in a fixed frame specified in some particular way) but can be expressed also through quantities measured in an arbitrary frame $\mathcal{O}$. On the other hand, the number of events occuring in the same physical target and in the corresponding time interval $(\Delta t)_{\mathcal{O}}$ counted in the frame $\mathcal{O}$ in which the target volume is $V_{\mathcal{O}}$, must be the same. These two facts will lead to the result (10.32).

We thus express the number of events measured in the LAB frame as

$$\begin{aligned}&(\text{number of events of a specified kind produced in } V \text{ in } \Delta t)_{\rm LAB}\\ &\qquad\qquad\qquad\qquad = \Sigma \cdot (n_1 \cdot n_2 \cdot V \cdot \Delta t)_{\mathcal{O}}\,,\end{aligned}$$

where n_1, n_2, V and Δt are now all measured in $\mathcal{O}$. Σ is a proportionality factor which has to be determined. The product $V \cdot \Delta t$ is by itself Lorentz invariant. Expressing the densities n_1 and n_2 measured in $\mathcal{O}$ in terms of the (by their very nature Lorentz invariant) densities $n_1^{(0)}$ and $n_2^{(0)}$ measured in the respective rest frames of particles 1 and 2:

$$n_i = \frac{n_i^{(0)}}{\sqrt{1-\mathbf{v}_i^2}} = n_i^{(0)}\,\frac{E_i}{m_i}\,, \tag{10.33}$$

we can write

$$\begin{aligned}&(\text{number of events of a specified kind produced in } V \text{ in } \Delta t)_{\mathrm{LAB}}\\ &\qquad = \Sigma\,\frac{E_1E_2}{(k_1\cdot k_2)}\left[(k_1\cdot k_2)\,\frac{n_1^{(0)}n_2^{(0)}}{m_1m_2}\cdot V\cdot\Delta t\right].\end{aligned} \tag{10.34}$$

The expression in the square brackets is Lorentz invariant. It then follows that the prefactor must be Lorentz invariant too. In the frame in which the particles 2 are at rest (the Laboratory frame) the expression in the square brackets reduces to $n_1 \cdot n_2 \cdot V \cdot \Delta t$ (measured in the Laboratory frame). Thus,

$$\Sigma\,\frac{E_1E_2}{(k_1\cdot k_2)} = (d\sigma\,|\mathbf{v}_1|)_{\mathrm{LAB}}\,. \tag{10.35}$$

It remains to express $|\mathbf{v}_1|_{\mathrm{LAB}}$ through the velocities $\mathbf{v}_1$ and $\mathbf{v}_1$ measured in $\mathcal{O}$. To this end we notice, that in the Laboratory frame in which $k_2^\mu = (m_2, \mathbf{0})$,

$$k_1\cdot k_2 = \frac{m_1m_2}{\sqrt{1-|\mathbf{v}_1|^2_{\mathrm{LAB}}}}\,.$$

Hence,

$$|\mathbf{v}_1|_{\mathrm{LAB}} = \frac{m_1m_2}{(k_1\cdot k_2)}\sqrt{\frac{(k_1\cdot k_2)^2}{m_1^2m_2^2}-1}\,. \tag{10.36}$$

Expressing under the square root four-momenta k_i^μ in the frame $\mathcal{O}$ as $(m_i, m_i\mathbf{v}_i)/\sqrt{1-\mathbf{v}_i^2}$,

$$\begin{aligned}|\mathbf{v}_1|_{\mathrm{LAB}} &= \frac{m_1m_2}{(k_1\cdot k_2)\sqrt{(1-\mathbf{v}_1^2)(1-\mathbf{v}_2^2)}}\sqrt{(1-\mathbf{v}_1\cdot\mathbf{v}_2)^2-(1-\mathbf{v}_1^2)(1-\mathbf{v}_2^2)}\,,\\ &\equiv \frac{E_1E_2}{(k_1\cdot k_2)}\sqrt{(1-\mathbf{v}_1\cdot\mathbf{v}_2)^2-(1-\mathbf{v}_1^2)(1-\mathbf{v}_2^2)}\,,\end{aligned} \tag{10.37}$$

where E_1 and E_2 in the last line are the energies measured in the frame $\mathcal{O}$. From (10.35) it then follows that (writing the expression under the root in the equivalent form) $\Sigma = d\sigma_{\mathrm{LAB}}\sqrt{|\mathbf{v}_1-\mathbf{v}_2|^2-|\mathbf{v}_1\times\mathbf{v}_2|^2}$. Combining this with the formula (10.35) and using (10.33) we finally get

$$\begin{aligned}&(\text{number of events of a specified kind produced in } V \text{ in } \Delta t)_{\mathrm{LAB}}\\ &\qquad = d\sigma_{\mathrm{LAB}}\cdot n_1\cdot n_2\cdot V\cdot\Delta t\cdot\sqrt{|\mathbf{v}_1-\mathbf{v}_2|^2-|\mathbf{v}_1\times\mathbf{v}_2|^2}\,,\end{aligned} \tag{10.38}$$

where all the quantities on the right hand side, except for $d\sigma_{\rm LAB}$ are measured in the frame $\mathcal{O}$. (In the nonrelativistic case the term $|\mathbf{v}_1 \times \mathbf{v}_2|^2$ under the square root is absent because it is suppressed by $1/c^2$ compared to the first term.) Since, as has been said, the number of events produced in the target in the period in which it was exposed to the flux is the same in any frame, $d\sigma_{\rm LAB}$ must be equal to the cross section $d\sigma$ measured in the frame $\mathcal{O}$ because the remaining factors on the right hand side are just those which are used to define $d\sigma$ in $\mathcal{O}$. This proves the assertion (10.32).

To compute $d\sigma$ theoretically one can therefore work in the frame $\mathcal{O}$. The volume in which the reaction occurs can be then identified with the volume V in which the state-vectors $|\alpha_0\rangle_V$, and $|\beta_0\rangle_V$ (10.13) are properly normalized. (Recall again, that although the formula (10.21) is expressed in terms of our standard states $|\alpha_0\rangle$ and $|\beta_0\rangle$, it gives the probability per unit time of the transition between the properly normalized states $|\alpha_0\rangle_V$, and $|\beta_0\rangle_V$). The differential cross section $d\sigma$, as follows from (10.38), is given by dividing the corresponding differential reaction density[10]

$$\frac{dw(\alpha\to\beta)}{V} = \frac{1}{2E_{\mathbf{k}_1}V2E_{\mathbf{k}_2}V}\,|\mathcal{A}_{\beta\alpha}|^2(2\pi)^4\delta^{(4)}(P_\beta - k_1 - k_2)\prod_{j=1}^{n_\beta} d\Gamma_{\mathbf{p}_j}\,, \tag{10.39}$$

by $n_1 n_2\sqrt{|\mathbf{v}_1-\mathbf{v}_2|^2 - |\mathbf{v}_1\times\mathbf{v}_2|^2}$. Since with the adopted normalization of the state-vectors $|\alpha_0\rangle_V$, and $|\beta_0\rangle_V$ the particle densities (in the frame $\mathcal{O}$) are just $n_1 = n_2 = 1/V$,

$$d\sigma = \frac{1}{F}\,|\mathcal{A}_{\beta\alpha}|^2\,(2\pi)^4\delta^{(4)}(\sum_i p_i - k_1 - k_2)\prod_{i=1}^{n}\frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\,, \tag{10.40}$$

where $F = 4E_{\mathbf{k}_1}E_{\mathbf{k}_2}\sqrt{|\mathbf{v}_1-\mathbf{v}_2|^2 - |\mathbf{v}_1\times\mathbf{v}_2|^2}$. The factor F can be written in a manifestly Lorentz invariant form by combining the formulae (10.36) and (10.37) and using them in the "opposite" direction:

$$F = 4\sqrt{(k_1\cdot k_2)^2 - m_1^2 m_2^2}\,, \tag{10.41}$$

so that in the formula (10.40) all the three factors: the one related to the initial state, the matrix element $\mathcal{A}$ squared and the final state phase factor, are separately Lorentz invariant. Moreover, the flux factor written in the form (10.41) is correct also in the case of non-parallel $\mathbf{k}_1$ and $\mathbf{k}_2$.

Finally, as in computing the decay widths, if in the final state there are n identical particles, the expression for the cross section obtained by integrating (10.40) over the whole phase space has to be divided by the factor $n!$

[10] The rate $dw(\alpha\to\beta)$ should again be interpreted within the statistical ensemble of identical systems: it is the fraction of the number of systems in the ensemble which in the unit time make transitions from the initial state to the specified final state.

10.3 Cross sections and the phase shifts

As it was seen in Section 7.6, processes of collisions of two particles are most efficiently analyzed in their ceenter of mass (CMS) frame. In this frame the three-momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ of the colliding particles of masses m_1 and m_2 are opposite, $\mathbf{k}_1 = -\mathbf{k}_2 \equiv \mathbf{k}$ and, if the product of the reaction is another two particles of masses m_a and m_b, the same is true of the two final state three-momenta $\mathbf{p}_a = -\mathbf{p}_b \equiv \mathbf{p}$. To the Mandelstamm variable $s \equiv (k_1 + k_2)^2$ the absolute values of the three-momenta $\mathbf{k}$ and $\mathbf{p}$ are related by

$$|\mathbf{k}| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_1^2, m_2^2)\,, \qquad |\mathbf{p}| = \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, m_a^2, m_b^2)\,, \tag{10.42}$$

where $\lambda(x, y, z)$ is the function defined in (**??**). In other words, in the case of a binary scattering proces $1 + 2 \to a + b$ the delta function in the formula (10.40) fixes completely the lenght of the final particles three-momenta, leaving undetermined only their direction. In the CMS system the flux factor (10.41) can be written as

$$F = 4|\mathbf{k}|\sqrt{s} = 2\lambda^{1/2}(s, m_1^2, m_2^2)\,, \tag{10.43}$$

and the CMS differential cross section takes the form

$$\frac{d\sigma}{d\Omega_{\hat{\mathbf{p}}}} = \frac{1}{64\pi^2 s}\,\frac{\lambda^{1/2}(s, m_a^2, m_b^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)}\;|\mathcal{A}_{\beta\alpha}|^2 = \frac{1}{64\pi^2 s}\,\frac{|\mathbf{p}|}{|\mathbf{k}|}\;|\mathcal{A}_{\beta\alpha}|^2\,. \tag{10.44}$$

Expressing the amplitude $\mathcal{A}_{\beta\alpha}$ using the partial wave amplitudes defined in (7.120) we get

$$\frac{d\sigma}{d\Omega_{\hat{\mathbf{p}}}} = \frac{4}{s}\frac{|\mathbf{p}|}{|\mathbf{k}|}\left|\sum_j^{\infty}(2j+1)\,\mathcal{T}^{(j)}_{\lambda_a,\lambda_b;\lambda_1,\lambda_2}(s)D^{(j)*}_{\lambda_1-\lambda_2,\lambda_a-\lambda_b}(\Omega_{\hat{\mathbf{p}}})\right|^2. \tag{10.45}$$

The angular dependence of the differential cross section can also be expressed in terms of the Legendre polynomials by using the decomposition (4.122) of the products of two D-functions and the Clebsch-Gordan coefficients (4.111)

$$\begin{aligned} D^{(j_1)}_{m_1' m_1} D^{(j_2)*}_{m_2' m_2} = (-1)^{m_2-m_2'} &\sum_j C_{j_1 j_2}(m_1', -m_2'|j, m_1'-m_2') \\ &\times C_{j_1 j_2}(m_1, -m_2|j, m_1-m_2) D^{(j)}_{m_1'-m_2', m_1-m_2}\,, \end{aligned} \tag{10.46}$$

Since $D^{(l)}_{0,0}(\phi, \theta, 0) = P_l(\cos\theta)$ (j_1, j_2 are either both integer or both half-integer), where P_l are the Legendre polynomials, this shows that the differential cross section for transitions between definite helicity states (in which case in (10.46) $m_1' - m_2' = m_1 - m_2 = 0$) does not depend on the azimuthal angle ϕ.

From the form (10.45) of the differential cross section, using the formula (6.109), one arrives at the integrated cross section for transitions between definite helicity states

$$\sigma_{\lambda_a,\lambda_b;\lambda_1,\lambda_2} \equiv \sum_j \sigma^{(j)}_{\lambda_a,\lambda_b;\lambda_1,\lambda_2} = \frac{16\pi}{s}\frac{|\mathbf{p}|}{|\mathbf{k}|}\sum_j (2j+1)\left|\mathcal{T}^{(j)}_{\lambda_a,\lambda_b;\lambda_1,\lambda_2}(s)\right|^2. \tag{10.47}$$

The quantities $\sigma^{(j)}_{\lambda_a,\lambda_b;\lambda_1,\lambda_2}$ are called *partial cross sections.* They are constrained by the unitarity bounds derived in Section 7.6. Indeed, from (7.140) one finds that the partial cross section of any transition between states of two different particles, inelastic or elastic one but with the helicities flipped, satisfies the bound

$$\sigma^{(j)}_{\lambda_a,\lambda_b;\lambda_1,\lambda_2} \le \frac{\pi}{|\mathbf{k}|^2}\,(2j+1)\,, \tag{10.48}$$

while the partial cross section of an elastic process with no helicity flip in view of (7.139) is bounded by

$$\sigma^{(j)}_{\lambda_1,\lambda_2;\lambda_1,\lambda_2} \le \frac{4\pi}{|\mathbf{k}|^2}\,(2j+1)\,. \tag{10.49}$$

The same bounds apply also to partial cross sections averaged over all helicity states of the initial particles

$$\sigma^{(j)}_{\lambda_a,\lambda_b} = \frac{1}{(2s_1+1)(2s_2+1)}\sum_{\lambda_1,\lambda_2}\sigma^{(j)}_{\lambda_a,\lambda_b;\lambda_1,\lambda_2} \tag{10.50}$$

giving rates of transitions from unpolarized states, and get multiplied by factors $(2s_a+1)(2s_b+1)$ for cross sections summed over helicity states of final particles. Bounds on partial cross sections with identical particles in the initial and/or final state can be derived similarly using the unitarity bounds (7.140) and (7.139).

The differential cross section of elastic scattering with no change of helicities, $\lambda_a \equiv \lambda_1' = \lambda_1$ and $\lambda_b \equiv \lambda_2' = \lambda_2$, can be expressed through the phase shifts using the formula (7.131):

$$\frac{d\sigma_{\rm el}}{d\Omega_{\hat{\mathbf{p}}}} = \frac{1}{4|\mathbf{k}|^2}\left|\sum_j^{\infty}(2j+1)\left(e^{2i\delta_j(s)-2\beta_j(s)}-1\right)D^{(j)*}_{\lambda_1-\lambda_2,\lambda_1-\lambda_2}(\Omega_{\hat{\mathbf{p}}})\right|^2 . \tag{10.51}$$

For $\lambda_1-\lambda_2=0$, e.g. when applied to the scattering of two spinless particles, or longitudinally polarized massive vector bosons, this reduces to the result familiar from nonrelativistic quantum mechanics

$$\frac{d\sigma_{\rm el}}{d\Omega_{\hat{\mathbf{p}}}} = \frac{1}{4|\mathbf{k}|^2}\left|\sum_{j=0}^{\infty}(2j+1)\left(e^{2i\delta_j(s)-2\beta_j(s)}-1\right)P_j(\cos\theta)\right|^2 , \tag{10.52}$$

where $P_j(x)$ are Legendre polynomials and θ is the scattering angle.

Integrating (10.51) over the solid angle with the help of the relation (6.109) one obtains the elastic (with no helicity change) scattering cross section (of distinct particles)

$$\sigma_{\rm el} = \frac{\pi}{|\mathbf{k}|^2}\sum_j^{\infty}(2j+1)\left|\,1-e^{2i\delta_j(s)-2\beta_j(s)}\,\right|^2 \equiv \sum_j \sigma^{(j)}_{\rm el}\,. \tag{10.53}$$

Elastic scattering cross sections with helicities of the final state different than those of the initial one can be given similar representations in terms of the phase shifts $\delta_j^k(s) - 2\beta_j^k(s)$ and the matrix diagonalizing the partial wave amplitude $\mathcal{T}^{(j)}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}$ in the spin space. The maximum of $\sigma_{\rm el}^{(j)}$ reached for $e^{-2\beta_j(s)} = 1$ (from (7.132) and (7.126) it follows that $e^{-2\beta_j(s)} \leq 1$) and $e^{2i\delta_j(s)} = -1$ agrees, course of, with the bound (10.49).

Using in the optical theorem (7.115) the partial wave expansion (7.120) of $\mathcal{A}_{\alpha\alpha}$ with (corresponding to the forward scattering $\theta = \phi = 0$) $D^{(j)}_{\lambda_1-\lambda_2,\lambda_1-\lambda_2} = 1$, the total scattering scattering cross section (for production of any final state) takes the form

$$\sigma_{\rm tot} = \frac{2\pi}{|\mathbf{k}|^2}\sum_j^\infty (2j+1)\left(1 - {\rm Re}\, e^{2i\delta_j(s)-2\beta_j(s)}\right). \tag{10.54}$$

Hence, the total inelastic (absorptive) cross section (for production of anything except for the same particles as in the initial state with unchanged helicities) is

$$\sigma_{\rm abs} \equiv \sigma_{\rm tot} - \sigma_{\rm el} = \frac{\pi}{|\mathbf{k}|^2}\sum_j^\infty (2j+1)\left(1 - e^{-4\beta_j(s)}\right). \tag{10.55}$$

10.4 Muon decay

We apply first the formulae derived in section 10.2 to the calculation of the width of the $\mu^- \to e^-\bar\nu_e\nu_\mu$ decay.

This example serves to illustrate techniques which allow to efficiently compute $|\mathcal{A}|^2$ summed over spin projections of the particles in the initial and final states. The first order (in the perturbative expansion) term of the amplitude $-i\mathcal{A}$ of this process was written down in Section 9.2. Since in the expression

$$\begin{aligned} -i\mathcal{A} = -i\,\frac{G_F}{\sqrt{2}}\left[\bar u_{(e)}(\mathbf{p},\sigma_p)\,\gamma^\lambda(1-\gamma^5)\,v_{(\nu_e)}(\mathbf{k}_1,\sigma_1)\right] \\ \times\left[\bar u_{(\nu_\mu)}(\mathbf{k}_2,\sigma_2)\,\gamma_\lambda(1-\gamma^5)\,u_{(\mu)}(\mathbf{q},\sigma_q)\right]. \end{aligned} \tag{10.56}$$

each term is a c-number (not a matrix), its modulus squared $|\mathcal{A}|^2$ can be written as

$$\begin{aligned} |\mathcal{A}|^2 = \frac{G_F^2}{2}\left[\bar u_{(e)}\gamma^\lambda(1-\gamma^5)v_{(\nu_e)}\right]\left[\bar u_{(e)}\gamma^\kappa(1-\gamma^5)v_{(\nu_e)}\right]^* \\ \times\left[\bar u_{(\nu_\mu)}\gamma_\lambda(1-\gamma^5)u_{(\mu)}\right]\left[\bar u_{(\nu_\mu)}\gamma_\kappa(1-\gamma^5)u_{(\mu)}\right]^*. \end{aligned}$$

Using now the fact that taking the complex conjugation of c-numbers can be replaced by taking their Hermitian conjugation, we can write (recall: $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, $\gamma^{0\dagger} = \gamma^0$, $\gamma^{5\dagger} = \gamma^5$)

$$\begin{aligned} \left[\bar u\gamma^\kappa(1-\gamma^5)v\right]^* = \left[\bar u\gamma^\kappa(1-\gamma^5)v\right]^\dagger = v^\dagger(1-\gamma^5)^\dagger\gamma^{\kappa\dagger}\gamma^0 u \\ = v^\dagger(1-\gamma^5)\gamma^0\gamma^\kappa u = \bar v\gamma^\kappa(1-\gamma^5)u\,. \end{aligned}$$

Proceeding similarly with the second complex conjugation we obtain

$$|\mathcal{A}|^2 = \frac{G_F^2}{2}\,[\bar{u}(\mathbf{p},\sigma_p)\,\gamma^\lambda(1-\gamma^5)v(\mathbf{k}_1,\sigma_1)]\,[\bar{v}(\mathbf{k}_1,\sigma_1)\,\gamma^\kappa(1-\gamma^5)\,u(\mathbf{p},\sigma_p)]$$
$$\times[\bar{u}(\mathbf{k}_2,\sigma_2)\,\gamma_\lambda(1-\gamma^5)\,u(\mathbf{q},\sigma_q)]\,[\bar{u}(\mathbf{q},\sigma_q)\,\gamma_\kappa(1-\gamma^5)\,u(\mathbf{k}_2,\sigma_2)]\,.$$

Usually one is not interested in the spin variables because polarizations of the final particles are not measured and the polarization of the decaying μ^- is also unknown. In this case one has to sum $|\mathcal{A}|^2$ over the spins of the final state particles and average it over two possible spin projections of the muon:

$$|\mathcal{A}|^2 \to \frac{1}{2}\sum_{\sigma_q}\sum_{\sigma_p}\sum_{\sigma_1}\sum_{\sigma_2}|\mathcal{A}|^2\,. \tag{10.57}$$

The necessary summations in (10.57) can be efficiently performed with the help of the formulae (8.105). Following carefully the contractions of the spinor indices, one realizes that

$$\sum_{\sigma_q}\sum_{\sigma_2}\big[\bar{u}(\mathbf{k}_2,\sigma_2)\,\gamma_\lambda(1-\gamma^5)\,u(\mathbf{q},\sigma_q)\big]\big[\bar{u}(\mathbf{q},\sigma_q)\,\gamma_\kappa(1-\gamma^5)\,u(\mathbf{k}_2,\sigma_2)\big]$$
$$= \mathrm{tr}\big[\not{k}_2\,\gamma_\lambda\,(1-\gamma^5)(\not{q}+m_\mu)\gamma_\kappa\,(1-\gamma^5)\big]\,,$$

$$\sum_{\sigma_p}\sum_{\sigma_1}\big[\bar{u}(\mathbf{p},\sigma_p)\,\gamma^\lambda(1-\gamma^5)\,v(\mathbf{k}_1,\sigma_1)\big]\big[\bar{v}(\mathbf{k}_1,\sigma_1)\,\gamma^\kappa(1-\gamma^5)\,u(\mathbf{p},\sigma_p)\big]$$
$$= \mathrm{tr}\big[\not{p}\,\gamma^\lambda(1-\gamma^5)\,\not{k}_1\gamma^\kappa\,(1-\gamma^5)\big]\,,$$

where the traces are taken over the spinor indices. In writing these formulae we have (for simplicity) set to zero all particle masses except for the muon mass.[11] Thus,

$$\frac{1}{2}\sum_{\sigma_q}\sum_{\sigma_p}\sum_{\sigma_1}\sum_{\sigma_2}|\mathcal{A}|^2 = \frac{G_F^2}{4}\,\mathrm{tr}\big[\not{k}_2\,\gamma_\lambda\,(1-\gamma^5)(\not{q}+m_\mu)\,\gamma_\kappa\,(1-\gamma^5)\big]$$
$$\times\mathrm{tr}\big[\not{p}\,\gamma^\lambda\,(1-\gamma^5)\,\not{k}_1\,\gamma^\kappa\,(1-\gamma^5)\big]\,.$$

Using now the fact that $\mathbf{P}_L = \frac{1}{2}(1-\gamma^5)$ is a projection matrix ($\mathbf{P}_L^2 = \mathbf{P}_L$), the rules $\not{a}\,\not{b}\,\mathbf{P}_L = \mathbf{P}_L\,\not{a}\,\not{b}$, $\not{a}\,\mathbf{P}_L = \mathbf{P}_R\,\not{a}$, etc. and taking into account that $\mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\mathbf{P}_L) = 0$, the above expression can be simplified to

$$\frac{1}{2}\sum_{\sigma_q}\sum_{\sigma_p}\sum_{\sigma_1}\sum_{\sigma_2}|\mathcal{A}|^2 = \frac{G_F^2}{4}\,16\,\mathrm{tr}[\not{k}_2\,\gamma_\lambda\,\not{q}\,\gamma_\kappa\,\mathbf{P}_L]\,\mathrm{tr}\big[\not{p}\,\gamma^\lambda\,\not{k}_1\,\gamma^\kappa\,\mathbf{P}_L\big]\,.$$

[11] If the neutrino masses are set to zero, the electron mass drops out from these formulae, in the same way as does the muon mass. So in fact the electron mass can still be treated as nonzero.

Next, using the formulae

$$\begin{aligned}&\mathrm{tr}(\not a\,\not b\,\not c\,\not d) = 4\,[(a\cdot b)(c\cdot d) + (a\cdot d)(c\cdot b) - (a\cdot c)(b\cdot d)]\,,\\ &\mathrm{tr}(\not a\,\not b\,\not c\,\not d\,\gamma^5) = 4i\,\epsilon_{\mu\nu\lambda\rho}\,a^\mu b^\nu c^\lambda d^\rho\,,\end{aligned}$$

in which $\epsilon_{0123} = -1$ (i.e. $\epsilon^{0123} = +1$), after a bit tedious but straightforward algebra[12] one arrives at the surprisingly simple expression:

$$\frac{1}{2}\sum_{\sigma_q}\sum_{\sigma_p}\sum_{\sigma_1}\sum_{\sigma_2}|\mathcal{A}|^2 = 64G_F^2(q\cdot k_1)(p\cdot k_2)\,.$$

As usually, one is not interested in the probability of a transition to a particular state in which the final particles have definite momenta, but rather in the probability of a transition to a group of final states with the three-momenta $\mathbf{p}$, $\mathbf{k}_1$ and $\mathbf{k}_2$ in some intervals around some specified values. The fully differential decay rate given by the formula (10.23) takes here the form

$$\begin{aligned}d\Gamma = \frac{1}{2m_\mu}\,&64G_F^2(q\cdot k_1)(p\cdot k_2)\,(2\pi)^4\delta^{(4)}(q-p-k_1-k_2)\\ &\times\frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\,\frac{d^3\mathbf{k}_1}{(2\pi)^3 2|\mathbf{k}_1|}\,\frac{d^3\mathbf{k}_2}{(2\pi)^3 2|\mathbf{k}_2|}\,.\end{aligned}\tag{10.58}$$

The neutrino masses have been neglected here by setting $E_{\mathbf{k}_i} = |\mathbf{k}_i|$.

We now take into account that experimental setups usually do not register neutrinos and all what can be measured is the distribution of energies of electrons prodeced in such decays, i.e. the quantity $d\Gamma/dE_{\mathbf{p}}$. Therefore we first integrate the fully differential rate (10.58) over the neutrino momenta $\mathbf{k}_1$ and $\mathbf{k}_2$. To this end we evaluate first the integral

$$I^{\mu\nu}(Q) = \int\frac{d^3\mathbf{k}_1}{2|\mathbf{k}_1|}\,\frac{d^3\mathbf{k}_2}{2|\mathbf{k}_2|}\,k_1^\mu k_2^\nu\,\delta^{(4)}(Q-k_1-k_2)\,,\tag{10.59}$$

in which Q^μ is a four-vector. Since the measures $d^3\mathbf{k}_i/2|\mathbf{k}_i|$ as well as the four-dimensional delta function are Lorentz invariant, $I^{\mu\nu}$ is a symmetric second rank tensor in the indices $\mu\nu$, that is, if $Q' = \Lambda\cdot Q$, then

$$\begin{aligned}I'^{\mu\nu}(Q') &\equiv \int\frac{d^3\tilde{\mathbf{k}}_1}{2|\tilde{\mathbf{k}}_1|}\,\frac{d^3\tilde{\mathbf{k}}_2}{2|\tilde{\mathbf{k}}_2|}\,\tilde{k}_1^\mu\tilde{k}_2^\nu\,\delta^{(4)}(Q'-\tilde{k}_1-\tilde{k}_2)\\ &= \int\frac{d^3\mathbf{k}_1}{2|\mathbf{k}_1|}\,\frac{d^3\mathbf{k}_2}{2|\mathbf{k}_2|}\,(\Lambda\cdot k_1)^\mu(\Lambda\cdot k_2)^\nu\,\delta^{(4)}(\Lambda\cdot(Q'-k_1-k_2))\end{aligned}$$

[12] One needs also the formula

$$\epsilon_{\mu\nu\lambda\rho}\epsilon^{\lambda\rho\kappa\sigma} = -2\,(g_\mu{}^\kappa g_\nu{}^\sigma - g_\mu{}^\sigma g_\nu{}^\kappa)\,,$$

and the fact that $\epsilon_{\mu\nu\lambda\rho}p^\mu q^\nu k_1^\lambda k_2^\rho = 0$ because the four four-vectors q, p, k_1 and k_2 are not linearly independent ($q = p + k_1 + k_2$).

and, therefore, $I'^{\mu\nu}(Q') = \Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\sigma}I^{\mu\nu}(Q)$. (We have substituted $\tilde{k}_i^{\mu} = (\Lambda\cdot k_i)^{\mu}$ and used the Lorentz invariance of the measures and the fact that

$$\delta^{(4)}(\Lambda\cdot(Q' - k_1 - k_2)) = |\mathrm{det}\Lambda|^{-1}\delta^{(4)}(Q' - k_1 - k_2) = \delta^{(4)}(Q' - k_1 - k_2)\,,$$

because $|\mathrm{det}\Lambda| = 1$ - see Section 6.1). $I^{\mu\nu}(Q)$ must therefore be of the general form

$$I^{\mu\nu}(Q) = g^{\mu\nu}A + Q^{\mu}Q^{\nu}B\,, \tag{10.60}$$

with A and B some Lorentz scalar functions of the four-vector Q^{μ}. It is clear that if $Q^0 < 0$, the integral is zero (the integration in (10.59) is over positive $k_i^0 = |\mathbf{k}_i|$ only) and it vanishes also if $Q^2 < 0$, because (recall that $k_1^2 = k_2^2 = 0$)

$$(k_1 + k_2)^2 = 2\,k_1\cdot k_2 = 2\,|\mathbf{k}_1||\mathbf{k}_2|\,(1 - \cos\vartheta) \geq 0\,,$$

where ϑ is the angle between the three-vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. The functions A and B must be therefore proportional to $\theta(Q^0)$ and, apart from this dependence on the sign of Q^0 (which is Lorentz invariant if, as it is here, $Q^2 \geq 0$), it can depend (including the proportionality to $\theta(Q^2)$) only on the only available Lorentz scalar Q^2. The dependence of A and B on Q^2 can be established in the following way. One contracts first $I^{\mu\nu}$ in two different ways: with $Q_{\mu}Q_{\nu}$ and with $g_{\mu\nu}$. This gives two relations

$$\begin{aligned} Q^2A + Q^4B &= \int \frac{d^3\mathbf{k}_1}{2|\mathbf{k}_1|}\frac{d^3\mathbf{k}_2}{2|\mathbf{k}_2|}\,(k_1\cdot Q)(k_2\cdot Q)\,\delta^{(4)}(Q - k_1 - k_2)\,, \\ 4A + Q^2B &= \int \frac{d^3\mathbf{k}_1}{2|\mathbf{k}_1|}\frac{d^3\mathbf{k}_2}{2|\mathbf{k}_2|}\,(k_1\cdot k_2)\,\delta^{(4)}(Q - k_1 - k_2)\,. \end{aligned} \tag{10.61}$$

Next, these equalities are evaluated in the Lorentz frame in which $Q^{\mu} = (Q^0, 0, 0, 0)$. In this frame $\delta^{(4)}(Q - k_1 - k_2) = \delta(Q^0 - |\mathbf{k}_1| - |\mathbf{k}_2|)\,\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2)$. The three dimensional delta function allows to take one of the integrals, say over $d^3\mathbf{k}_2$. After this operation, $k_1^{\mu} = (|\mathbf{k}_1|,\,\mathbf{k}_1)$ and $k_2^{\mu} = (|\mathbf{k}_1|,\,-\mathbf{k}_1)$, so that $k_1\cdot k_2 = 2|\mathbf{k}_1|^2$. We thus have

$$\begin{aligned} (Q^0)^2A + (Q^0)^4B &= \int \frac{d^3\mathbf{k}_1}{4|\mathbf{k}_1|^2}\,Q^0Q^0\,|\mathbf{k}_1|^2\,\delta(Q^0 - 2|\mathbf{k}_1|) = \frac{\pi}{8}\,(Q^0)^4\,\theta(Q^0)\,, \\ 4A + (Q^0)^2B &= \int \frac{d^3\mathbf{k}_1}{4|\mathbf{k}_1|^2}\,2|\mathbf{k}_1|^2\,\delta(Q^0 - 2|\mathbf{k}_1|) = \frac{\pi}{4}\,(Q^0)^2\,\theta(Q^0)\,. \end{aligned} \tag{10.62}$$

This gives $A(Q^0) = (\pi/24)(Q^0)^2\theta(Q^0)$ and $B(Q^0) = (\pi/12)\theta(Q^0)$. In the original frame the theta functions obtained in the above way are interpreted as $\theta(Q^2)\theta(Q^0)$. Hence,

$$I^{\mu\nu}(Q) = \frac{\pi}{24}\left(Q^2g^{\mu\nu} + 2Q^{\mu}Q^{\nu}\right)\theta(Q^0)\theta(Q^2)\,. \tag{10.63}$$

In the muon rest frame

$$\theta(Q^2) = \theta(m_{\mu}^2 - 2m_{\mu}E_{\mathbf{p}}) = \theta(m_{\mu} - 2E_{\mathbf{p}})\,, \tag{10.64}$$

($\theta(Q^0)$ imposes a weaker condition) and the decay rate differential with respect to the outgoing electron variables is

$$d\Gamma = \frac{64G_F^2}{2m_\mu}\frac{\pi}{24}\, q_\mu p_\nu \big[(q-p)^2 g^{\mu\nu} + 2(q-p)^\mu (q-p)^\nu\big]\, \frac{\theta(m_\mu - 2E_{\mathbf{p}})}{(2\pi)^2}\frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\,,$$

Using now the equalities $q^2 = m_\mu^2$, $q\cdot p = m_\mu E_{\mathbf{p}}$, $d^3\mathbf{p} = |\mathbf{p}|^2 d|\mathbf{p}|\, d\Omega_{\mathbf{p}} = |\mathbf{p}| E_{\mathbf{p}}\, dE_{\mathbf{p}}\, d\Omega_{\mathbf{p}}$ and neglecting the electron mass (i.e. setting $p^2 = 0$, that is $E_{\mathbf{p}} = |\mathbf{p}|$) we arrive at the electron energy distribution

$$\frac{d\Gamma}{dE_{\mathbf{p}}} = \frac{1}{12\pi^3}\, G_F^2 m_\mu^2 \left(3 - 4\frac{E_{\mathbf{p}}}{m_\mu}\right) E_{\mathbf{p}}^2\, \theta(m_\mu/2 - E_{\mathbf{p}})\,. \tag{10.65}$$

Integrating finally over the electron energy $E_{\mathbf{p}}$ (from 0 to $m_\mu/2$) we obtain the muon decay width:

$$\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) = \frac{G_F^2 m_\mu^5}{192\pi^3}\,. \tag{10.66}$$

Comparison of the muon lifetime (the muon width) $\tau_\mu = \hbar/\Gamma_{\rm tot} = 2\times 10^{-6}$ sec with the theoretical result is normally used to determine the value of the Fermi coupling constant G_F. Including the one loop correction from the virtual photon exchange between the muon and electron lines in Figure 9.1 and the decau width $\Gamma(\mu \to e\nu_\mu\bar{\nu}_e\gamma)$ one obtais $G_F = 1.16627(1)\times 10^{-5}$ GeV^{-2}.

10.5 Simple cross section calculations

Next we consider simplest examples of the cross section calculation. Suppose we take a theory of a single massive neutral spinless particle described by the field operator

$$\varphi(x) = \int d\Gamma_{\mathbf{k}} \left(a(\mathbf{k})\, e^{-ik\cdot x} + a^\dagger(\mathbf{k})\, e^{+ik\cdot x}\right), \tag{10.67}$$

and the interaction Hamiltonian $\mathcal{H}_{\rm int}(x) = (\lambda/4!)\varphi^4(x)$ and we compute the cross section for elastic scattering. The initial and final states are then

$$\begin{aligned} |\alpha_0\rangle &= a^\dagger(\mathbf{k}_2)\, a^\dagger(\mathbf{k}_1)|\Omega_0\rangle\,, \\ \langle\beta_0| &= \langle\Omega_0|\, a(\mathbf{p}_1)\, a(\mathbf{p}_2)\,. \end{aligned} \tag{10.68}$$

The $S_{\beta\alpha}$ matrix element is given by

$$\begin{aligned} S_{\beta\alpha} &= \langle\beta_0|1 - i\int d^4x\, \mathcal{H}_{\rm int}(x) + \frac{(-i)^2}{2!}\int d^4x \int d^4y\, T\left[\mathcal{H}_{\rm int}(x)\mathcal{H}_{\rm int}(y)\right] + \dots |\alpha_0\rangle \\ &= \delta_{\beta\alpha} - \frac{i\lambda}{4!}\int d^4x\, \langle\beta_0|\varphi^4(x)|\alpha_0\rangle + \dots \\ &= \delta_{\beta\alpha} - \frac{i\lambda}{4!}\int d^4x\, \langle\beta_0|\left[\varphi^{(+)}(x) + \varphi^{(-)}(x)\right]^4 |\alpha_0\rangle + \dots \end{aligned} \tag{10.69}$$

where we restrict the calculation to the first nontrivial term. Since the initial and final states are two-particle states, only the term containing two positive frequency $\varphi^{(+)}(x)$ and two negative frequency $\varphi^{(-)}(x)$ operators, i.e. the term with two creation and two annihilation operators, can contribute to the connected part of the amplitude (that is to the part containing a single delta function expressing the conservation of the total four-momentum). This term has the binomial factor of 6, so we get

$$S_{\beta\alpha} = \delta_{\beta\alpha} - \frac{i\lambda}{4}\int d^4x\, \langle\beta_0|\varphi^{(-)}(x)\varphi^{(-)}(x)\varphi^{(+)}(x)\varphi^{(+)}(x)|\alpha_0\rangle + \dots\,, \tag{10.70}$$

where we have neglected terms with commutators arising in the course of placing the $\varphi^{(-)}(x)$'s to the left of $\varphi^{(+)}(x)$'s as they do not contribute to the connected part of the S-matrix. Inserting the expressions for the field operators and integrating over x we get

$$\begin{aligned}
(S-1)_{\beta\alpha} &= -\frac{i\lambda}{4}\int d\Gamma_{\mathbf{q}_1} d\Gamma_{\mathbf{q}_2} d\Gamma_{\mathbf{r}_1} d\Gamma_{\mathbf{r}_2}\,(2\pi)^4\delta^{(4)}(q_1+q_2-r_1-r_2)\\
&\qquad \times\langle\beta_0|a^\dagger(\mathbf{q}_1)a^\dagger(\mathbf{q}_2)a(\mathbf{r}_1)a(\mathbf{r}_2)|\alpha_0\rangle + \dots\\
&= -\frac{i\lambda}{4}\int d\Gamma_{\mathbf{q}_1} d\Gamma_{\mathbf{q}_2} d\Gamma_{\mathbf{r}_1} d\Gamma_{\mathbf{r}_2}\,(2\pi)^4\delta^{(4)}(q_1+q_2-r_1-r_2)\\
&\qquad \times[\delta_\Gamma(\mathbf{p}_1-\mathbf{q}_1)\,\delta_\Gamma(\mathbf{p}_2-\mathbf{q}_2)+\delta_\Gamma(\mathbf{p}_1-\mathbf{q}_2)\,\delta_\Gamma(\mathbf{p}_2-\mathbf{q}_1)]\\
&\qquad \times[\delta_\Gamma(\mathbf{k}_1-\mathbf{r}_1)\,\delta_\Gamma(\mathbf{k}_2-\mathbf{r}_2)+\delta_\Gamma(\mathbf{k}_1-\mathbf{r}_2)\,\delta_\Gamma(\mathbf{k}_2-\mathbf{r}_1)]\,.
\end{aligned}$$

In the last step we have used (10.68), performed the commutations necessary to get rid of all operators and finally used $\langle\Omega_0|\Omega_0\rangle = 1$. After integrations over $d\Gamma$'s all the four terms give the same and finally one obtains

$$(S-1)_{\beta\alpha} = -i\lambda\,(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2) + \dots\,, \tag{10.71}$$

where the dots stand for higher order terms. Thus, to this order, using (7.81) we get $-i\mathcal{A}_{\beta\alpha} = -i\lambda$.

In this simple example the same can be also quickly obtained from the old-fashioned perturbation calculus (7.62) with time-independent $V_{\rm int}$: to this order $T^+_{\beta\alpha} = \langle\beta_0|V_{\rm int}|\alpha_0\rangle$ where $V_{\rm int}$ is given by

$$V_{\rm int} = \int d^3\mathbf{x}\,\mathcal{H}(0,\mathbf{x})\,,$$

i.e. it is expressed in terms of operators taken at $t=0$. This leads to

$$t_{\beta\alpha} = (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}_1-\mathbf{k}_2)\,\lambda\,,$$

and, through (7.82), to the same $\mathcal{A}_{\beta\alpha}$ as previously.

In the center of mass system (CMS) the differential cross section of the $2\to 2$ scattering is given by

$$d\sigma(\theta,\phi) = \frac{1}{64\pi^2 s}\frac{|\mathbf{p}_f|}{|\mathbf{k}_i|}\,|\mathcal{A}|^2\,d\Omega_f\,. \tag{10.72}$$

where $\mathbf{k}_i = \mathbf{k}_1 = -\mathbf{k}_2$ and $\mathbf{p}_f = \mathbf{p}_1 = -\mathbf{p}_2$ are the CMS three-momenta of the initial and final particles, respectively, and the Mandelstam variable $s = (k_1 + k_2)^2 = (p_1 + p_2)^2$. For the elastic scattering $|\mathbf{p}_f| = |\mathbf{k}_i|$ and the differential cross section is

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2 s}\,\lambda^2 \,. \tag{10.73}$$

In the example considered the differential cross section is isotropic and falls of with energy of the colliding particles as $1/s$. To obtain the full elastic scattering cross section we integrate (10.73) over the solid angle, which in this trivial case reduces to multiplication by 4π. However, since the particles in the final state are identical, the two configurations with the particle momenta interchanged are indistinguishable one from another and represent therefore the same quantum state. Integration over the full solid angle counts then twice each possible final state. Hence, the result should be divided by a factor of 2:

$$\sigma = \frac{1}{32\pi s}\,\lambda^2 \,. \tag{10.74}$$

In general, if there are n identical particles in the final state the cross section obtained by integrating over all angles must be divided by $n!$ to compensate for multiple counting of the same final states.

More interesting is production of the heavy particles in a collision of two light particles. To illustrate such a process let us take $a^\dagger(\mathbf{k}_2)a^\dagger(\mathbf{k}_1)|\Omega_0\rangle$ with $k_1^2 = k_2^2 = m^2$ as the initial state, $\langle\Omega_0|A(\mathbf{p}_1)A(\mathbf{p}_2)$ with $p_1^2 = p_2^2 = M^2 > m^2$ as the final one and let the interaction have the simplest possible form $\mathcal{H}_{\rm int} = (\lambda/4)\varphi^2(x)\phi^2(x)$ in which $\varphi(x)$ (containing the operators a and $a^\dagger$) and $\phi(x)$ (containing the operators A and $A^\dagger$) are the field operators of the light (mass m) and heavy (mass M) neutral spin zero particles, respectively. Proceeding as in the previous case we get $-i\mathcal{A}_{\beta\alpha} = -i\lambda$ and to write down the formula for $\sigma(\theta,\phi)$ we express $\mathbf{k}_i$ and $\mathbf{p}_f$ through $s = (k_1 + k_2)^2 = (p_1 + p_2)^2$. It is easy to find

$$|\mathbf{k}_i| = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m^2}{s}}\,, \qquad |\mathbf{p}_f| = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4M^2}{s}}\,, \tag{10.75}$$

so that from (10.72) we obtain the differential cross section

$$\frac{d\sigma}{d\Omega} = \sigma(\theta,\phi) = \frac{\lambda^2}{64\pi^2 s}\sqrt{\frac{s - 4M^2}{s - 4m^2}}\,\theta(\sqrt{s} - 2M)\,, \tag{10.76}$$

in which we have reinstated the theta function (omitted in the formula (10.72)) which arises from the integral over the final state phase space. Since $\sigma(\theta,\phi)$ is isotropic, the full production cross section is given by multiplying (10.76) by $4\pi/2 = 2\pi$. It is zero if the energy of the colliding particles is not sufficient, starts to grow from $s = 4M^2$ and then falls off as $1/s$ for $s \gg M^2$.

Since we are using the system of units in which $\hbar = c = 1$, the cross section has the dimension $[M]^{-2}$. To convert it to normal units one has to multiply it by $(\hbar c)^2 =$

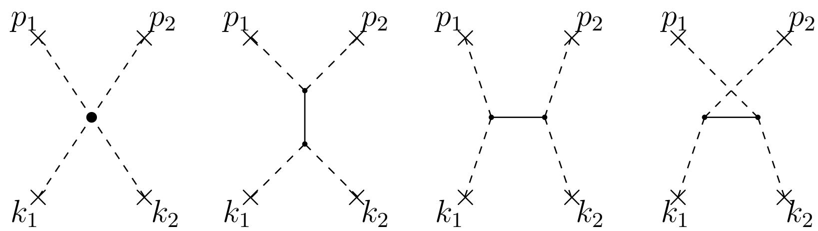

Figure 10.15: Four diagrams contributing to the amplitude of the elastic scattering of spinless neutral particles. Diagrams 2, 3 and 4 represent respectively the s, t and u channel exchanges of a virtual "meson".

0.38937966 GeV^2·mb (1 mb= 10^{-3}b, 1 b= 10^{-28}m^2 = 10^{-24}cm^2). For example, for $\lambda = 0.01$ (dimensionless coupling constant), $\sqrt{s} = 10$ GeV, $M = 1$ GeV and $m \approx 0$ we get from (10.76)

$$\sigma \approx \frac{\lambda^2}{32\pi s} \times \hbar^2 c^2 \approx 4 \cdot 10^{-9}\text{mb} = 4 \cdot 10^{-12}\text{b} = 4\,\text{pb}. \tag{10.77}$$

Experimentally the luminosity $\mathcal{L}$ defined in (10.29) is known for a given accelerator and its energy $\sqrt{s}$. For example, the LEP luminosity (at $\sqrt{s} = 91$ GeV) was $\mathcal{L} \sim 25 \cdot 10^{30}$ cm^{-2}sec^{-1} = $25/(10^{-30}$ cm^2·sec) = $25/(10^{-6}$ b·sec)= $25 \cdot 10^{-6}$pb^{-1}· sec^{-1}. Therefore, one year (1 year$\approx \pi \cdot 10^7$ sec; the "effective accelerator year" is usually shorter by the factor of π) of such an accelerator running at this energy is equivalent to

$$\int dt\ \mathcal{L} \approx 10^7 \times 25 \cdot 10^{-6}\ \text{pb}^{-1}. \tag{10.78}$$

Experimental groups usually announce "we have collected 250 inverse picobarns of data", which means that with the cross section $\sigma \approx 4$pb as in the example above they should see some 10^3 events of heavy scalars pair production (slightly less if the experimental acceptance - the factor which accounts for the fact that only a fraction of produced pairs can actually be identified experimentally due to some limitations inherent in experimental techniques - of the detector is taken into account). If no pair production has been seen, the mass of the heavy particles is - if they exists at all - larger that initially expected by theoreticians or its interactions are weaker...

10.6 Elastic scattering of spinless particles

As the second example, intended to illustrate how to deal with kinematics, we consider a model of interaction of the two kinds of spinless particles. Let interaction Hamiltonian have the simple form

$$\mathcal{H}_{\text{int}}(x) = \frac{\kappa}{2}\,\varphi^2\phi + \frac{\lambda}{4!}\,\varphi^4\,, \tag{10.79}$$

in which the operator φ corresponds to some neutral spin zero particles, call them π's (although they may have nothing to do with real pions!) of mass m and ϕ corresponds to another kind of spinless particles η of mass M and compute the cross section of the elastic scattering process

$$\pi(\mathbf{k}_1)\,\pi(\mathbf{k}_2) \to \pi(\mathbf{p}_1)\,\pi(\mathbf{p}_2)\,.$$

In the lowest order there are four Feynman diagrams shown in figure 10.15 contributing to the amplitude of this process. According to the general formulae (10.40) and (10.41) the fully differential cross section is given by

$$d\sigma = \frac{1}{4\sqrt{(k_1\cdot k_2)^2 - m^4}}\,|\mathcal{A}|^2\,(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_1 - k_2)\,d\Gamma_{\mathbf{p}_1} d\Gamma_{\mathbf{p}_2}\,. \tag{10.80}$$

In the lowest order the invariant amplitude, expressed through the Lorentz invariant Mandelstam variables

$$\begin{aligned} s &= (k_1 + k_2)^2 = (p_1 + p_2)^2\,,\\ t &= (p_1 - k_1)^2 = (p_2 - k_2)^2\,,\\ u &= (p_1 - k_2)^2 = (p_2 - k_1)^2\,, \end{aligned} \tag{10.81}$$

which, because $k_1^2 = m^2$ $k_2^2 = m^2$, $p_1^2 = m^2$ and $p_2^2 = m^2$, satisfy the identity $s + t + u = 2m^2 + 2m^2$, reads

$$\begin{aligned} -i\mathcal{A} &= -i\,(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4)\\ &\equiv -i\lambda + (-i\kappa)^2\frac{i}{s - M^2} + (-i\kappa)^2\frac{i}{t - M^2} + (-i\kappa)^2\frac{i}{u - M^2}\,. \end{aligned} \tag{10.82}$$

The formulae (10.80)-(10.81) can be used in any Lorentz frame. We will work in the CMS defined by the condition

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}\,. \tag{10.83}$$

In this frame the momenta k_1^μ and k_2^μ can be taken in the form

$$k_1^\mu = (E_\mathbf{k},\,0,\,0,\,\mathbf{k})\,,\qquad k_2^\mu = (E_\mathbf{k},\,0,\,0,\,-\mathbf{k})\,, \tag{10.84}$$

with $E_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}$, so that $s = (E_\mathbf{k} + E_\mathbf{k})^2 = 4E_\mathbf{k}^2$ and

$$E_\mathbf{k} = \frac{1}{2}\sqrt{s}\,,\qquad |\mathbf{k}| = \sqrt{\frac{s}{4} - m^2} = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m^2}{s}}\,. \tag{10.85}$$

$\sqrt{s} = 2E_\mathbf{k}$ is called therefore the CMS collision energy.

In the CSM the four-dimensional delta function in (10.80) takes the form

$$\delta^{(4)}(p_1 + p_2 - k_1 - k_2) = \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2)\,\delta(E_{\mathbf{p}_1} + E_{\mathbf{p}_2} - \sqrt{s}\,)\,. \tag{10.86}$$

Performing the integral over $d\Gamma_{\mathbf{p}_2}$ and over the length of $\mathbf{p}_1$ amounts then to expressing $\mathcal{A}$ in terms of[13]

$$\begin{aligned}
p_1 &= (E_{\mathbf{p}}, 0, |\mathbf{p}|\sin\theta_{\mathbf{p}}\,, |\mathbf{p}|\cos\theta_{\mathbf{p}})\,,\\
p_2 &= (E_{\mathbf{p}}, 0, -|\mathbf{p}|\sin\theta_{\mathbf{p}}\,, -|\mathbf{p}|\cos\theta_{\mathbf{p}})\,,
\end{aligned}\tag{10.87}$$

where (because the scattering is elastic)

$$E_{\mathbf{p}} = \frac{1}{2}\sqrt{s}\,, \qquad |\mathbf{p}| = \sqrt{\frac{s}{4} - m^2} = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m^2}{s}}\,. \tag{10.88}$$

Thus, the amplitude $\mathcal{A}$ can be expressed in terms of s and the scattering angle $\theta_{\mathbf{p}}$:

$$\begin{aligned}
t &= p_1^2 + k_1^2 - 2p_1\!\cdot\! k_1 = 2m^2 - \frac{s}{2}\left(1 - \left(1 - \frac{4m^2}{s}\right)\cos\theta_{\mathbf{p}}\right),\\
u &= p_1^2 + k_2^2 - 2p_1\!\cdot\! k_2 = 2m^2 - \frac{s}{2}\left(1 + \left(1 - \frac{4m^2}{s}\right)\cos\theta_{\mathbf{p}}\right),
\end{aligned}\tag{10.89}$$

The differential cross section in this case is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left|\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4\right|^2, \tag{10.90}$$

where on the right hand side we have omitted the factor $|\mathbf{p}|\,/\,|\mathbf{k}|$ which in the case of elastic scatterings equals unity. With the formulae (10.89) the differential cross section can be expressed through s and $\theta_{\mathbf{p}}$.

To get the full elastic scattering cross section one has to integrate the 10 terms of (10.90) over $d\Omega = d\phi_{\mathbf{p}} d\theta_{\mathbf{p}} \sin\theta_{\mathbf{p}}$ and to divide the result by a factor of 2 (because the final state particles are indistinguishable). The integrals over $d\phi_{\mathbf{p}}$ are trivial and yield 2π. The integrals over $d\theta_{\mathbf{p}}$ are a little bit more complicated. The necessary master integrals are collected in Appendix H. In the considered case the general formulae of Appendix H give

$$\int d\Omega_{\hat{\mathbf{p}}}\left|\mathcal{A}_1 + \mathcal{A}_2\right|^2 = 4\pi\left(\lambda + \frac{\kappa^2}{s - M^2}\right)^2.$$

The integrals of $|\mathcal{A}_3|^2$ and $|\mathcal{A}_4|^2$ are of the type (H.1) with $a = 2m^2 - (s/2) - M^2$ and $b = \pm(s/2)(1 - 4m^2/s)$. Together they give

$$\int d\Omega_{\hat{\mathbf{p}}}\left(|\mathcal{A}_3|^2 + |\mathcal{A}_4|^2\right) = \frac{8\pi\kappa^4}{(2m^2 - s/2 - M^2)^2 - (s^2/4)(1 - 4m^2/s)^2}\,.$$

The terms corresponding to the interference of the s-channel diagrams (diagrams 1 and 2 in figure 10.15) with the other two are integrated using (H.4) and give

$$\begin{aligned}
&\int d\Omega_{\hat{\mathbf{p}}}\, 2\mathrm{Re}\left[(\mathcal{A}_1 + \mathcal{A}_2)^*\left(\mathcal{A}_3 + \mathcal{A}_4\right)\right]\\
&\qquad = 8\pi\left(\lambda + \frac{\kappa^2}{s - M^2}\right)\frac{2\kappa^2}{s(1 - 4m^2/s)}\ln\frac{M^2}{M^2 + s - 4m^2}\,.
\end{aligned}$$

[13] More generally one could write $p_{1,2} = (E_{\mathbf{p}}, \pm|\mathbf{p}|\sin\theta_{\mathbf{p}}\cos\phi_{\mathbf{p}}, \pm|\mathbf{p}|\sin\theta_{\mathbf{p}}\sin\phi_{\mathbf{p}}, \pm|\mathbf{p}|\cos\theta_{\mathbf{p}})$ but the angle $\phi_{\mathbf{p}}$ would cancel out in $\mathcal{A}$.

The terms corresponding to the interference of the t and u-channel diagrams (3 and 4 in figure 10.15) is integrated with the help of (H.7) with $c = a$ and $d = -b$:

$$\int d\Omega_{\hat{\mathbf{p}}}\, 2\mathrm{Re}\,(\mathcal{A}_3^* \mathcal{A}_4) = \frac{8\pi\kappa^4}{s(1-4m^2/s)(2m^2 - s/2 - M^2)} \ln \frac{M^2}{M^2 + s - 4m^2}\,.$$

Collecting all terms one gets

$$\begin{aligned}\sigma = \frac{1}{2}\frac{1}{16\pi s}\Bigg\{ & \left(\lambda + \frac{\kappa^2}{s - M^2}\right)^2 \\ & + \frac{2\kappa^4}{(2m^2 - s/2 - M^2)^2 - (s^2/4)(1 - 4m^2/s)^2} \\ & + \left(\lambda + \frac{\kappa^2}{s-M^2}\right) \frac{4\kappa^2}{s(1-4m^2/s)} \ln \frac{M^2}{M^2 + s - 4m^2} \\ & + \frac{2\kappa^4}{s(1-4m^2/s)(2m^2 - s/2 - M^2)} \ln \frac{M^2}{M^2 + s - 4m^2} \Bigg\}\,.\end{aligned} \tag{10.91}$$

Obe can now consider the limit $s \ll M^2$. For simplicity we will set $m = 0$. Expanding the successive terms in powers of s/M^2 we get

$$\begin{aligned}\lambda + \frac{\kappa^2}{s - M^2} &= \lambda - \frac{\kappa^2}{M^2} - \frac{\kappa^2 s}{M^4} + \ldots\,, \\ \frac{2\kappa^4}{(-s/2 - M^2)^2 - s^2/4} &= \frac{2\kappa^4}{M^4 + sM^2} = \frac{2\kappa^4}{M^4} - \frac{2\kappa^4 s}{M^6} + \ldots\,, \\ \frac{4\kappa^2}{s} \ln \frac{M^2}{M^2 + s} &= -\frac{4\kappa^2}{s}\left(\frac{s}{M^2} - \frac{s^2}{2M^4} + \ldots\right), \\ \frac{2\kappa^4}{s(s/2 + M^2)} \ln \frac{M^2 + s}{M^2} &= \frac{2\kappa^4}{sM^2}\left(1 - \frac{s}{2M^2} + \ldots\right)\left(\frac{s}{M^2} - \frac{s^2}{2M^4} + \ldots\right).\end{aligned}$$

Thus, for $s \ll M^2$ the integrated cross section can be approximated by

$$\sigma \;=\; \frac{1}{2}\frac{1}{16\pi s}\left\{\left(\lambda - 3\frac{\kappa^2}{M^2}\right)^2 + \mathcal{O}\left(\frac{s}{M^2}\right)\right\} = \frac{1}{2}\frac{\lambda_{\text{eff}}^2}{16\pi s} + \ldots \tag{10.92}$$

i.e. by the cross section which would follow from the second term of (10.79) alone, but with λ replaced by $\lambda_{\text{eff}} = \lambda - 3\kappa^2/M^2$. This is an example of the *decoupling of particles*: in most cases their physical effects, can - up to terms suppressed by inverse powers of their masses - be absorbed into redefinitions of parameters of the effective model from which the heavy particles are absent.

Another remark concerning the result (10.91) is that the cross section becomes singular as $s \to M^2$. This results from the singularity at $s = M^2$ of the contribution $\mathcal{A}_2$ to the scattering amplitude of the s-channel exchange of the virtual η particle (the second diagram in figure 10.15); the contributions of the remaining diagrams are not singular because the Mandelstamm variables t and u are negative (see (10.89)) when $s > 4M_\pi^2$ which is the physical domain of the s variable in the case of the elastic $\pi^+\pi^- \to \pi^+\pi^-$ scattering. As a result of this singularity near $s \approx M^2$ the partial amplitude $\mathcal{T}^{l=0}(s)$ seems to violate the bounds derived rigorously from the unitarity of the S-matrix in Section 7.6. The problem is automatically cured by including higher order contributions to the scattering amplitude (this is the perturbative unitarization mentioned in section 2.2 in figure 2.2). With the corrections included, the the denominator of the effective propagator (of the two-point Green's function) develops an imaginary part related to the fact that the heavy particle η is unstable and, therefore, can decay into $\pi\pi$. Near $s = M^2$ this effective propagator can be (if $\Gamma_\eta \ll M_\eta$ - the η particle is long-living, narrow resonance) approximated by

$$\frac{i}{s - M^2 + iM\Gamma_\eta}\,, \tag{10.93}$$

in which Γ_η is just the η decay width into $\pi\pi$ which in this case is also its total decay width. The pole of the amplitude is therefore shifted from the real s axis and $|\mathcal{T}^{(l=0)}|^2$ (and with it the integrated cross section) exhibits near $s = M^2$ only a finite bump which is interpreted as creation of an unstable resonance η. It can be checked that in the leading order in $1/\Gamma_\eta$ after the replacement of the original η propagator by (10.93) the unitarity relation (7.126) is restored at $s = M_\eta^2$.

10.7 Antifermion-spin zero particle elastic scattering

As the last illustrative example we consider a theory of neutral spin zero particles π (of mass M) interacting with fermions f and their antifermions $\bar{f}$ (having mass m). Let the interaction have the form

$$\mathcal{H}_{\rm int} = ih\,\bar{\psi}\gamma^5\psi\,\varphi\,, \tag{10.94}$$

with some real coupling constant h, and compute the cross section of the elastic antifermion-π scattering

$$\bar{f}(\mathbf{p}_1)\,\pi(\mathbf{k}_1) \to \bar{f}(\mathbf{p}_2)\,\pi(\mathbf{k}_2)\,. \tag{10.95}$$

We take π to be a pseudoscalar (i.e. having negative intrinsic parity) because the calculations are then somewhat simpler. The two lowest order diagrams contributing to the scattering amplitude are shown in figure 10.16. The corresponding expressions follow from the Feynman rules and read

$$\begin{aligned} -i\mathcal{A}_1 &= h^2\,\bar{v}(\mathbf{p}_1, s_1)\,\gamma^5\,\frac{i}{-\not{p}_1 - \not{k}_1 - m + i0}\,\gamma^5\,v(\mathbf{p}_2, s_2)\,,\\ -i\mathcal{A}_2 &= h^2\,\bar{v}(\mathbf{p}_1, s_1)\,\gamma^5\,\frac{i}{-\not{p}_1 + \not{k}_2 - m + i0}\,\gamma^5\,v(\mathbf{p}_2, s_2)\,. \end{aligned} \tag{10.96}$$

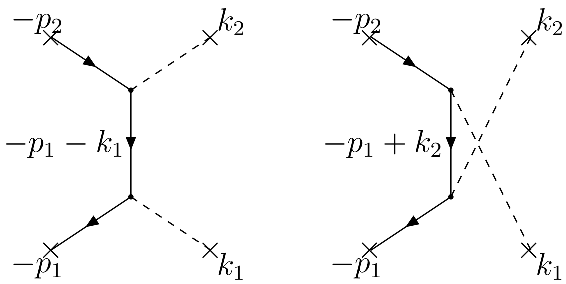

Figure 10.16: Two diagrams contributing to the elastic scattering of spinzero particles and antifermions interacting through the Yukawa coupling.

(Notice how the four-momenta are ascribed to the internal lines of the diagrams shown in figure 10.16 and compare with the form of the corresponding propagators in the amplitudes (10.96)!)

The case of the Yukawa coupling with γ^5 gives an opportunity to demonstrate a trick allowing to simplify the amplitude. Let us rewrite (10.96) in the form

$$\mathcal{A}_1 = \frac{h^2}{(p_1+k_1)^2 - m^2 + i0}\,\bar{v}(\mathbf{p}_1, s_1)\,\gamma^5\,(\not{p}_1 + \not{k}_1 - m)\,\gamma^5\,v(\mathbf{p}_2, s_2)\,,$$
$$\mathcal{A}_2 = \frac{h^2}{(p_1-k_2)^2 - m^2 + i0}\,\bar{v}(\mathbf{p}_1, s_1)\,\gamma^5\,(\not{p}_1 - \not{k}_2 - m)\,\gamma^5\,v(\mathbf{p}_2, s_2)\,.$$

In both terms one can now write

$$\bar{v}(\mathbf{p}_1, s_1)\,\gamma^5 \not{p}_1 = -\bar{v}(\mathbf{p}_1, s_1)\,\not{p}_1\,\gamma^5 = \bar{v}(\mathbf{p}_1, s_1)\,\gamma^5\,m\,, \tag{10.97}$$

upon using the (hermitian conjugation of the) property (8.101)

$$\bar{v}(\mathbf{p})\,(\not{p} + m) = 0\,. \tag{10.98}$$

The two amplitudes then simplify then to

$$\mathcal{A}_1 = -\frac{h^2}{(p_1+k_1)^2 - m^2 + i0}\,\bar{v}(\mathbf{p}_1, s_1)\,\not{k}_1\,v(\mathbf{p}_2, s_2)\,,$$
$$\mathcal{A}_2 = \frac{h^2}{(p_1-k_2)^2 - m^2 + i0}\,\bar{v}(\mathbf{p}_1, s_1)\,\not{k}_2\,v(\mathbf{p}_2, s_2)\,.$$

The terms with masses in the numerators have disappeared due to the trick (10.98) after which we have used $\gamma^5 \not{k}_i \gamma^5 = -\not{k}_i$.

We will compute the cross section of the scattering of unpolarized antifermions assuming that the spin projection of the final state antifermions are not measured. Tho thos end we need

$$\sum_{s_1}\sum_{s_2} |\mathcal{A}_1 + \mathcal{A}_2|^2 = \sum_{s_1}\sum_{s_2} \left(|\mathcal{A}_1|^2 + |\mathcal{A}_2|^2 + 2\,\mathrm{Re}(\mathcal{A}_1\mathcal{A}_2^*)\right). \tag{10.99}$$

Using the standard trace technique we find

$$\sum_{\text{spins}} |\mathcal{A}_1|^2 = \frac{h^4}{(s-m^2)^2}\,\mathrm{tr}\,[(\not p_1 - m)\not k_1(\not p_2 - m)\not k_1]$$
$$= \frac{4h^4}{(s-m^2)^2}\left[2(p_1\cdot k_1)(p_2\cdot k_1) - M^2(p_1\cdot p_2) + m^2M^2\right],$$
$$\sum_{\text{spins}} |\mathcal{A}_2|^2 = \frac{4h^4}{(u-m^2)^2}\left[2(p_1\cdot k_2)(p_2\cdot k_2) - M^2(p_1\cdot p_2) + m^2M^2\right],$$

and ($\mathcal{A}_2$ differs from $\mathcal{A}_1$ only by the replacement $k_1 \to -k_2$)

$$\sum_{\text{spins}} \mathcal{A}_1\mathcal{A}_2^* = -\frac{h^4}{(s-m^2)(u-m^2)}\,\mathrm{tr}\,[(\not p_1 - m)\not k_1(\not p_2 - m)\not k_2]$$
$$= -\frac{4h^4}{(s-m^2)(u-m^2)}\big[(p_1\cdot k_1)(p_2\cdot k_2) + (p_1\cdot k_2)(p_2\cdot k_1)$$
$$-(p_1\cdot p_2)(k_1\cdot k_2) + m^2(k_1\cdot k_2)\big].$$

We have used the mass-shell conditions $k_1^2 = k_2^2 = M^2$. The scalar products can be now expressed in terms of the Mandelstam variables $s = (p_1 + k_1)^2$, $u = (p_1 - k_2)^2$ and $t = (p_2 - p_1)^2$:

$$p_1\cdot k_1 = \frac{1}{2}(s - m^2 - M^2) = p_2\cdot k_2\,,$$
$$p_1\cdot p_2 = -\frac{1}{2}(p_2 - p_1)^2 + m^2 = m^2 - \frac{t}{2}\,,$$
$$k_1\cdot k_2 = -\frac{1}{2}(k_2 - k_1)^2 + M^2 = M^2 - \frac{t}{2}\,,$$
$$p_1\cdot k_2 = -\frac{1}{2}(k_2 - p_1)^2 + \frac{1}{2}m^2 + \frac{1}{2}M^2 = \frac{1}{2}(m^2 + M^2 - u) = p_2\cdot k_1\,.$$

Using these relations we find

$$\sum_{\text{spins}} |\mathcal{A}_1|^2 = \frac{2h^4}{(s-m^2)^2}\left[(s+u)(m^2+M^2) - (m^2+M^2)^2 - su + M^2t\right],$$
$$\sum_{\text{spins}} |\mathcal{A}_2|^2 = \frac{2h^4}{(u-m^2)^2}\left[(s+u)(m^2+M^2) - (m^2+M^2)^2 - su + M^2t\right],$$
$$\sum_{\text{spins}} \mathcal{A}_1\mathcal{A}_2^* = -\frac{2h^4}{(s-m^2)(u-m^2)}\Big[\frac{1}{2}(s - m^2 - M^2)^2$$
$$+\frac{1}{2}(u - m^2 - M^2)^2 + t\,(M^2 - \frac{t}{2})\Big].$$

One of the Mandelstamm variables, e.g. t, can be eliminated using the identity

$$s + u + t = 2m^2 + 2M^2\,, \qquad (10.100)$$

so that

$$\sum_{\text{spins}} |\mathcal{A}_1|^2 = \frac{2h^4}{(s-m^2)^2} \left[m^2(s+u) - (m^4 - M^4) - su\right],$$
$$\sum_{\text{spins}} |\mathcal{A}_2|^2 = \frac{2h^4}{(u-m^2)^2} \left[m^2(s+u) - (m^4 - M^4) - su\right],$$
$$\sum_{\text{spins}} \mathcal{A}_1\mathcal{A}_2^* = -\frac{2h^4}{(s-m^2)(u-m^2)} \left[m^2(s+u) - (m^4 - M^4) - su\right].$$

It follows that

$$\sum_{\text{spins}} |\mathcal{A}|^2 = 2h^4 \left(\frac{1}{s-m^2} - \frac{1}{u-m^2}\right)^2 \left[M^4 - m^4 + m^2(s+u) - su\right].$$

The CMS differential cross section can be now written using the general formula (10.72) as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\frac{1}{2}\sum_{\text{spins}} |\mathcal{A}|^2, \tag{10.101}$$

where we have already used the fact that for elastic scattering processes in the general formula $|\mathbf{p}_f|/|\mathbf{k}_i| = 1$. The factor 1/2 comes from averaging over the spin states of the initial antifermion. The CMS kinematics is accounted for by writing the four-momenta explicitly (defining thereby the CMS scattering angle $\theta_{\mathbf{p}}$):

$$p_1 = (E_m(\mathbf{p}),\, 0,\, 0,\, |\mathbf{p}|), \qquad p_2 = (E_m(\mathbf{p}),\, 0,\, |\mathbf{p}|\sin\theta_{\mathbf{p}},\, |\mathbf{p}|\cos\theta_{\mathbf{p}}),$$
$$k_1 = (E_M(\mathbf{p}),\, 0,\, 0,\, -|\mathbf{p}|), \qquad k_2 = (E_M(\mathbf{p}),\, 0,\, -|\mathbf{p}|\sin\theta_{\mathbf{p}},\, -|\mathbf{p}|\cos\theta_{\mathbf{p}}),$$

where

$$\begin{aligned} |\mathbf{p}| &= \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2s(m^2+M^2) + (M^2-m^2)^2} \equiv \frac{1}{2\sqrt{s}}\,\lambda^{1/2}(s, M^2, m^2), \\ E_m &= \frac{1}{2\sqrt{s}}\left(s + m^2 - M^2\right), \\ E_M &= \frac{1}{2\sqrt{s}}\left(s + M^2 - m^2\right). \end{aligned} \tag{10.102}$$

The Mandelstam variable u is then related to s and $\theta_{\mathbf{p}}$ by

$$\begin{aligned} u &= m^2 + M^2 - 2(E_m E_M + |\mathbf{p}|^2\cos\theta_{\mathbf{p}}) \\ &= \frac{1}{s}(m^2 - M^2)^2 - \frac{\lambda}{2s}(1 + \cos\theta_{\mathbf{p}}), \end{aligned} \tag{10.103}$$

Instead of writing the differential cross section in terms of the angle $\theta_{\mathbf{p}}$ one can put it in a manifestly Lorentz invariant form by using the relation

$$du = -2|\mathbf{p}|^2 d\cos\theta_{\mathbf{p}} = -\frac{1}{2s}\,\lambda(s, M^2, m^2)\, d(\cos\theta_{\mathbf{p}}).$$

The differential cross section (integrated over $d\phi_{\mathbf{p}}$) then reads

$$\frac{d\sigma}{du} = -\frac{h^4}{16\pi\lambda(s, M^2, m^2)} \left(\frac{1}{s-m^2} - \frac{1}{u-m^2} \right)^2 \left[M^4 - m^4 + m^2(s+u) - su \right] .$$

In the CMS the differential cross section can be written explicitly in the form

$$\frac{d\sigma}{d\Omega_{\mathbf{p}}} = \frac{h^4}{64\pi^2 s} \left(\frac{1}{s-m^2} - \frac{1}{u-m^2} \right)^2 \left[M^4 - m^4 + m^2(s+u) - su \right] ,$$

in which now u is treated as a function of $\cos\theta_{\mathbf{p}}$ and s given by (10.103).

Appendix G Boltzmann H-theorem

The formula (10.22) for the differential probability per unit time per unit volume of the transition between the properly normalized (in the box of volume V) multiparticle states $|\alpha_0\rangle_V$ and $|\beta_0\rangle_V$ written in the form

$$\frac{1}{V}\,\frac{dw(\alpha\to\beta)}{d\beta} = \frac{1}{c_\alpha}\,|\mathcal{A}_{\beta\alpha}|^2\,(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)\,, \tag{G.1}$$

in which

$$c_\alpha = \prod_{j=1}^{n_\alpha}\left(2E_{\mathbf{k}_j}V\right), \qquad \text{and} \qquad d\beta = \prod_{i=1}^{n_\beta}\frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\,, \tag{G.2}$$

and combined with the relation following from the unitarity of the S-matrix (see Section 7.6) can be used to prove in full generality the Boltzmann H-theorem which plays the central role in the *classical* kinetic theory. Within this classical theory it is, however, possible to treat particles not entirely classically by allowing them to take part in processes of inelastic scattering and to decay by using rates of these processes computed in the framework of a relativistic quantum theory (these rates play then the role of an external input to the kinetic theory).

Starting with the S-matrix unitarity condition written in the alternative to (7.43) form

$$\int d\beta\, S_{\gamma\beta}(S^\dagger)_{\beta\alpha} \equiv \int d\beta\, S_{\gamma\beta}S^*_{\alpha\beta} = \delta_{\gamma\alpha}\,,$$

and repeating the steps which led to the relation (7.110) one obtains

$$-i\left(\mathcal{A}^*_{\alpha\gamma} - \mathcal{A}_{\gamma\alpha}\right) = \int d\beta\,(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)\mathcal{A}_{\gamma\beta}\mathcal{A}^*_{\alpha\beta}\,. \tag{G.3}$$

This, taken for $\gamma = \alpha$ and combined with (7.111) taken for $\beta = \alpha$ (and renaming then in it γ to β), shows that

$$\int d\beta\,(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)|\mathcal{A}_{\beta\alpha}|^2 = \int d\beta\,(2\pi)^4\delta^{(4)}(P_\beta - P_\alpha)|\mathcal{A}_{\alpha\beta}|^2\,, \tag{G.4}$$

which, in turn, applied to (G.1) integrated over $d\beta$ and to the integrated over $d\beta$ relation obtained from (G.1) by interchanging the labels α and β allows to write the equality

$$\int d\beta\, c_\alpha\,\frac{dw(\alpha\to\beta)}{d\beta} = \int d\beta\, c_\beta\,\frac{dw(\beta\to\alpha)}{d\alpha}\,. \tag{G.5}$$

Let now P_α be[1] the probability density of finding a set of classical particles in a (classical) state α characterized by the distribution of their momenta and spin variables.

[1] As the total four-momentum of the state $|\alpha_0\rangle$ will no more appear in this Appendix, this should not cause any confusion.

In other words, P_α is the phase space probability density of the classical micro-state α. As a result of particle interactions P_α changes in time:

$$\frac{d}{dt}\,P_\alpha = \int d\beta\, P_\beta\,\frac{dw(\beta\to\alpha)}{d\alpha} - P_\alpha\int d\beta\,\frac{dw(\alpha\to\beta)}{d\beta}\,. \tag{G.6}$$

The first term on the right hand side of this equality accounts for the increase the population of particles in the state α as a result of all possible transitions to this state from all other states β and the second term accounts for the decrease this population as a result of transitions from the state α to all other states. Of course

$$\frac{d}{dt}\int d\alpha\, P_\alpha = \int d\alpha\,\frac{d}{dt}\,P_\alpha = 0, \tag{G.7}$$

as can easily be seen by integrating (G.6) over $d\alpha$ and renaming in the second term the the integration variables $\alpha \leftrightarrow \beta$.

The probability density P_α is related to the classical statistical phase space probability distribution $\rho_\alpha \equiv \rho_{\sigma_1,\dots,\sigma_{n_\alpha}}(\mathbf{x}_1,\mathbf{p}_1,\dots,\mathbf{x}_{n_\alpha},\mathbf{p}_{n_\alpha})$ by

$$\rho_\alpha = \frac{1}{c_\alpha}\,P_\alpha\,. \tag{G.8}$$

Indeed, in classical statistical mechanics the probability of finding the system of n_α particles (for simplicity we assume there is only one kind of identical particles) in the classical microstate α characterized by n_α their momenta (taking into account in the classical way their indistinguishability) anywhere in the volume V is

$$d\Gamma^{n_\alpha}_{\rm stat}\left(\prod_{i=1}^{n_\alpha}\int_V d^3\mathbf{x}_i\right)\rho_\alpha = \frac{1}{n_\alpha!}\prod_{i=1}^{n_\alpha}\frac{d^3\mathbf{p}_i}{(2\pi)^3}\left(\prod_{i=1}^{n_\alpha}\int_V d^3\mathbf{x}_i\right)\rho_\alpha = \frac{V^{n_\alpha}}{n_\alpha!}\prod_{i=1}^{n_\alpha}\frac{d^3\mathbf{p}_i}{(2\pi)^3}\,\rho_\alpha\,,$$

(it is assumed that ρ_α is independent of the space positions $\mathbf{x}_i$) whereas the same probability probability expressed through P_α used in (G.5) is given by

$$d\alpha\, P_\alpha \equiv \frac{1}{n_\alpha!}\prod_{i=1}^{n_\alpha}\frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\,P_\alpha\,.$$

The kinetic entropy[2] $S_{\rm kin} = -k_{\rm B}H$, where H is the famous Boltzmann H-function, of a (nonequilibrium) statistical state of a system of interacting classical particles given by the universal formula $S = -k_{\rm B}\overline{\ln\rho}$ (the bar means statistical averaging with respect to the classical probability distribution ρ) can be therefore written as

$$S_{\rm kin} = -k_{\rm B}\int d\alpha\, P_\alpha \ln\frac{P_\alpha}{c_\alpha}\,.$$

[2]It should not be identified with the thermodynamical entropy which can be ascribed only to equilibrium states of the system.

Using (G.6) we get

$$\begin{aligned}\frac{d}{dt}S_{\rm kin} &= -k_{\rm B}\int d\alpha \left(1+\ln\frac{P_\alpha}{c_\alpha}\right)\frac{dP_\alpha}{dt}\\ &= -k_{\rm B}\int d\alpha\, \ln\frac{P_\alpha}{c_\alpha}\left(\int d\beta\, P_\beta\,\frac{dw(\beta\to\alpha)}{d\alpha} - P_\alpha\int d\beta\,\frac{dw(\alpha\to\beta)}{d\beta}\right),\end{aligned}$$

(the term with “1” in the bracket in the first line drops out due to the relation (G.7)). The right hand side, after interchanging in the second term the labels, $\alpha\leftrightarrow\beta$, gives

$$\frac{d}{dt}S_{\rm kin} = k_{\rm B}\int d\alpha\int d\beta\, c_\beta\,\frac{P_\beta}{c_\beta}\,\ln\!\left(\frac{P_\beta\, c_\alpha}{c_\beta P_\alpha}\right)\frac{dw(\beta\to\alpha)}{d\alpha}\,. \tag{G.9}$$

One can now use the inequality[3] $x\ln(x/y)\geq x-y$ valid for positive x and y to write

$$\begin{aligned}\frac{d}{dt}S_{\rm kin} &\geq k_{\rm B}\int d\alpha\int d\beta\left(\frac{P_\beta}{c_\beta}-\frac{P_\alpha}{c_\alpha}\right)c_\beta\,\frac{dw(\beta\to\alpha)}{d\alpha}\\ &= k_{\rm B}\int d\alpha\int d\beta\,\frac{P_\alpha}{c_\alpha}\left(c_\alpha\,\frac{dw(\alpha\to\beta)}{d\beta} - c_\beta\,\frac{dw(\beta\to\alpha)}{d\alpha}\right).\end{aligned}$$

(In the last step the labels α and β in the first term have again been interchanged.) Since by (G.5) the expression in the bracket vanishes upon integration over $d\alpha$ we get the Boltzmann H-theorem

$$\frac{d}{dt}\,S_{\rm kin}\geq 0\,. \tag{G.10}$$

From (G.9) it follows that $S_{\rm kin}=$ const when $P_\alpha/c_\alpha = P_\beta/c_\beta$. Replacing in the first term of (G.6) P_β by $P_\alpha(c_\beta/c_\alpha)$ and using (G.5) shows that in this case the probability P_α does not change in time: the system is in equilibrium.

[3] It is equivalent to the inequality $\eta-\ln\eta\geq 1$ (with nonnegative $\eta=y/x$) which is easily seen to be true: it becomes the equality for $\eta=1$ and the derivative of its left hand side is positive for $\eta>1$ and negative for $\eta<1$.