Quantum Field Theory of Fundamental Interactions. Problems, set 10.

Problem 1. Show that the vacuum polarization $\Pi^{\rm ON}_{\gamma}$ (renormalized using the On-Shell scheme) in electrodynamics of spin 1/2 particles can be written as

$$\Pi^{\rm ON}_{\gamma}(q^2) = \frac{(eQ)^2}{16\pi^2} 8 \int_0^1 dx x(1-x) \ln\left(1 - x(1-x)\frac{q^2}{m^2 - i0}\right)$$

Compute also the imaginary part of $\Pi^{\rm ON}_{\gamma}(q^2)$.
Hint: Start from the original expression for vacuum polarization and perform the integral over virtual momentum directly (without reducing this integral to standard functions $b_0(q^2, m, m)$ etc.).

Problem 2. Calculate the vacuum polarization $\Pi^{\rm ON}_{\gamma}$ in electrodynamics of spinless particles. Check that it has the form

$$\Pi_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\tilde{\Pi}_{\gamma\gamma}(q^2)$$

Derive the asymptotic forms of the photon polarization tensor $\Pi_{\gamma\gamma}(q^2) = q^2\tilde{\Pi}_{\gamma}(q^2)$ for $q^2 \to 0$ and for $q^2 \gg M^2$ both in the MS scheme and in the On Shell scheme (in the latter the counterterm Z_3 is such that the renormalized $\Pi_{\gamma}(0) = 0$).

Problem 3. Using the form the photon self energy

$$\Pi_{\mu\nu}(q) = \Pi_{\gamma\gamma}(q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$

show that summing one perticle reducible diagrams with 0,1,2,... insertions of $i\Pi_{\mu\nu}$ one obtains the full (connected) photon two point function in the form

$$iD^{\rm full}_{\mu\nu}(q^2) = \frac{-i}{q^2(1-\tilde{\Pi}_{\gamma}(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - i\xi\frac{q_\mu q_\nu}{q^4}$$

Prove (using formal arguments with renormalization constants) that in quantum electrodynamics

$$e^2_{\rm inv}(q^2) \equiv \frac{e^2_{\rm R}}{1-\tilde{\Pi}^{\rm R}_{\gamma}(q^2)}$$

is independent of the renormalization scheme adopted, i.e. the subscript R (renormalized) can mean any scheme like On-Shell, MS, etc. (hence the name "invariant" charge). Use the asymptotic form of $\tilde{\Pi}_\gamma(q^2)$ in the MS scheme to show that for $|q^2| \to \infty$ the invariant charge $e^2_{\rm inv}(q^2)$ becomes identical to the charge $\hat{e}^2(\mu)$ in the MS scheme for $\mu^2 = |q^2|$. Use the limit $q^2 \to 0$ of $\tilde{\Pi}_\gamma(q^2)$ to argue that the electric charge defined in the On-Shell scheme is the same as $\hat{e}(\mu)$ in the $\overline{MS}$ scheme for $\mu = m_e$.

Problem 4. Consider quantum electrodynamics describing all known elementary charged fermions (leptons and quarks). Appealing to the Appelquist-Carrazzone decoupling theorem express the scale at which the MS scheme coupling diverges in terms of $\hat{e}(m_e)$ and fermion masses.[1] Ignore the existence of the charged vector bosons $W^\pm$.

Problem 5. Consider electrodynamics of spin 1/2 particles coupled to a scalar field

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \bar{\psi}\left(i\,\partial\!\!\!/ - eQ\,A\!\!\!/ - m\right)\psi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 - g\bar{\psi}\psi\varphi$$

Show by direct calculation that $Z_1 = Z_2$ in the MS and in the On-Shell schemes.

Problem 6. As in the preceding problem but with the pseudoscalar-electron coupling of the form

$$-ig\bar{\psi}\gamma^5\psi\varphi$$

Problem 7. Some theories aiming at solving the so-called strong CP problem postulate the existence of a light CP-odd (pseudoscalar) particle a^0 called axion, which couples to electrons throght the interaction term

$$\mathcal{L}_{int} = -i\lambda a^0 \bar{\psi}_e\gamma^5\psi_e$$

Find the axion contribution to the formfactor $F_2(0)$, i.e. the contribution to $a_e = (g-2)/2$, as a function of its mass m_a and coupling λ.

Problem 8. Find the contribution to the formfactor $F_2(0)$ of a scalar neutral particle h^0 coupled to electrons through the interaction term

$$\mathcal{L}_{int} = -\lambda h^0 \bar{\psi}_e\psi_e$$

In the Standard Model the Yukawa coupling λ of the Higgs particle h^0 to electrons is proportional to m_e: $\lambda = (e/\sqrt{2}s_W)(m_e/M_W)$ where $e = \sqrt{4\pi\alpha_{\rm EM}}$, $s_W = 0.23$ and

[1] As a result of strong interactions of light quarks their contribution cannot be computed perturbatively and in practice it is extracted from the measured total cross section $\sigma(e^+e^- \to$hadrons) by using a dispersion relation.

$M_W = 80.4$ GeV. Find the lower bound on M_h implied by the fact that

$$|a^e_{exp} - a^e_{QED}| < 1 \times 10^{-10}$$

(a^e is just $(g-2)/2$ of the electron). Is it stronger than the direct LEP bound $M_h > 114$ GeV? The h^0 couplings to muons is $\lambda = (e/\sqrt{2}s_W)(m_\mu/M_W)$ and

$$|a^\mu_{exp} - a^\mu_{QED}| < 3 \times 10^{-8}$$

Does this give a better bound on M_h?

Problem 9. Assume that electron couples to a neutral fermion N of mass m_N and a charged scalar $H^\pm$ (of mass M)

$$\mathcal{L}_{int} = -\lambda H^\dagger \bar{\psi}_e \psi_N + \text{H.c.}$$

where H annihilates positively charged scalar (i.e. $Q_H = +1$). λ is a coupling constant. Find the corresponding contribution to $(g-2)/2$.

Problem 10. Add to the Lagrangian of spin 1/2 particles electrodynamics the photon mass term $-\frac{1}{2}\lambda^2_\gamma A_\mu A^\mu$ and compute the electron self energy $-i\Sigma(\not{p}) = -i\left(\Sigma_V(p^2)\,\not{p} + \Sigma_m(p^2)\right)$ in arbitrary gauge ξ keeping only terms nonvanishing (i.e. also the singular ones) as $\lambda_\gamma \to 0$. Compute then the external lines renormalization factor $\mathcal{Z}_\psi$ in the MS scheme (in $\xi = 1$ gauge, if too difficult in general). Compare with the result obtained in the dimensional regularization. What is the substitution rule allowing to switch between the latter approacg (with $1/\epsilon_{\rm IR}$ singularities) and the one with λ_γ?

Problem 11. ('$Z_1 = Z_2$' in the On Shell scheme) In the On Shell scheme renormalization of the vertex (i.e. the Z_1 renormalization constant) of the vertex

$$ie\gamma^\mu + ie\Lambda^\mu_{\rm R}(p, p+q)$$

is defined so that $\Lambda^\mu_{\rm R}(p,p) = 0$, i.e.

$$\gamma^\mu(Z_2^{\rm OS} - 1) = -\Lambda^\mu(p,p)$$

On the other hand, we have shown (in the lecture) that in the MS scheme

$$\gamma^\mu(\mathcal{Z}_\psi^{\rm MS} - 1) = -\Lambda^\mu_{\rm MS}(p,p)$$

Since $\mathcal{Z}_\psi^{\rm MS} = Z_1^{\rm OS}|_{\text{no } 1/\epsilon}$ and $\Lambda^\mu_{\rm MS}(p,p+q) = \Lambda^\mu_{\text{no } 1/\epsilon}(p,p+q)$ it follows that $Z_1^{\rm OS} = Z_2^{\rm OS}$. Check this by using the explicit formulae (derived in the Lecture) for $\Lambda^\mu_{\rm R}(p,p)$ and $Z_2^{\rm OS}$ (this is a nontrivial check of these formulae).