Quantum Field Theory of Fundamental Interactions. Problems, set 12.

Problem 1. Show that parity invariance of strong and electromagnetic interactions forbids the term $\propto \epsilon_{\mu\nu\lambda\rho}P^{\lambda}q^{\rho}$ in the tensor $W_{\mu\nu}$

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\,\frac{1}{2}\sum_{\sigma}\int d^4x\ e^{iqx}\langle\mathbf{P},\sigma|J_{\mu}^{\mathrm{EM\ had}}(x)J_{\nu}^{\mathrm{EM\ had}}(0)|\mathbf{P},\sigma\rangle$$
$$= \frac{1}{4\pi}\,\frac{1}{2}\sum_{\sigma}\int d^4x\ e^{iqx}\langle\mathbf{P},\sigma|\left[J_{\mu}^{\mathrm{EM\ had}}(x),\ J_{\nu}^{\mathrm{EM\ had}}(0)\right]|\mathbf{P},\sigma\rangle$$

describing electron-nucleon scattering.

Problem 2. Show that with

$$W_{\mu\nu} = -W_1(q^2,\nu)\left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right) + \frac{W_2(q^2,\nu)}{M^2}\left(P_{\mu} - \frac{P\cdot q}{q^2}q_{\mu}\right)\left(P_{\nu} - \frac{P\cdot q}{q^2}q_{\nu}\right)$$

(whre $\nu = P\cdot q$) the differential scattering cross section takes the form

$$\frac{d\sigma}{dE'd\Omega'} = \frac{\alpha^2}{q^4}\frac{4E'^2}{M}\left[2W_1(q^2,\nu)\sin^2\frac{\theta}{2} + W_2(q^2,\nu)\cos^2\frac{\theta}{2}\right]$$

where E' and $d\Omega'$ refer to the final charged lepton.

Problem 3. Show by direct calculation that with

$$\langle\mathbf{P}',\sigma'|J_{\mu}^{\mathrm{EM\ had}}(0)|\mathbf{P},\sigma\rangle = \bar{u}(\mathbf{P}'\sigma')\left[\gamma_{\mu}F_1(q^2) + \frac{i}{2M}\sigma_{\mu\nu}q^{\nu}F_2(q^2)\right]u(\mathbf{P}\sigma)$$

the structure functions $W_1(q^2,\nu)$ and $W_2(q^2,\nu)$ take the form

$$W_1(q^2,\nu) = \delta(q^2+2\nu)\frac{-q^2}{2}G_M^2(q^2)$$
$$W_2(q^2,\nu) = \delta(q^2+2\nu)\frac{2M^2}{1-\frac{q^2}{4M^2}}\left[G_E^2 - \frac{q^2}{4M^2}G_M^2\right]$$

where

$$G_M(q^2) = F_1(q^2) + F_(q^2) \qquad G_E(q^2) = F_1(q^2) + \frac{q^2}{4M^2}F_(q^2)$$

Problem 4. Introducing the variables

$$x_{\rm Bj} = \frac{-q^2}{2P\cdot q} \equiv \frac{-q^2}{2\nu}\ , \qquad\qquad y = \frac{2P\cdot q}{2P\cdot k}$$

express the differential scattering cross section obtained in Problem 2 as $d^2\sigma/dydx_{\rm Bj}$ using $F_1(q^2,\nu) \equiv W_1(q^2,\nu)$ and $F_2(q^2,\nu) \equiv (P\cdot q/M^2)W_2(q^2,\nu)$.

Problem 5. For the weak neutrino-nucleon interaction Lagrangian

$$\mathcal{L}_{\rm int} = -\frac{G_F}{\sqrt{2}}\left(J^{\rm lept\dagger}_\mu J^\mu_{\rm had} + {\rm H.c.}\right)$$

where $J^{\rm lept}_\mu = \bar\psi_\nu\gamma_\mu(1-\gamma^5)\psi_e$ the most general form of the tensors

$$\begin{aligned}
W^{(\nu N)}_{\mu\nu} &= \frac{1}{4\pi}\,\frac{1}{2}\sum_\sigma\int d^4x\ e^{iq\cdot x}\langle \mathbf{P},\sigma|\left[J^{\rm had\dagger}_\mu(x),\ J^{\rm had}_\nu(0)\right]|\mathbf{P},\sigma\rangle\\
W^{(\bar\nu N)}_{\mu\nu} &= \frac{1}{4\pi}\,\frac{1}{2}\sum_\sigma\int d^4x\ e^{iq\cdot x}\langle \mathbf{P},\sigma|\left[J^{\rm had}_\mu(x),\ J^{\rm had\dagger}_\nu(0)\right]|\mathbf{P},\sigma\rangle
\end{aligned}$$

is

$$\begin{aligned}
W_{\mu\nu} &= -W_1 g_{\mu\nu} + W_2\frac{P_\mu P_\nu}{M^2} + iW_3\epsilon_{\mu\nu\lambda\rho}\frac{P^\lambda q^\rho}{M^2}\\
&+\ W_4\frac{q_\mu q_\nu}{M^2} + W_5\frac{P_\mu q_\nu + P_\nu q_\mu}{M^2} + iW_6\frac{P_\mu q_\nu - P_\nu q_\mu}{M^2}
\end{aligned}$$

Using this form with $W^{(\nu N)}_i$ and $W^{(\bar\nu N)}_i$ compute the differential cross sections

$$\frac{d^2\sigma^{(\nu N)}}{d\Omega' dE'}\ , \qquad\qquad \frac{d^2\sigma^{(\bar\nu N)}}{d\Omega' dE'}\ ,$$

(only W_1, W_2 and W_3 should contribute; the two cross sections should differ by a sign of the W_3 term).

Problem 7. In the Laboratory system in which $P^\mu = (M,0,0,0)$ and $q^\mu = (q^0,0,0,q^3)$ introduce the polarization vectors

$$\epsilon^\mu_{(S)} = \frac{1}{\sqrt{-q^2}}(q^3,0,0,q^0)\ , \qquad \epsilon^\mu_{(R,L)} = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$$

and define $W_X \equiv \epsilon^*_{(X)} \cdot W \cdot \epsilon_{(X)}$ $(X = L, R, S)$. Show that in the Bjorken limit $-q^2 \to \infty$, $\nu = P \cdot q \to \infty$ with $x_{\rm Bj} = -q^2/2\nu$ fixed for the electron-nucleon scattering

$$W_{L,R} \to F_{L,R} \equiv F_1 \qquad \text{and} \qquad W_S \to \frac{1}{2} F_S \equiv -F_1 + \frac{1}{2x_{\rm Bj}} F_2$$

whereas for for the (anti)neutrino-nucleon scattering

$$W_{L,R} \to F_{L,R} \equiv F_1 \mp \frac{1}{2} F_3 \qquad \text{and} \qquad W_S \to F_S \equiv -2F_1 + \frac{1}{x_{\rm Bj}} F_2$$

(recall: in the Bjorken limit $W_1(q^2, \nu) \to F_1(x)$ and $(\nu/M^2) W_2(q^2, \nu) \to F_2(x)$, $(\nu/M^2) W_3(q^2, \nu) \to F_3(x)$ in all cases). Show that in the the Bjorken limit the differential cross sections for neutrino and antineutrino-nucleon scatterings can be written as

$$\begin{aligned}
\frac{d^2\sigma^{(\nu N)}}{dy dx_{\rm Bj}} &= \frac{G_F^2 M E}{\pi} x_{\rm Bj} \left[(1-y) F_S^{(\nu N)} + F_L^{(\nu N)} + (1-y)^2 F_R^{(\nu N)} \right] \\
\frac{d^2\sigma^{(\bar\nu N)}}{dx dy} &= \frac{G_F^2 M E}{\pi} x_{\rm Bj} \left[(1-y) F_S^{(\bar\nu N)} + (1-y)^2 F_L^{(\bar\nu N)} + F_R^{(\bar\nu N)} \right]
\end{aligned}$$

Problem 8. In the parton model the electron nucleon scattering cross section is given by

$$d\sigma^{\rm had}(P, k, k') = \int_0^1 d\xi \sum_i f_i(\xi)\ d\sigma^{\rm parton}(\xi P, k, k')$$

which leads to

$$\begin{aligned}
F_1^{\rm had}(x, q^2) &= \int_0^1 \frac{d\xi}{\xi} \sum_i Q_i^2 f_i(\xi)\ F_1^{\rm parton}\left(\frac{x}{\xi}, q^2\right) \\
\frac{1}{x} F_2^{\rm had}(x, q^2) &= \int_0^1 \frac{d\xi}{\xi} \sum_i Q_i^2 f_i(\xi)\ \frac{\xi}{x} F_2^{\rm parton}\left(\frac{x}{\xi}, q^2\right)
\end{aligned}$$

Assuming that charged partons are spin 1/2 particles we have found $2F_1^{\rm parton}(y) = \delta(1-y)$, $(1/y) F_2^{\rm parton}(y) = \delta(1-y)$. Find $F_1^{\rm had}(x, q^2)$ and $(1/x) F_2^{\rm had}(x, q^2)$ assuming that partons are spin zero particles.

Problem 9. Assuming that the at the parton level the weak charged current $J^\mu_{\rm had}$ is given by

$$J^\mu_{\rm had} = \bar{\psi}_u \gamma^\mu (1-\gamma^5)\psi_d$$

(neglecting the Cabibbo suppressed admixture of the strange quark) compute the functions $F_i^{(\nu N){\rm had}}(x,q^2)$ and $F_i^{(\bar{\nu} N){\rm had}}(x,q^2)$, for $i = 1,2,3$ and also the structure functions $F_{(X)}^{(\nu N){\rm had}}(x,q^2)$ and $F_{(X)}^{(\bar{\nu} N){\rm had}}(x,q^2)$, for $X = L, R, S$ from Problem 5.

Problem 10. Derive the so-called Gottfried sum rule

$$\int_0^1 \frac{dx}{x}\left[F_2^p(x) - F_2^n(x)\right] = \frac{1}{3}$$

using the standard assumptions (isospin invariance, equality of the sea quark distributions, etc.) about the quark distribution functions in nucleons.

Problem 11. Consider the so-called Drell-Yan process $pp \to X\ell^+\ell^-$ where X is any hadronic state. Using the parton model write down the differential cross section

$$\frac{d\sigma}{dq^2}$$

for producing the lepton pair with invariant mass $\sqrt{q^2}$.
Hint: Work in the CM system in which protons have four-momenta $(P,0,0,P)$ and $(P,0,0,-P)$ so that $s = 4P^2$ (P^2 here it is not $P^\mu P_\mu$!). Show that the colliding partons have $\hat{s} = 4\xi_1\xi_2 P^2$ where ξ_1 and ξ_2 are the momentum fractions of the initial two protons carried by the colliding partons ($\hat{s}$ means here the kinematic Mandelstam variable for partons) and that

$$\frac{d\sigma(q_i\bar{q}_i \to \ell^+\ell^-)}{dq^2} = \frac{4\pi\alpha^2}{3q^2} Q_i^2 \delta(q^2 - \xi_1\xi_2 s)$$

Then convolute appropriately the parton level cross section with the particle distribution functions $f_i(\xi)$

Problem 12. Prove (by direct calculation) that

$$i\Delta_F(x, m=0) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i0}e^{-ik\cdot x} = \frac{-1}{4\pi^2}\frac{1}{x^2-i0}$$

and that

$$\int d^4x \; e^{iq\cdot x}\delta(x^2)\varepsilon(x^0) = 4\pi^2 i\delta(q^2)\varepsilon(q^0)$$