Quantum Field Theory of Fundamental Interactions. Problems, set 11.

Problem 1. Calculate the imaginary part of the loop function

$$b_0(s,m_1,m_2) = \int_0^1 dx\ \ln\frac{x(x-1)s + xm_1^2 + (1-x)m_2^2}{4\pi\mu^2}$$

Problem 2. Consider a scalar particle interacting with fermions

$$\mathcal{L}_{\rm int} = \bar{\psi}_2(c_L\mathbf{P}_L + c_R\mathbf{P}_R)\psi_1\phi + \text{H.c.}$$

Assuming that the scalar mass M is greater than m_1+m_2 calculate the imaginary part of the self energy $\Sigma_\phi(p^2)$ for $p^2 = M^2 > (m_1+m_2)^2$ directly and using the Cutcosky-Landau rules. Relate it through the unitarity condition to the width $\Gamma(\phi\to f_2\bar{f}_1)$.

Problem 3. Do the same for a massive vector boson V^μ interacting with fermions

$$\mathcal{L}_{\rm int} = \bar{\psi}_2\gamma^\mu(c_L\mathbf{P}_L + c_R\mathbf{P}_R)\psi_1 V_\mu + \text{H.c.}$$

Of course in this case one has to contract the vector boson self energy $\Pi_{\mu\nu}(q^2)$ with the appropriate polarization vectors. The optical theorem should hold for each polarization separately.

Problem 4. The corrected Coulomb potential is given by

$$V_{\rm Coul}(r) = ZQe^2\int\frac{d^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{1}{\mathbf{q}^2}\left[1+\tilde{\Pi}_\gamma(-\mathbf{q}^2)\right]$$

where $r = |\mathbf{x}|$. The first term gives the Coulomb term $Ze/4\pi r$. In the crudest approximation expand $\tilde{\Pi}_\gamma(q^2)$ around $\mathbf{q}^2 = 0$ and keeping only the first term in the expansion compute the correction. Does this correction split the $2S_{1/2}$ and $2P_{1/2}$ states (they are degenerate even in relativistic Dirac equation treatment of the Hydrogen atom)? By how much?

Problem 5. Derive the formula

$$\delta V_{\rm Coul} = \frac{ZQ\alpha_{\rm ph}}{r}\,\frac{2}{3}\frac{Q^2\alpha_{\rm ph}}{\pi}\int_{2m}^{+\infty}\frac{dz}{z}\ e^{-zr}\sqrt{1-\frac{4m^2}{z^2}}\left(1+\frac{2m^2}{z^2}\right)$$

by performing first the integral over dx in $\tilde{\Pi}_\gamma(q^2)$ by parts and substituting $v = 2x-1$. The integral over $d^3\mathbf{q}$ can be then done (just in the same way which allows to get the Coulomb term). Find the modification of the Coulomb potential for $mr \ll 1$.

Problem 6. Consider the elastic scattering $e^- p \to e^- p$ in the frame in which proton is initially at rest. Using the most general form of the proton-photon vertex (with formfactors F_1 and F_2) derive the Rosenbluth formula

$$\frac{d\sigma}{d\Omega} = \alpha_{\rm EM}^2 \frac{(F_1^2 - \frac{q^2}{4m^2}F_2^2)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}}{4E^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)\sin^4\frac{\theta}{2}}$$

where θ is the electron deflection angle, m is the proton mass (the electron mass can be neglected) and E is the energy of the initial electron.

Problem 7. Derive the 1-loop β function

$$\beta_g(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}C_2(R_G) - \frac{2}{3}S_2(R_F) - \frac{1}{6}S_2(R_S)\right]$$

in a nonabelian gauge theory with gauge bosons coupled to Weyl fermions in the representation R_F and scalars in the (real) representation R_S (For scalas in a complex representation the factor 1/6 should be replaced by 1/3)

Problem 8. Compute the tree level cross section for the process $e^+e^- \to q\bar{q}g$ where g is a gluon using QED+QCD. (In QCD massless gluons couple to quarks through the vertex $-ig_s\gamma^\mu T^a$ where T^a are the $SU(3)$ generators in the fundamental representation). Show first that for massless quarks the three body final phase space integrated over the variables of the final gluon (irrespectively of its mass) is

$$\int \prod_{i=1}^{3} \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^{(4)}(q - \sum_i p_i) = \frac{q^2}{128\pi^3}\int dx_1 dx_2$$

p_1, p_2 and p_3 are the momenta of q, $\bar{q}$ and g, respectively and the variables x_i are defined as $x_i = 2p_i \cdot q/q^2$. Find the integration region if the gluon has a mass λ (necessary for infrared regulation).

Furthermore, since $\sum_{\rm spins}|\mathcal{M}|^2 = L_{\mu\nu}(k_1,k_2)H^{\mu\nu}(p_1,p_2,p_3)$. where by gauge invariance $q^\mu H_{\mu\nu} = q^\nu H_{\mu\nu} = 0$ (and also $q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0$) one must have

$$\int \prod_{i=1}^{3} \frac{d^3\mathbf{k}_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^{(4)}(q - \sum_i k_i)H^{\mu\nu} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{q^2}{128\pi^3}\int dx_1 dx_2\ H(x_1,x_2)$$

Use this to simplify the calculation and derive the formula

$$\frac{d\sigma}{dx_1 dx_2}(e^+e^- \to q\bar{q}g) = \frac{8\alpha_{\rm EM}^2}{3s}\, Q_q^2\, \frac{g_s^2}{4\pi}\, \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

in the limit $\lambda \to 0$ (no gluon mass). Finally, include the appropriate factor arising from summation over colours of the final state particles.

For finite $\lambda^2 \ll q^2$ integrate the differential cross section over dx_1dx_2 (over the appropriate region). Keep only terms nonvanishing as $\lambda \to 0$

Problem 9. Consider QCD corrections to the electromagnetic process $e^+e^- \to q\bar{q}$ where quarks are treated as massless. Compute cross section regulating the infrared divergences by the gluon mass λ. Sum over colours of the final state quarks. The integral over the Feynman parameters (introduced to compute the vertex correction) has singularities for $q^2 > 0$. Evaluate it for $q^2 < 0$ and then continue analytically to $q^2 > 0$ remembering that $q^2 \to q^2 + i0$.

Combine this cross section with the one calculated in the Problem 6. The infrared divergences should cancel out, and the result should read

$$\sigma(e^+e^- \to q\bar{q}) + \sigma(e^+e^- \to q\bar{q}g) = \frac{4\pi\alpha}{3s}\, 3Q_q^2 \left(1 + \frac{3g^2}{16\pi^2} + \dots\right) .$$

Problem 10. Consider the following simple model defined by the Lagrangian

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\sum_{l=1}^{K}\left(\partial_\mu\varphi_l\partial^\mu\varphi_l - M_l^2\varphi_l^2\right) + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \\ -\frac{1}{2}\sum_{l=1}^{K}(g_l\varphi_l\phi\phi + g_l^*\varphi_l\phi^*\phi^*) + \text{other terms,}\end{aligned}$$

where the K real scalar fields φ_l describe K spinless particles N_l of masses M_l and the complex field ϕ describes a spinless particle b and its antiparticle $\bar{b}$ of mass m. Show that if $K \geq 2$ and the coupling constants g_l are complex numbers, C and CP are violated. Assuming that $M_l > 2m$, so that the decays $N_l \to bb$ and $N_l \to \bar{b}\bar{b}$ are kinematically allowed, compute in the lowest nontrivial order the CP asymmetry of these decays, i.e. the quantity

$$\mathcal{A}_{\rm CP}^{(l)} = \frac{\Gamma(N_l \to bb) - \Gamma(N_l \to \bar{b}\bar{b})}{\Gamma(N_l \to bb) + \Gamma(N_l \to \bar{b}\bar{b})} .$$

Suppose the particles b and $\bar{b}$ play the role of baryons and antibaryons and N_l's are some heavy particles which were present at the early stage of the Universe thermal evolution. Discuss (qualitatively) if the $N_l \to bb$ and $N_l \to \bar{b}\bar{b}$ processes can be a source of the cosmological baryon asymmetry?

Hint 1: Since the decay amplitudes $\mathcal{M}(N_l \to bb)$ and $\mathcal{M}(N_l \to \bar{b}\bar{b})$ are nonzero already in the lowest order, the mass of N_l defined as the pole of the one-loop propagator and the corresponding $\mathcal{Z}_{N_l}$ factor are complex. In computing the N_l decay amplitude at one loop their imaginary parts have to be discarded, however, because in such a calculation N_l is treated as a stable particle.

Hint 2: To evaluate the imaginary part of the vertex correction Feynman diagram use the Cutcosky rule.