Quantum Field Theory of Fundamental Interactions. Problems, set 13.

Problem 1. Consider the Lagrangian

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2 \ ,$$

with $m^2 < 0$, so that the $U(1)$ symmetry is broken and the theory describes a massless Goldstone boson G and a massive spinless particle A. Using the linear parametrization $\phi = \frac{1}{\sqrt{2}}(v + A + iG)$ compute the one-loop self energy $\Sigma_G(p^2)$ of the Goldstone boson and show that it vanishes for $p^2 = 0$, i.e. the Goldstone boson is masless even with corrections taken into account. To this end, write down all relevant one loop diagrams (do not forget tadpoles!) and show that their sum vanishes for $p^2 = 0$. Do this without any regularization to make it clear that also quadratic divergences (which would be automatically removed by Dimensional Regularization) as well the infrared divergences cancel out.

Problem 2. Study at one-loop order the renormalization of the model from Problem 1. Assume first that $m^2 > 0$ and find at one loop δZ_ϕ, δm^2 and $\delta\lambda$. Next, set $m^2 < 0$, shift the fields $\phi = \frac{1}{\sqrt{2}}(v + A + iG)$ in the renormalized Lagrangian and show that the counterterms generated in this way (with δZ_ϕ, δm^2 and $\delta\lambda$ computed in the symmetric phase) do renormalize Green's functions in the broken symmetry phase.

Problem 3. In the theory of Problem 1 show using the linear parametrization $\phi = \frac{1}{\sqrt{2}}(v + A + iG)$ that the amplitude of the elastic scattering $G(k_1)G(k_2) \to G(k_3)G(k_4)$ vanishes at the threshold, i.e. for $k_1 \to 0$ or $k_2 \to 0$. Show the same for the process $G(k_1)A(p_1) \to G(k_2)A(p_2)$ (the amplitude should vanish in the limit $k_1 \to 0$).

Problem 4. Use the nonlinear parametrization

$$\phi(x) = \frac{1}{\sqrt{2}}(v + A(x)) \exp\left(\frac{iG}{v}\right)$$

in the Lagrangian of Problem 1. and calculate the amplitudes of the elastic scattering processes $G(k_1)A(p_1) \to G(k_2)A(p_2)$ and $G(k_1)G(k_2) \to G(k_3)G(k_4)$. Show that on mass shell these amplitudes agree with the ones computed in Problem 3.

Problem 5. Include fermions in the Problem 1

$$\begin{aligned}\mathcal{L} = &\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2 \\ &+ i\bar{\lambda}_1 \bar{\sigma}^\mu \partial_\mu \lambda_1 + i\bar{\lambda}_2 \bar{\sigma}^\mu \partial_\mu \lambda_2 - g(\phi \lambda_1 \lambda_2 + \phi^* \bar{\lambda}_1 \bar{\lambda}_2)\end{aligned}$$

where the $U(1)$ charges of the two Weyl twocomponent spinor fields λ_1, λ_2 are $Q_1 = -1$, $Q_2 = 0$, respectively, and $Q_\phi = +1$. Assume that $m^2 < 0$ and show that fermions λ_1 and λ_2 combine into a Dirac spinor describing a massive spin 1/2 particle.[1] Use the linear parametrization $\phi = \frac{1}{\sqrt{2}}(v + A + iG)$ to compute the one loop corrections to the Goldstone boson self energy and show that $\Sigma_G(p^2 = 0) = 0$.

Problem 6. In Problem 5 use the nonlinear parametrization $\phi = \frac{1}{\sqrt{2}}(v+A)\exp(iG/v)$ to show that $\Sigma_G(p^2 = 0) = 0$.

Problem 7. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i - \frac{\lambda}{4}(\phi_i\phi_i)^2$$

where $i = 1, \dots, N$. Assume that $m^2 < 0$ so that the $SO(N)$ global symmetry is broken spontaneously down to $SO(N-1)$. Show that the tree-level amplitude of the elastic scattering $\pi_i(k_i)\pi_j(k_j) \to \pi_l(k_l)\pi_k(k_k)$ of Goldstone bosons vanishes at the threshold, i.e. for $\mathbf{k}_i \to 0$. Show by direct one loop calculation that the Goldstone bosons remain massless.

Problem 8. Add to the Lagrangian of the previous Problem the term $-a\phi_N$ explicitly breaking the $SO(N)$ symmetry. Compute now the (tree level) masses of π_i's both, directly and using the Dashen's theorem and compare. What is now the threshold value of the elastic $\pi\pi$ scattering amplitude?

Problem 9. Consider a theory with a global $SU(3)$ symmetry. What symmetry group can remain unbroken if $SU(3)$ is broken by the vacuum expectation value of an octet of scalars (i.e. fields transforming as an adjoint representations of $SU(3)$)?

Problem 10. Suppose left-chiral (Weyl) fermions in a theory transform as $\overline{\mathbf{5}}$ and $\mathbf{10}$ of $SU(5)$ (the representation $\mathbf{10}$ is an antisymmetric tensor ψ_{ij} transforming according to the rule $\psi'_{ij} = U_{ik}U_{jl}\psi_{kl}$). Assume that $SU(5)$ is spontaneously broken by the vacuum expectation value of a scalar field $\mathbf{\Phi} \equiv \sum_a \phi^a T^a$ in the adjoint

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[1] When the $U(1)$ symmetry is broken one can combine the two Weyl fields into a single Dirac spinor

$$\Psi = \begin{pmatrix} \lambda_1 \\ \bar{\lambda}_2 \end{pmatrix}, \qquad \overline{\Psi} = (\lambda_2, \bar{\lambda}_1)$$

using the chiral representation for the gamma matrices.

representation

$$\langle \mathbf{\Phi} \rangle = V \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} .$$

What subgroup of $SU(5)$ is unbroken? How do the fermions transform under the unbroken subgroup of $SU(5)$? Can they make up one generation of known matter fields (leptons e, ν_e and quarks u, d)?

Problem 11. Consider the Lagrangian of the so-called σ-model

$$\begin{aligned} \mathcal{L} &= \bar{\psi}_N i \not\partial \psi_N - g_{\pi N} \bar{\psi}_N \left(\sigma + i\tau^a \pi^a \gamma^5 \right) \psi_N \\ &+ \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2} m^2 (\sigma^2 + \pi^a\pi^a) - \frac{\lambda}{4}(\sigma^2 + \pi^a \pi^a)^2 \ , \end{aligned}$$

(notice the wrong sign of the scalar fields mass term!) where τ^a are the three Pauli matrices and

$$\psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} , \qquad \pi^a = (\pi^2, \pi^2\pi^3) \ .$$

Show that $\mathcal{L}$ is invariant under the following infinitesimal vector

$$\begin{aligned} \psi_N &\rightarrow \psi_N - i\theta_V^a \frac{\tau^a}{2} \psi_N \ , \\ \pi^a &\rightarrow \pi^a + \epsilon^{abc}\theta_V^b \pi^c \ , \\ \sigma &\rightarrow \sigma \ , \end{aligned}$$

and axial

$$\begin{aligned} \psi_N &\rightarrow \psi_N - i\theta_A^a \frac{\tau^a}{2} \gamma^5 \psi_N \ , \\ \pi^a &\rightarrow \pi^a + \theta_A^a \sigma \ , \\ \sigma &\rightarrow \sigma - \theta_A^a \pi^a \ , \end{aligned}$$

transformations. Are there any other symmetries of the Lagrangian? Find the Noether currents V_μ^a and A_μ^a corresponding to these symmetries. Show that the algebra of charges $Q_V^a = \int d^3x \, V_0^a$ and $Q_A^a = \int d^3x \, A_0^a$ is that of $SU(2) \times SU(2)$. What is the complete symmetry group of tthe classical Lagrangian?

Splitting the field ψ_N into the right- and left-chiral parts write the Lagrangian in the form which makes its $SU(2)\times SU(2)$ symmetry manifest.

Assuming that $\langle\sigma\rangle = v \neq 0$ as a result of the wrong sign of the mass squared term of the scalar fields shift the field $\sigma \to v+\sigma$ and study the mass spectrum of the model. Show that π^a remain massless Goldstone bosons after inclusion of one-loop corrections. How is the pion decay constant f_π related to v? Add to $\mathcal{L}$ a small perturbation $-a\sigma$ and study the mass spectrum again. What are now the masses of π^a? Find the divergence $\partial^\mu A^a_\mu$ of the axial currents using (naively) the operator equations of motion.

Problem 12. ("Be like Jack Steinberger") Extend the linear σ-model Lagrangian of the preceding Problem by coupling proton (ψ_p) and charged pion fields ($\pi^\pm = (\pi^1 \mp i\pi^2)/\sqrt{2}$) to the electromagnetic field and compute the amplitude

$$-i\mathcal{M} = i\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\,T^{\mu\nu}(k_1,k_2) = i\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\,\epsilon^{\mu\nu\kappa\rho}k_{1\kappa}k_{2\rho}\,T(q^2=M^2_\pi)\ ,$$

(where $q = k_1 + k_2$) of the $\pi^0 \to \gamma\gamma$ decay. Show that $T(0)\neq 0$ (naive current algebra arguments, which neglect the anomaly, lead to $T(0) = 0$ - see Problem 14). Read off from the amplitude the coefficient h in the effective Lagrangian

$$\mathcal{L}_{\rm eff} = -h\,\pi^0\epsilon_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\ .$$

Compute the decay width.

Problem 13. Show that

$$i\int\frac{d^4p}{(2\pi)^4}\left[\frac{p^\mu+a^\mu}{[(p+a)^2-m^2]^2}-\frac{p^\mu}{[p^2-m^2]^2}\right] = -\frac{1}{32\pi^2}\,a^\mu\ ,$$

$$\begin{aligned}i\int\frac{d^4p}{(2\pi)^4}\,{\rm tr}\Bigg[&\frac{1}{\not p-\not k_1+\not a-m}\gamma^\mu\frac{1}{\not p+\not a-m}\gamma^\nu\frac{1}{\not p+\not k_2+\not a-m}\gamma^\lambda\gamma^5\\ &-\frac{1}{\not p-\not k_1-m}\gamma^\mu\frac{1}{\not p-m}\gamma^\nu\frac{1}{\not p+\not k_2-m}\gamma^\lambda\gamma^5\Bigg] = \frac{i}{8\pi^2}\epsilon^{\lambda\mu\nu\kappa}\,a_\kappa\ ,\end{aligned}$$

$$\begin{aligned}i\int\frac{d^4p}{(2\pi)^4}{\rm tr}\Bigg[&\frac{1}{\not p-\not k_1-m}\gamma^\mu\frac{1}{\not p+\not k_2-m}\gamma^\lambda\gamma^5\\ &-\frac{1}{\not p-\not k_2-m}\gamma^\mu\frac{1}{\not p+\not k_1-m}\gamma^\lambda\gamma^5\Bigg] = -\frac{i}{4\pi^2}\epsilon^{\lambda\mu\kappa\rho}\,k_{1\kappa}k_{2\rho}\ ,\end{aligned}$$

Problem 14. Using the PCAC assumption and the photon field equations of motion in the LSZ formalism express the $\pi^0 \to \gamma\gamma$ amplitude through the Fourier transform $T^{\lambda\mu\nu}(k_1, k_2)$ of the vacuum expectation values of the (covariant) chronological product (see Problem VII.6)

$$T^{\lambda\mu\nu}(k_1, k_2) = ie^2 \int d^4x_1 d^4x_2 \, e^{-k_1\cdot x_1} \, e^{-k_2\cdot x_2} \, \langle \Omega_- | T^* \left\{ A^3_\lambda(0) J^\mu_{\rm EM}(x_1) J^\nu_{\rm EM}(x_2) \right\} | \Omega_+ \rangle \; ,$$

of two electromagnetic current operators and the axial current. Make the assumption that the equal-time commutators (ETCs) of the currents are as derived naively (i.e. ignoring the composite nature of these operators) from the canonical ETCs of a free-field realization of the $SU(2)_L \times SU(2)_R$ current algebra. Write down the most general (for $k_i^2 = 0$ but $q^2 = (k_1 + k_2)^2$ arbitrary) form of $T^{\lambda\mu\nu}(k_2, k_1)$ consistent with the parity invariance and electromagnetic currents conservation and show that (under the usual assumptions of the PCAC hypothesis) the formfactor $T(q^2)$ defined in Problem 12 should vanish for $q^2 = 0$.

Next use the linear σ-model of Problems 11 and 12 (with the $SU(2) \times SU(2)$ symmetry broken spontaneously and explicitly) to compute the vacuum matrix element of the chronological product of currents in question explicitly in the lowest nontrivial order in the fine structure constant. Notice that there are two different types of contributions to $T^{\lambda\mu\nu}(k_1, k_2)$: one coming from the term in A^3_λ linear in the pion field and the second one from the fermionic part of the current A^3_λ. Show that the latter depends crucially on the precise way of assigning the momenta in the relevant triangle diagrams. Check that if the assignement is done in such a way that the electromagnetic currents are conserved, $T(0)$ does indeed vanish in conflict with the result of Problem 12. Find the origin of this discrepancy.

Problem 15. Suppose that the global $SU(2)_R \times SU(2)_L$ symmetry of QCD is broken to its diagonal $SU(2)_V$ subgroup both by the vacuum and, in addition, explicitly by the $SU(2)_V$ symmetric quark mass term (i.e. $m_u = m_d$). The lowest order (in derivatives and pion mass, considering $M \propto (m_u + m_d)$ to be of the same order as ∂_μ), $\mathcal{O}(p^2)$ chiral effective Lagrangian describing pion interactions reads (see Problem 11 set 6)

$$\mathcal{L}^{(2)} = \frac{f^2}{4} \, {\rm tr} \left(\partial_\mu U \partial_\mu U^{-1} \right) + \frac{f^2 M^2}{4} \, {\rm tr} \left(U + U^{-1} \right) \; .$$

Write down the order $\mathcal{O}(p^4)$ Lagrangian $\mathcal{L}^{(4)}$ and fix infinite parts of two of its parameters by requiring that the physical pion mass M_π and the physical pion decay constant f_π (measured in $\pi^- \to \mu^- \bar{\nu}_\mu$ decay) determined in terms of the Lagrangian parameters M and f are finite with one loop corrections taken into account.

Problem 16. Use the order $\mathcal{O}(p^4)$ effective Lagrangian derived in the preceding Problem to find the $\pi^a\pi^b \to \pi^c\pi^d$ scattering amplitude including the one-loop corrections. Show that the corrections unitarize the amplitude so that the unitarity relation (for inelastic channels closed) for isospin partial amplitudes

$$\left|\mathcal{T}_J^I(s)\right|^2 = -\frac{2}{\sqrt{1-4M^2/s}}\,\mathrm{Im}\mathcal{T}_J^I(s)$$

is satisfied up to higher order terms, i.e. with the tree level amplitude in the left hand side.

$-i\Sigma_G =$

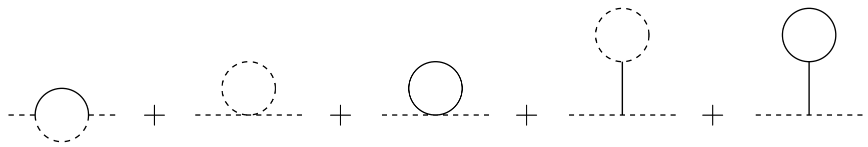


Figure 1: Diagrams contributing to the self energy of the Goldstone boson G. Dashed lines denote G propagators and continuous ones - propagators of A. The first three diagrams give $-i\Sigma_G^{1-\mathrm{PI}}$.

Solution to Problem 1.
After shifting the field ϕ by the real[2] tree level VEV $v^2 = -4m^2/\lambda$ (the position of the minimum of the potential $V(\phi)$) the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu G\partial^\mu G + \frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}M_A^2 A^2 - \frac{1}{4}\lambda v A(A^2+G^2) - \frac{\lambda}{16}(A^4 + 2G^2A^2 + G^4)$$

where $M_A^2 = \frac{1}{2}\lambda v^2$. The contribution of the diagrams shown in figure 1 is

$$\begin{aligned}-i\Sigma_G(p^2) = -i\frac{1}{4}\lambda^2 v^2 \int \frac{i}{(k+p)^2(k^2-M_A^2)} - i\frac{3}{4}\lambda \int \frac{i}{k^2} - i\frac{1}{4}\lambda \int \frac{i}{k^2-M_A^2} \\ -i\frac{1}{2}\lambda v \frac{i}{-M_A^2}\left(-i\frac{\lambda}{4}v\right)\left\{\int \frac{i}{k^2} + 3\int \frac{i}{k^2-M_A^2}\right\}\end{aligned}$$

where $\int$ stands for $\int d^dk/(2\pi)^d$. The first integral gives a function of p^2 (because it is Lorentz invariant). Hence, to compute $\Sigma_G(p^2=0)$ we can simply set $p^\mu = 0$. The first integral can be then decomposed

$$\int \frac{i}{k^2(k^2-M_A^2)} = \frac{1}{M_A^2}\int\left(\frac{i}{k^2-M_A^2} - \frac{i}{k^2}\right)$$

Hence, using $M_A^2 = \frac{1}{2}\lambda v^2$ we get

$$\begin{aligned}-i\Sigma_G(0) = -i\left\{\frac{1}{2}\lambda\int\frac{i}{k^2-M_A^2} - \frac{1}{2}\lambda\int\frac{i}{k^2}\right. \\ +\frac{3}{4}\lambda\int\frac{i}{k^2} + \frac{1}{4}\lambda\int\frac{i}{k^2-M_A^2} \\ \left.-\frac{1}{4}\lambda\int\frac{i}{k^2} - \frac{3}{4}\lambda\int\frac{i}{k^2-M_A^2}\right\} = 0\end{aligned}$$

Hence, the Goldstone boson remains massless also at the one-loop level.

[2]Because of the $U(1)$ symmetry all choices of the phase of v are physically equivalent.

Solution to Problem 2.

We first compute the one-loop counterterms δm^2 and $\delta\lambda$ (in this approximation $\delta Z_\phi = 0$) assuming that $m^2 > 0$ so that it is the true mass of the complex ϕ field quanta - the particle and its antiparticle. In the $\overline{\rm MS}$ scheme this gives

$$\delta\lambda = -\frac{1}{16\pi^2}\frac{5\lambda^2}{2\epsilon}\,, \qquad \delta m^2 = -\frac{1}{16\pi^2}\lambda m^2\frac{1}{\epsilon}$$

Since the $\overline{\rm MS}$ scheme automatically drops quadratic divergences (divergences of the form Λ^2 which can arise when integrals are cut-off for Euclidean momenta $|k| = \Lambda$) we will prefer to write

$$\begin{aligned}\delta\lambda &= -\frac{5}{2}\lambda^2\int\frac{i}{[k^2-m^2]^2} + \text{const}\\ \delta m^2 &= -\lambda\int\frac{i}{k^2-m^2} + \text{const}\end{aligned}$$

Thus $\delta\lambda$ is only logarithmically divergent whereas δm^2 has both, quadratic and logarithmic divergence. The finite constant depends on the renormalization prescription and is inessential for further considerations.

There are two approaches. In the first one the field ϕ is shifted by the tree level VEV $v = -4m^2/\lambda$ we get the Lagrangian

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\partial_\mu G\partial^\mu G + \frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}M_A^2A^2 - \frac{1}{4}\lambda vA(A^2+G^2) - \frac{\lambda}{16}(A^4+2G^2A^2+G^4)\\ -v\left(\delta m^2 + \frac{1}{4}v^2\delta\lambda\right)A - \frac{1}{2}\left(\delta m^2+\frac{3}{4}v^2\delta\lambda\right)A^2 - \frac{1}{2}\left(\delta m^2+\frac{1}{4}v^2\delta\lambda\right)G^2\\ -\frac{1}{16}\delta\lambda\left(4vA(A^2+G^2)+(A^2+G^2)^2\right)\end{aligned}$$

where $M_A^2 = \frac{1}{2}\lambda v^2 = -2m^2$.

We can now check that at one loop all divergences are removed by the counterterms. Since the theory is renormalizable, one has to consider only 1,2,3 and 4-point 1-PI functions.

Let us consider first the 1-PI one-point function

$$i\tilde{\Gamma}_{\rm R}^{(1)} = -i\mathcal{T}_A - iv\left(\delta m^2 + \frac{1}{4}v^2\delta\lambda\right)$$

where $-i\mathcal{T}_A$ is the sum of all 1-PI diagrams with one A external leg (without the propagator corresponding to this external line). This is related to the correction to the A vacuum expectation value

$$\langle A\rangle = \frac{i}{-M_A^2} i\tilde{\Gamma}_{\mathrm{R}}^{(1)}$$

For $i\tilde{\Gamma}_{\mathrm{R}}^{(1)}$ we have

$$\begin{aligned} & -\frac{i}{4}\lambda v\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)-iv\left(\delta m^2+\frac{1}{4}v^2\delta\lambda\right) \\ = & -i\lambda v\left(\frac{3}{4}\int\frac{i}{k^2-M_A^2}+\frac{1}{4}\int\frac{i}{k^2}-\int\frac{i}{k^2-m^2}-\frac{5\lambda}{8}v^2\int\frac{i}{[k^2-m^2]^2}-\mathrm{const}\right)\end{aligned}$$

This can be written as a single integral

$$-\frac{i}{4}\lambda v\int i\left(\frac{3}{k^2-M_A^2}+\frac{1}{k^2}-\frac{4}{k^2-m^2}-\frac{5M_A^2}{[k^2-m^2]^2}\right)$$

(we have dropped the finite constant). For large k^2 the individual terms can be expanded

$$-\frac{i}{4}\lambda v\int i\left\{\frac{3}{k^2}\left(1+\frac{M_A^2}{k^2}+\dots\right)+\frac{1}{k^2}-\frac{4}{k^2}\left(1+\frac{m^2}{k^2}+\dots\right)-\frac{5M_A^2}{k^4}-\dots\right\}$$

It is now clear that the quadratic divergences (terms $\propto 1/k^2$ in the integrand) cancel out between the three first terms. The logarithmic divergences (terms $\propto 1/k^4$) cancel too, being proportional to

$$3M_A^2-4m^2-5M_A^2=-2M_A^2-4m^2=0$$

Let us consider now the two point functions. For the 1-PI G-self energy we have (see figure 1)

$$\Sigma_G^{1-\mathrm{PI}}=\frac{1}{4}\lambda^2v^2\int\frac{i}{(k+p)^2(k^2-M_A^2)}+\frac{3}{4}\lambda\int\frac{i}{k^2}+\frac{1}{4}\lambda\int\frac{i}{k^2-M_A^2}+\delta m^2+\frac{1}{4}v^2\delta\lambda$$

or, as a single integral

$$=\frac{1}{4}\lambda\int i\left(\frac{2M_A^2}{(k+p)^2(k^2-M_A^2)}+\frac{3}{k^2}+\frac{1}{k^2-M_A^2}-\frac{4}{k^2-m^2}-\frac{5M_A^2}{[k^2-m^2]^2}\right)$$

Quadratic divergences are again seen to cancel out between the three middle terms and the terms which for $k \to \infty$ behave as $1/k^4$ (leading to logarithmic divergences) are proportional to

$$2M_A^2 + M_A^2 - 4m^2 - 5M_A^2 = 0$$

In a similar fashion we consider $\Sigma_A^{1-\mathrm{PI}}$ (see figure 5)

$$\Sigma_A^{1-\mathrm{PI}} = \frac{1}{2}\left(\frac{3}{2}\lambda v\right)^2 \int \frac{i}{[k^2 - M_A^2][(k+p)^2 - M_A^2]} + \frac{1}{2}\left(\frac{1}{2}\lambda v\right)^2 \int \frac{i}{k^2[k+p]^2}$$
$$+\frac{3}{4}\lambda \int \frac{i}{k^2 - M_A^2} + \frac{1}{4}\lambda \int \frac{i}{k^2} + \delta m^2 + \frac{3}{4} v^2 \delta\lambda$$

or, as a single integral

$$= \frac{1}{4}\lambda \int i \left(\frac{9M_A^2}{[k^2 - M_A^2][(k+p)^2 - M_A^2]} + \frac{M_A^2}{k^2[k+p]^2} \right.$$
$$\left. + \frac{3}{k^2 - M_A^2} + \frac{1}{k^2} - \frac{4}{k^2 - m^2} - \frac{15M_A^2}{[k^2 - m^2]^2} \right)$$

Again the terms $1/k^2$ cancel out and the terms which for $k \to \infty$ behave as $1/k^4$ are proportional to

$$9M_A^2 + M_A^2 + 3M_A^2 - 4m^2 - 15M_A^2 = -2M_A^2 - 4m^2 = 0$$

It is also wothwile to write down the total counterterm to Σ_G:

$$\delta m^2 + \frac{1}{4} v^2 \delta\lambda + \frac{1}{2}\lambda v \frac{i}{-M_A^2}\left(-iv\delta m^2 - i\frac{1}{4} v^3 \delta\lambda\right) = 0$$

after using $M_A^2 = \frac{1}{2}\lambda v^2$. The Goldstone bosons are massless irrespectively of the renormalization scheme used to fix δm^2 and $\delta\lambda$.

There are only two three point functions to consider. $\Gamma_{GGA}^{(3)}$ and $\Gamma_{AAA}^{(3)}$. The divergent diagrams contributing to $\Gamma_{GGA}^{(3)}$ are the left two diagrams in figure 7 and the last diagram of figure 8. The remaining contributions to $\Gamma_{GGA}^{(3)}$ are finite. The divergent diagrams together with the counterterms give

$$-i\frac{1}{2}\left(\frac{1}{2}\lambda\right)\left(\frac{3}{2}\lambda v\right) \int \frac{i}{[k^2 - M_A^2]^2} - i\frac{1}{2}\left(\frac{3}{2}\lambda\right)\left(\frac{1}{2}\lambda v\right) \int \frac{i}{k^4}$$
$$-i2 \times \left(\frac{1}{2}\lambda v\right)\left(\frac{1}{2}\lambda\right) \int \frac{i}{k^2[k^2 - M_A^2]} - i2v\delta\lambda$$

(the contribution of the diagram from figure 8 is multiplied by two because there is such a diagram for each of the two external G legs). This is

$$-i\frac{1}{8}\lambda^2 v\int i\left(\frac{3}{[k^2-M_A^2]^2}+\frac{3}{k^4}+\frac{4}{k^2[k^2-M_A^2]}-\frac{10}{[k^2-m^2]^2}\right)$$

and it is clear that the integral is finite. Finitness of $\Gamma^{(3)}_{AAA}$ can be checked similarly. Finally, the four point functions $\Gamma^{(4)}_{AAAA}$, $\Gamma^{(4)}_{AAGG}$ and $\Gamma^{(4)}_{GGGG}$ have divergences exactly as in the unbroken symmetry case and the counterterms are also unmodified. Hence, the counterterms from the symmetric phase renormalize the Green's functions also in the broken symmetry phase.

Another approach is to take

$$v^2=-4\frac{m^2+\delta m^2}{\lambda+\delta\lambda}=-4\frac{m^2}{\lambda}\left(1+\frac{1}{2}\frac{\delta m^2}{m^2}-\frac{1}{2}\frac{\delta\lambda}{\lambda}+\ldots\right)=v^2_{\rm tree}+\text{counterterms}$$

i.e. determined by the bare parameters of the Lagrangian. The Lagrangian then takes the form

$$\begin{aligned}\mathcal{L}&=\frac{1}{2}\partial_\mu G\partial^\mu G+\frac{1}{2}\partial_\mu A\partial^\mu A-\frac{1}{2}(-2m^2-2\delta m^2)A^2\\&-\frac{1}{4}(\lambda+\delta\lambda)vA(A^2+G^2)-\frac{1}{16}(\lambda+\delta\lambda)(A^4+2G^2A^2+G^4)\end{aligned}$$

There is no counterterm to the tadpole diagrams nor to the 1-PI G self-energy $\Sigma_G^{1-\mathrm{PI}}$. But of course, the full G self-energy (including tadpoles) is finite (as is clear from the calculation done in Problem 1). The tadpole diagrams, in turn, can be interpreted as the correction to the VEV of A. To one loop accuracy we have

$$\begin{aligned}\langle A\rangle&=\frac{i}{-M_A^2}\left(-i\mathcal{T}_A\right)=-i\frac{2}{\lambda v^2_{\rm tree}}\left(-\frac{i}{4}\lambda v_{\rm tree}\right)\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)\\&=-\frac{1}{2v_{\rm tree}}\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)\end{aligned}$$

This is infinite, but the total 1-loop VEV of the original ϕ field is finite:

$$\begin{aligned}\sqrt{2}\langle\phi\rangle=v+\langle A\rangle&=v_{\rm tree}\left(1+\frac{1}{2}\frac{\delta m^2}{m^2}-\frac{1}{2}\frac{\delta\lambda}{\lambda}+\ldots\right)+\langle A\rangle\\&=v_{\rm tree}-v_{\rm tree}\frac{\lambda}{2m^2}\int\frac{i}{k^2-m^2}+v_{\rm tree}\frac{5\lambda}{4}\int\frac{i}{[k^2-m^2]^2}\\&\quad-\frac{1}{2v_{\rm tree}}\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)\end{aligned}$$

The correction can be rewritten as

$$-\frac{1}{2v_{\text{tree}}}\left(-\frac{\lambda}{m^2}v_{\text{tree}}^2\int\frac{i}{k^2-m^2}+\frac{5}{2}\lambda v_{\text{tree}}^2\int\frac{i}{[k^2-m^2]^2}-3\int\frac{i}{k^2-M_A^2}-\int\frac{i}{k^2}\right)$$

With $v_{\text{tree}}^2=-4m^2/\lambda$ the content of the bracket is

$$4\int\frac{i}{k^2}\left(1+\frac{m^2}{k^2}+\dots\right)-10m^2\int\frac{i}{[k^2-m^2]^2}-3\int\frac{i}{k^2}\left(1+\frac{M_A^2}{k^2}+\dots\right)-\int\frac{i}{k^2}$$

It is now clear that both, quadratic and logarithmic divergences cancel out.

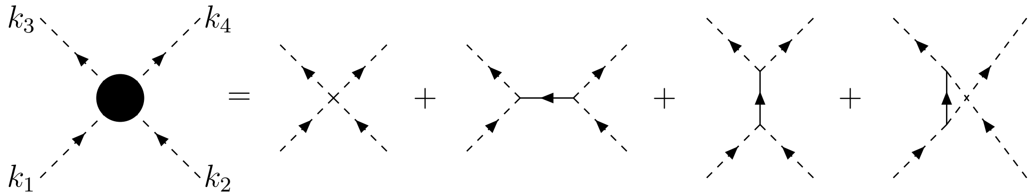


Figure 2: Diagrams for elastic scattering of two Goldstone bosons in the linear parametrization. Arrows show directions of the momenta.

Solution to Problem 3

Four diagrams shown in figure 2 contribute to the amplitude of the process $G(k_1)G(k_2) \to G(k_3)G(k_4)$. They give

$$\begin{aligned}
\mathcal{A} &= -i\frac{3}{2}\lambda - i\frac{\lambda^2}{4}v^2\left\{\frac{1}{(k_1-k_3)^2 - M_A^2} + \frac{1}{(k_1-k_4)^2 - M_A^2} + \frac{1}{(k_1+k_2)^2 - M_A^2}\right\} \\
&= -i\frac{\lambda}{2}\left\{3 + \frac{\lambda}{2}v^2\left(\frac{1}{t-M_A^2} + \frac{1}{u-M_A^2} + \frac{1}{s-M_A^2}\right)\right\}
\end{aligned}$$

where $s=(k_1+k_2)^2$, $t=(k_1-k_3)^2$, $u=(k_1-k_4)^2$ are the Mandelstamm variables. For on shell particles, i.e. for $k_i^2=0$, this is

$$\begin{aligned}
\mathcal{A} &= -i\frac{\lambda}{2}\left\{3 - \frac{\lambda v^2}{2M_A^2}\left(\frac{1}{1+2k_1\cdot k_3/M_A^2} + \frac{1}{1+2k_1\cdot k_4/M_A^2} + \frac{1}{1-2k_1\cdot k_2/M_A^2}\right)\right\} \\
&= -i\frac{\lambda}{2}\left\{3 - \frac{\lambda v^2}{2M_A^2}\left(\frac{1}{1-t/M_A^2} + \frac{1}{1-u/M_A^2} + \frac{1}{1-s/M_A^2}\right)\right\}
\end{aligned}$$

For future use it will be convenient to rewrite this amplitude in the form

$$\mathcal{A} = -i\frac{m_\varphi^2}{v^2}\left\{\frac{t}{t-M_A^2} + \frac{u}{u-M_A^2} + \frac{s}{s-M_A^2}\right\} \to -3i\frac{M_A^2}{v^2} + \mathcal{O}\left(\frac{M_A^2}{E^2}\right)$$

where we have used $M_A^2 = \frac{1}{2}\lambda v^2$. It is now clear that the amplitude vanishes with vanishing momenta k_1 and k_2 of the colliding particles. Assuming that $E^2 \ll M_A^2$ and expanding the denominators the amplitude can be written as

$$\begin{aligned}
\mathcal{A} &= -i\frac{\lambda}{2}\left\{3 - \sum_{n=0}^{\infty}\left(\frac{t}{M_A^2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{u}{M_A^2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{s}{M_A^2}\right)^n\right\} \\
&= -i\frac{\lambda}{2}\left\{\frac{2k_1\cdot(k_3+k_4-k_2)}{M_A^2} - \frac{4}{M_A^4}\left[(k_1\cdot k_3)^2 + (k_1\cdot k_4)^2 + (k_1\cdot k_2)^2\right] + \dots\right\}
\end{aligned}$$

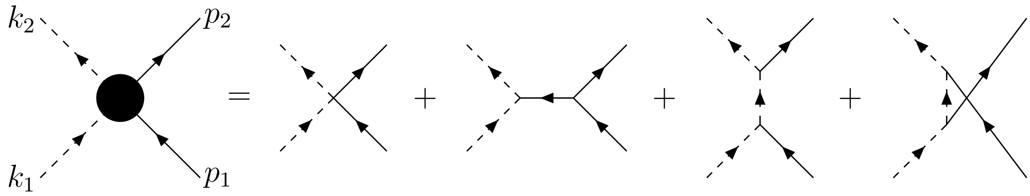


Figure 3: Diagrams for elastic Higgs boson Goldstone boson scattering in the linear parametrization.

The first term also vanishes because $k_3 + k_4 - k_2 = k_1$ and on-shell $k_1^2 = 0$ (in the form with the Mandelstamm variables this is also clear because for massless particles in the initial and final states $s + t + u = 0$). Thus

$$\mathcal{A} = +i\frac{8}{\lambda v^4}\left[(k_1 \cdot k_3)^2 + (k_1 \cdot k_4)^2 + (k_1 \cdot k_2)^2\right] + \ldots$$

The amplitude can be also written exactly in the form:

$$\mathcal{A} = +i\frac{2}{\lambda v^4}\left\{t^2 \sum_{n=0}^{\infty}\left(\frac{t}{M_A^2}\right)^n + u^2 \sum_{n=0}^{\infty}\left(\frac{u}{M_A^2}\right)^n + s^2 \sum_{n=0}^{\infty}\left(\frac{s}{M_A^2}\right)^n\right\}$$

It is again clear that the amplitude vanishes for $E \to 0$ i.e. in the limit of very soft Goldstone bosons.

For the second process we get from the diagrams shown in figure 3

$$\mathcal{A} = -i\frac{1}{2}\lambda - i\frac{3}{4}\lambda^2 v^2 \frac{1}{(k_1 - k_2)^2 - M_A^2} - i\frac{1}{4}\lambda^2 v^2\left(\frac{1}{(p_1 + k_1)^2} + \frac{1}{(k_1 - p_2)^2}\right)$$

For particles, on-shell, $k_i^2 = 0$, $p_i^2 = M_A^2 = \frac{1}{2}\lambda v^2$ this is

$$\begin{aligned}
\mathcal{A} &= -i\frac{\lambda}{2}\left\{1 - \frac{3}{2}\lambda v^2 \frac{1}{2k_1 \cdot k_2 + M_A^2} + \frac{\lambda}{2}v^2\left(\frac{1}{M_A^2 + 2k_1 \cdot p_1} + \frac{1}{M_A^2 - 2k_1 \cdot p_2}\right)\right\} \\
&= -i\frac{\lambda}{2}\left\{1 - \frac{3M_A^2}{2k_1 \cdot k_2 + M_A^2} + \frac{M_A^2}{M_A^2 + 2k_1 \cdot p_1} + \frac{M_A^2}{M_A^2 - 2k_1 \cdot p_2}\right\} \\
&= -i\frac{\lambda}{2}\left\{\frac{6k_1 \cdot k_2}{2k_1 \cdot k_2 + M_A^2} - 2\frac{k_1 \cdot p_1}{M_A^2 + 2k_1 \cdot p_1} + 2\frac{k_1 \cdot p_2}{M_A^2 + 2k_1 \cdot p_2}\right\}
\end{aligned}$$

Expanding the denominators we find

$$\mathcal{A} = -i\frac{\lambda}{2}\left\{1 - 3\sum_{n=0}^{\infty}\left(-\frac{2k_1 \cdot k_2}{M_A^2}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{2k_1 \cdot p_1}{M_A^2}\right)^n + \sum_{n=0}^{\infty}\left(\frac{2k_1 \cdot p_2}{M_A^2}\right)^n\right\}$$

The constant term in the amplitude cancels out and one is left with

$$\mathcal{A} = -i\frac{\lambda}{2}\left\{\frac{12}{\lambda v^2}(k_1 \cdot k_2) - \frac{4}{\lambda v^2}(k_1 \cdot p_1) + \frac{4}{\lambda v^2}(k_1 \cdot p_2) + \ldots\right\}$$

However, from the kinematics $k_1 \cdot (p_2 - p_1) = k_1 \cdot (k_1 - k_2) = -(k_1 \cdot k_2)$ so that finally

$$\mathcal{A} = -i\frac{4}{v^2}(k_1 \cdot k_2) + \ldots$$

Again the amplitude vanishes for vanishing Goldstone boson momenta $k_i \to 0$.

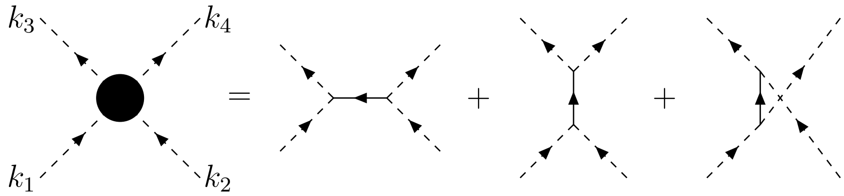


Figure 4: Diagrams for elastic scattering of two Goldstone bosons in the nonlinear parametrization.

Solution to Problem 4
With

$$\phi = \frac{1}{\sqrt{2}}(v + A) \exp\left(i\frac{\chi}{v}\right)$$

and the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial A)^2 + \frac{1}{2}(v^2 + 2vA + A^2)\partial_\mu e^{-i(\chi/v)}\partial^\mu e^{+i(\chi/v)} - \frac{1}{2}M_A^2 A^2 - \frac{\lambda}{4}vA^3 - \frac{\lambda}{16}A^4$$

In the case of the $U(1)$ group the differentiation of the exponential factors is trivial and the exact Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial A)^2 + \frac{1}{2}\left(1 + 2\frac{A}{v} + \frac{A^2}{v^2}\right)\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}M_A^2 A^2 - \frac{\lambda}{4}vA^3 - \frac{\lambda}{16}A^4$$

Diagrams contributing in this parametrization to the elastic scattering of Goldstone bosons are shown in figure 4. They give

$$\begin{aligned}
\mathcal{A} = i\left(2\frac{i}{v}\right)^2 &\left\{\frac{(k_1\cdot k_3)(k_2\cdot k_4)}{t - M_A^2} + \frac{(k_1\cdot k_4)(k_2\cdot k_3)}{u - M_A^2} + \frac{(k_1\cdot k_2)(k_3\cdot k_4)}{s - M_A^2}\right\} \\
&= -\frac{i}{v^2}\left\{\frac{t^2}{t - M_A^2} + \frac{u^2}{u - M_A^2} + \frac{s^2}{s - M_A^2}\right\} \\
= -\frac{i}{v^2}(s + t + u) - i\frac{M_A^2}{v^2}&\left\{\frac{t}{t - M_A^2} + \frac{u}{u - M_A^2} + \frac{s}{s - M_A^2}\right\}
\end{aligned}$$

Since for massless particles $s + t + u = 0$ we recover the amplitude obtained in Problem 3. Expanding we get

$$\mathcal{A} = \frac{i}{v^2 M_A^2}\left\{t^2\sum_{n=0}^{\infty}\left(\frac{t}{M_A^2}\right)^n + u^2\sum_{n=0}^{\infty}\left(\frac{u}{M_A^2}\right)^n + s^2\sum_{n=0}^{\infty}\left(\frac{s}{M_A^2}\right)^n\right\}$$

which is exactly the amplitude obtained in Problem 3.

Finally we compute the amplitude of elastic Higgs boson - Goldstone boson scattering. Two diagrams contributing to this amplitude in the nonlinear parametrization are the same as those shown in figure 3. However the vertices have now different Feynman rules. With these rules the diagrams of figure 3 give

$$\begin{aligned}\mathcal{A} &= 2i\frac{(k_1\cdot k_2)}{v^2} + 2\frac{i}{v}(k_1\cdot k_2)\frac{i}{(k_1-k_2)^2-M_A^2}\left(-i\frac{3}{2}\lambda v\right)\\ &+\left(2\frac{i}{v}\right)^2[k_1\cdot(k_1+p_1)][k_2\cdot(k_1+p_1)]\frac{i}{(k_1+p_1)^2}\\ &+\left(2\frac{i}{v}\right)^2[k_1\cdot(k_1-p_2)][k_2\cdot(k_1-p_2)]\frac{i}{(k_1-p_2)^2}\end{aligned}$$

Since $k_1+p_1=k_2+p_2$, $k_1-p_2=p_1-k_2$ and $k_i^2=0$ on shell we have $(k_1\cdot p_1)=(k_2\cdot p_2)$ and $(k_1\cdot p_2)=(k_2\cdot p_1)$ so that the amplitude is

$$\begin{aligned}\mathcal{A} &= 2i\frac{(k_1\cdot k_2)}{v^2} - 3i\lambda(k_1\cdot k_2)\frac{1}{M_A^2+2k_1\cdot k_2}\\ &-\frac{i}{v^2}\frac{(2k_1\cdot p_1)^2}{M_A^2+2k_1\cdot p_1}-\frac{i}{v^2}\frac{(2k_1\cdot p_2)^2}{M_A^2-2k_1\cdot p_2}\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\mathcal{A} &= i\lambda\frac{(k_1\cdot k_2)}{M_A^2} - 3i\lambda(k_1\cdot k_2)\frac{1}{M_A^2+2k_1\cdot k_2} - i\frac{(2k_1\cdot p_1)}{v^2}+i\frac{(2k_1\cdot p_2)}{v^2}\\ &+i\frac{M_A^2}{v^2}\frac{(2k_1\cdot p_1)}{M_A^2+2k_1\cdot p_1}-i\frac{M_A^2}{v^2}\frac{(2k_1\cdot p_2)}{M_A^2-2k_1\cdot p_2}\end{aligned}$$

Two last terms in the first line cancel the first term because $-i(2/v^2)k_1(\cdot p_1-p_2)=-i(\lambda/M_A^2)k_1\cdot k_2$ and we get

$$\mathcal{A} = -i\frac{\lambda}{2}\frac{6(k_1\cdot k_2)}{M_A^2+2k_1\cdot k_2}+i\frac{\lambda}{2}\frac{(2k_1\cdot p_1)}{M_A^2+2k_1\cdot p_1}-i\frac{\lambda}{2}\frac{(2k_1\cdot p_2)}{M_A^2-2k_1\cdot p_2}$$

and this is exactly the same amplitude as obtained in Problem 3.

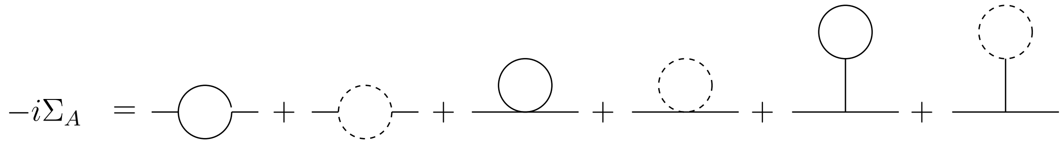


Figure 5: Diagrams contributing to the self energy of the massive scalar A. Dashed lines denote G propagators and continuous ones - propagators of A. The first three diagrams give $-i\Sigma_A^{1-\mathrm{PI}}$.

Solution to Problem 5.

The tree level amplitude vanishes at the threshold as shown in Problem 3. At one loop there are several contributions which we compute directly for zero momenta k_1 and k_2 of the incoming Goldstone bosons.

We first compute the A self energy. The relevant one loop diagrams are shown in figure 5. They give

$$-i\Sigma_A = -i\frac{1}{2}\left(\frac{3}{2}\lambda v\right)^2 \int \frac{i}{[k^2 - M_A^2]^2} - i\frac{1}{2}\left(\frac{1}{2}\lambda v\right)^2 \int \frac{i}{k^4} - i\frac{3}{4}\lambda \int \frac{i}{k^2 - M_A^2}$$
$$-i\frac{1}{4}\lambda \int \frac{i}{k^2} - i\left(\frac{3}{2}\lambda v\right)\frac{i}{-M_A^2}\left(-i\frac{1}{4}\lambda v\right)\left(3\int \frac{i}{k^2 - M_A^2} + \int \frac{i}{k^2}\right)$$

(successive terms correspond to successive diagrams in figure 5). Using $M_A^2 = \frac{1}{2}\lambda v^2$ the last two terms (from tadpol diagrams) can be combined with the middle two (from seagull diagrams) so that

$$\Sigma_A = \frac{9}{8}\lambda^2 v^2 \int \frac{i}{[k^2 - M_A^2]^2} + \frac{1}{8}\lambda^2 v^2 \int \frac{i}{k^4} - \frac{1}{2}\lambda \left(3\int \frac{i}{k^2 - M_A^2} + \int \frac{i}{k^2}\right)$$

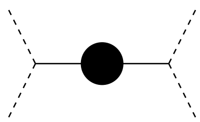


Figure 6: $t-$channel contribution of A self-energy to the elastic Goldstone boson scattering.

The correction to the amplitude corresponding to the diagram 6 is

$$\mathcal{A}_1 = \left(-i\frac{1}{2}\lambda v\right)^2 \frac{i}{-M_A^2}\left(-i\Sigma_A\right)\frac{i}{-m_A^2} = -i\frac{\lambda}{2M_A^2}\Sigma_A$$

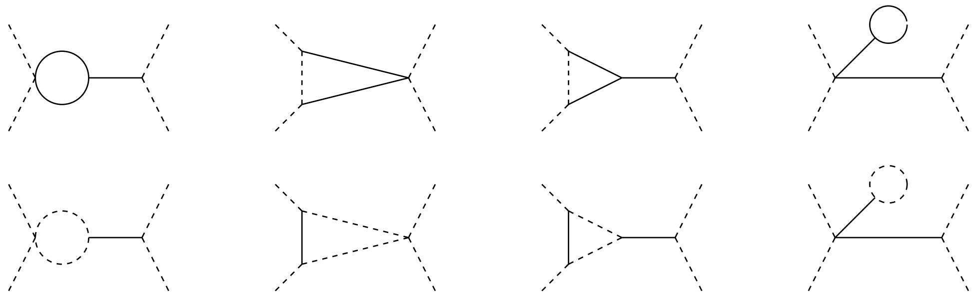


Figure 7: Corrections to the left vetex in the $t-$channel diagrams.

Hence

$$-i\mathcal{A}_1 = -i\frac{9}{16}\frac{\lambda^3 v^2}{M_A^2}\int\frac{i}{[k^2-M_A^2]^2} - i\frac{1}{16}\frac{\lambda^3 v^2}{M_A^2}\int\frac{i}{k^4} + i\frac{\lambda^2}{4M_A^2}\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)$$

Next we consider correction to the left vertex in the $t-$channel diagrams. The relevant diagrams are shown in figure 7. From left to right they give:

$$\frac{1}{2}\left(-i\frac{1}{2}\lambda\right)\left(-i\frac{3}{2}\lambda v\right)\left(-i\frac{1}{2}\lambda v\right)\frac{i}{-M_A^2}\int\frac{i^2}{[k^2-M_A^2]^2} + \frac{1}{2}\left(-i\frac{3}{2}\lambda\right)\left(-i\frac{1}{2}\lambda v\right)^2\frac{i}{-M_A^2}\int\frac{i^2}{k^4}$$

$$\left(-i\frac{1}{2}\lambda\right)\left(-i\frac{1}{2}\lambda v\right)^2\int\frac{i^3}{k^2[k^2-M_A^2]^2} + \left(-i\frac{3}{2}\lambda\right)\left(-i\frac{1}{2}\lambda v\right)^2\int\frac{i^3}{k^4[k^2-M_A^2]}$$

$$\left(-i\frac{1}{2}\lambda v\right)^3\left(-i\frac{3}{2}\lambda v\right)\int\frac{i^3}{k^2[k^2-M_A^2]^2} + \left(-i\frac{1}{2}\lambda v\right)^4\frac{i}{-M_A^2}\int\frac{i^3}{k^4[k^2-M_A^2]}$$

$$\left(-i\frac{1}{2}\lambda\right)\left(-i\frac{1}{2}\lambda v\right)\left(\frac{i}{-M_A^2}\right)^2\left(-i\frac{1}{4}\lambda v\right)\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)$$

where for pedagogical reasons we have shown all the vertex and combinatoric factors. Collecting terms we can write these four amplitudes as

$$-i\mathcal{A}_2 = +i\frac{3}{16}\frac{\lambda^3 v^2}{M_A^2}\left(\int\frac{i^2}{[k^2-M_A^2]^2}+\int\frac{i}{k^4}\right)$$
$$-i\mathcal{A}_3 = -i\frac{1}{8}\lambda^3 v^2\left(\int\frac{i}{k^2[k^2-M_A^2]^2}+3\int\frac{i}{k^4[k^2-M_A^2]}\right)$$

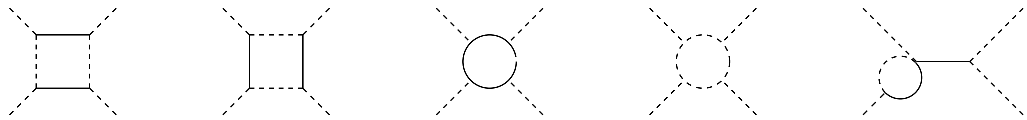


Figure 8: Remaining types of diagrams contributing at one loop to the elastic scattering of Goldstone bosons.

$$
\begin{aligned}
-i\mathcal{A}_4 &= +i\frac{1}{16}\frac{\lambda^4 v^4}{M_A^2}\left(3\int\frac{i}{k^2[k^2-M_A^2]^2}+\int\frac{i}{k^4[k^2-M_A^2]}\right)\\
-i\mathcal{A}_5 &= -i\frac{1}{16}\frac{\lambda^3 v^2}{M_A^4}\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)
\end{aligned}
$$

The remaining types of diagrams contributing at one loop are shown in figure 8. The first two together give

$$
-i\mathcal{A}_6 = 2\cdot\left(-i\frac{1}{2}\lambda v\right)^4\int\frac{i^4}{k^4[k^2-M_A^2]^2}
$$

The next two diagrams give

$$
-i\mathcal{A}_7 = \frac{1}{2}\left(-i\frac{1}{2}\lambda\right)^2\int\frac{i^2}{[k^2-M_A^2]^2}+\frac{1}{2}\left(-i\frac{3}{2}\lambda\right)^2\int\frac{i^2}{k^4}
$$

Finally the last diagram gives

$$
\left(-i\frac{1}{2}\lambda v\right)^2\left(-i\frac{1}{2}\lambda\right)\frac{i}{-M_A^2}\int\frac{i^2}{k^2[k^2-M_A^2]}
$$

and the corresponding amplitude (after using $M_A^2=\frac{1}{2}\lambda v^2$) is

$$
-i\mathcal{A}_8 = +i\frac{1}{4}\lambda^2\int\frac{i}{k^2[k^2-M_A^2]}
$$

Total one-loop contribution to the scattering amplitude for $k_1=k_2=0$ is

$$
\mathcal{A}_{\rm thr} = 3\mathcal{A}_1+6(\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+\mathcal{A}_5)+3(\mathcal{A}_6+\mathcal{A}_7)+12\mathcal{A}_8
$$

Since a given G legg can enter the vertex together with any of the other three G leggs each contribution (at the threshold) is multiplied by 3 (this corresponds to accounting for the t, s and $u-$channel digrams). Moreover, there are corrections, similar to those shown in figure 6, but to the right vertex. Therefore in total $\mathcal{A}_2$ to $\mathcal{A}_5$ have the coefficient of 6. Finally, the loop in the last diagram of figure 8 can be drawn on each of the four external leggs which gives an extra factor of 4 and, therefore, the contribution $\mathcal{A}_8$ has the factor of 12. Combining all contributions together we have

$$\begin{aligned}
\mathcal{A}_{\rm thr} &= 3\left\{\frac{1}{16}\frac{\lambda^3 v^2}{M_A^2}\int\frac{i}{k^4}+\frac{9}{16}\frac{\lambda^3 v^2}{M_A^2}\int\frac{i}{[k^2-M_A^2]^2}-\frac{\lambda^2}{4M_A^2}\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)\right\}\\
&+6\left\{-\frac{3}{16}\frac{\lambda^3 v^2}{M_A^2}\int\frac{i^2}{[k^2-M_A^2]^2}-\frac{3}{16}\frac{\lambda^3 v^2}{M_A^2}\int\frac{i}{k^4}\right.\\
&\quad+\frac{1}{8}\lambda^3 v^2\int\frac{i}{k^2[k^2-M_A^2]^2}+\frac{3}{8}\lambda^3 v^2\int\frac{i}{k^4[k^2-M_A^2]}\\
&\quad-\frac{3}{16}\frac{\lambda^4 v^4}{M_A^2}\int\frac{i}{k^2[k^2-M_A^2]^2}-\frac{1}{16}\frac{\lambda^4 v^4}{M_A^2}\int\frac{i}{k^4[k^2-M_A^2]}\\
&\quad\left.+\frac{1}{16}\frac{\lambda^3 v^2}{M_A^4}\left(3\int\frac{i}{k^2-M_A^2}+\int\frac{i}{k^2}\right)\right\}\\
&+3\left\{\frac{1}{8}\lambda^4 v^4\int\frac{i}{k^4[k^2-M_A^2]^2}\right\}\\
&+3\left\{\frac{1}{8}\lambda^2\int\frac{i}{[k^2-M_A^2]^2}+\frac{9}{8}\lambda^2\int\frac{i}{k^4}\right\}\\
&+12\left\{-\frac{1}{4}\lambda^2\int\frac{i}{k^2[k^2-M_A^2]}\right\}
\end{aligned}$$

We now have to show that $\mathcal{A}_{\rm thr}=0$. First of all, we notice that for $M_A^2=\frac{1}{2}\lambda v^2$ the two tadpole contributions (the last term in the first line and the fifth line) exactly cancel each other. Next, we observe that the potential infrared divergences cancel out. The dangerous integrals whose integrand behaves as $1/k^4$ for $k^2\to 0$ can be collected as follows

$$\begin{aligned}
&\frac{\lambda^4 v^4}{M_A^4}\left(\frac{3}{32}-\frac{18}{32}+\frac{27}{32}\right)\int\frac{i}{k^4}\\
&\qquad+\frac{\lambda^4 v^4}{M_A^2}\left(\frac{18}{16}-\frac{6}{16}\right)\int\frac{i}{k^4[k^2-M_A^2]}+\frac{3}{8}\lambda^4 v^4\int\frac{i}{k^4[k^2-M_A^2]^2}
\end{aligned}$$

where the the contributions to the first integral come from the first, second and next to last line, the contribution to the second integral - from the third and fourth lines and finally the last integral comes from the eighth line. Since the coefficients in brackets are 3/8 and 6/8 all these integrals can be combined to

$$\frac{3}{8}\lambda^4 v^4 \int \frac{i}{k^4}\left(\frac{1}{M_A^4} + \frac{2}{M_A^2[k^2 - M_A^2]} + \frac{1}{[k^2 - M_A^2]^2}\right) = \frac{3}{8}\frac{\lambda^4 v^4}{M_A^4}\int \frac{i}{[k^2 - M_A^2]^2}$$
$$= \frac{3}{2}\lambda^2 \int \frac{i}{[k^2 - M_A^2]^2}$$

which is not infrared divergent.

Thus, using $M_A^2 = \frac{1}{2}\lambda v^2$, one is left with

$$\left(\frac{27}{8} - \frac{18}{8} + \frac{3}{8} + \frac{3}{2}\right)\lambda^2 \int \frac{i}{[k^2 - M_A^2]^2}$$
$$+ \left(\frac{6}{8} - \frac{18}{8}\right)\lambda^3 v^2 \int \frac{i}{k^2[k^2 - M_A^2]^2} - 3\lambda^2 \int \frac{i}{k^2[k^2 - M_A^2]}$$

This is

$$3\lambda^2 \int \frac{i}{[k^2 - M_A^2]^2} - 3\lambda^2 \int \frac{iM_A^2}{k^2[k^2 - M_A^2]^2} - 3\lambda^2 \int \frac{i}{k^2[k^2 - M_A^2]} = 0$$

Thus, we have demonstrated that the one loop corrected amplitude of the elastic Goldstone boson scattering vanishes at the threshold.

Solution to Problem 5.
Shifting the field:

$$\phi = \frac{1}{\sqrt{2}}\left(v + A + iG\right)$$

we get the fermion mass term and scalar and pseudoscalar Yukawa couplings:

$$-g(\phi\lambda_1\lambda_2 + \phi^*\bar{\lambda}_1\bar{\lambda}_2) = -\frac{1}{\sqrt{2}}gv(\lambda_1\lambda_2 + \bar{\lambda}_1\bar{\lambda}_2)$$
$$-\frac{1}{\sqrt{2}}gA(\lambda_1\lambda_2 + \bar{\lambda}_1\bar{\lambda}_2) - \frac{i}{\sqrt{2}}gG(\lambda_1\lambda_2 - \bar{\lambda}_1\bar{\lambda}_2)$$
$$= -m_f\bar{\Psi}\Psi - \frac{1}{\sqrt{2}}gA\bar{\Psi}\Psi + \frac{i}{\sqrt{2}}gG\bar{\Psi}\gamma^5\Psi$$

where Ψ is the four-component Dirac spinor.

$-i\Delta\Sigma_G \quad =$

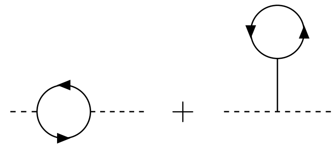


Figure 9: Additional diagrams contributing to the self energy of the Goldstone boson G in the linear parametrization. Continuous lines with arrows denote propagators of the Dirac fermion Ψ.

Two additional diagrams contributing to the Goldstone boson self-energy are shown in figure (9). The first of these diagrams give

$$-i\Delta^{(1)}\Sigma_G(p^2) = (-)^F \left(-\frac{g}{\sqrt{2}}\right)^2 i \int i \frac{\mathrm{tr}[\gamma^5(\not{k} + m_f)\gamma^5((\not{k}+ \not{p}) + m_f)]}{[k^2 - m_f^2][(k+p)^2 - m_f^2]}$$

and the second one

$$-i\Delta^{(2)}\Sigma_G(p^2) = \left(-i\frac{\lambda}{2}v\right) \frac{i}{-M_A^2} \left(-i\frac{g}{\sqrt{2}}\right) (-)^F \int i \frac{\mathrm{tr}(\not{k} + m_f)}{k^2 - m_f^2}$$

As $\Delta^{(1)}\Sigma_G$ depends on p^2 only, to get $\Delta^{(1)}\Sigma_G(0)$ one can set $p^\mu = 0$. The first integral gives then

$$-i\Delta^{(1)}\Sigma_G(0) = -i\frac{g^2}{2} \int 4i \frac{-k^2 + m_f^2}{[k^2 - m_f^2]^2} = +i2g^2 a(m_f)$$

whereas the second one, after using $M_A^2 = \frac{1}{2}\lambda v^2$ is

$$-i\Delta^{(2)}\Sigma_G(0) = \frac{1}{v}\left(-i\frac{g}{\sqrt{2}}\right) 4m_f a(m_f) = -i2g^2 a(m_f)$$

(in the last step $m_f = \frac{1}{\sqrt{2}}gv$ has been used. Thus, $\Delta^{(1)}\Sigma_G(0) = 0$.

Solution to Problem 6.
In the nonlinear parametrization $\phi = \frac{1}{\sqrt{2}}(v + A)e^{iG/v}$ we have

$$\begin{aligned}-g(\phi\lambda_1\lambda_2 + \phi^*\bar{\lambda}_1\bar{\lambda}_2) = &-\frac{1}{\sqrt{2}}gv\left(e^{iG/v}\lambda_1\lambda_2 + e^{-iG/v}\bar{\lambda}_1\bar{\lambda}_2\right)\\ &-\frac{1}{\sqrt{2}}gA\left(e^{iG/v}\lambda_1\lambda_2 + e^{-iG/v}\bar{\lambda}_1\bar{\lambda}_2\right)\end{aligned}$$

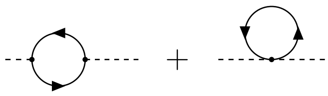
$-i\Delta\Sigma_G \;=$ 

Figure 10: Additional diagrams contributing to the self energy of the Goldstone boson G in the nonlinear parametrization. Continuous lines with arrows denote propagators of the Dirac fermion Ψ.

$$
\begin{aligned}
&= -m_f\bar{\Psi}\Psi + \frac{i}{\sqrt{2}}gG\bar{\Psi}\gamma^5\Psi + \frac{g}{2\sqrt{2}}\frac{GG}{v}\bar{\Psi}\Psi + \ldots \\
&\quad -\frac{1}{\sqrt{2}}gA\bar{\Psi}\Psi + \frac{i}{\sqrt{2}}g\frac{AG}{v}\bar{\Psi}\gamma^5\Psi + \ldots
\end{aligned}
$$

where Ψ is the four-component Dirac spinor. In this parametrization there is no tadpole graphs because there is no potential for G, i.e. there is no GGA vertex needed to create tadpols. Fermionic loop diagrams contributing in this realization to $-i\Sigma_G$ are those in figure 10. The first one is as previously whereas the second one gives

$$
-i\Delta^{(2)}\Sigma_G(0) = 2(-)^F\frac{i}{2\sqrt{2}}\frac{g}{v}\int i\frac{\mathrm{tr}(\not{k}+m_f)}{k^2-m_f^2} = -\frac{i}{\sqrt{2}}\,\frac{g}{v}\,4m_f a(m_f)
$$

where the factor of 2 in front comes from combinatorics. This is the same as the contribution of the fermionic tadpole diagram in the linear parametrization. Hence one again gets $\Sigma_G(0) = 0$. In fact, the second diagram of figure 10 can be viewed as the tadpole diagram with the massive A particle propagator contracted to a point.

Solution to Problem 11.

To check the invariance of $\mathcal{L}$ we consider first the vector transformations

$$
\begin{aligned}
\psi_N &\to \psi_N - i\theta_V^a T^a\psi_N \ , \\
\bar{\psi}_N &\to \bar{\psi}_N + i\bar{\psi}_N\theta_V^a T^a \ , \\
\pi^a &\to \pi^a + \epsilon^{abc}\theta_V^b\pi^c \ .
\end{aligned}
$$

The terms $\sigma^2 + \pi^a\pi^a$ and the scalar field kinetic terms are easily seen to be invariant similarly as the fermion kinetic term:

$$
\delta(\bar{\psi}_N i\not{\partial}\psi_N) = i\theta_V^a\bar{\psi}_N T^a i\not{\partial}\psi_N - i\theta_V^a\bar{\psi}_N i\not{\partial}T^a\psi_N = 0 \ ,
$$

(we have set $T^a = \frac{1}{2}\tau^a$). The term proportional to $g_{\pi N}$ is also invariant

$$
\delta(\bar{\psi}_N\tau^a\pi^a\gamma^5\psi_N) = \theta_V^b\epsilon^{abc}\bar{\psi}_N\tau^a\pi^c\gamma^5\psi_N + \frac{i}{2}\theta_V^b\bar{\psi}_N\tau^b\tau^a\pi^a\gamma^5\psi_N - \frac{i}{2}\theta_V^b\bar{\psi}_N\pi^a\tau^a\tau^b\gamma^5\psi_N \ ,
$$

Since $[\tau^b,\ \tau^a] = 2i\epsilon^{bac}\tau^c = 2i\epsilon^{cba}\tau^c$ this vanishes.

Under the axial transformations

$$\begin{array}{rcl} \psi_N & \to & \psi_N - i\theta_A^a T^a \gamma^5 \psi_N \ , \\ \bar{\psi}_N & \to & \bar{\psi}_N - i\theta_A^a \bar{\psi}_N T^a \gamma^5 \ , \\ \pi^a & \to & \pi^a + \theta_A^a \sigma \ , \\ \sigma & \to & \sigma - \theta_A^a \pi^a \ , \end{array}$$

the term $\sigma^2 + \pi^a\pi^a$ is easily seen not to change:

$$\delta(\sigma^2 + \pi^a\pi^a) = 2\theta_A^a \pi^a \sigma - 2\theta_A^a \sigma \pi^a = 0 \ .$$

Since the axial transformations are global, this proves also invariance of the scalar fields kinetic term. Also the fermion field kinetic term is invariant:

$$\delta(\bar{\psi}_N i \not\partial \psi_N) = -i\theta_A^a \bar{\psi}_N T^a \gamma^5 i \not\partial \psi_N - i\theta_A^a \bar{\psi}_N T^a i \not\partial \gamma^5 \psi_N = 0 \ ,$$

because $\gamma^5 \not\partial = - \not\partial \gamma^5$. Finally, the variation of the interaction term is

$$\begin{array}{rl} \delta[\bar{\psi}_N(\sigma + i\tau^a\pi^a\gamma^5)\psi_N] = & -\frac{i}{2}\theta_A^b \bar{\psi}_N \tau^b \gamma^5 (\sigma + i\tau^a\pi^a\gamma^5)\psi_N \\ & -\frac{i}{2}\theta_A^b \bar{\psi}_N (\sigma + i\tau^a\pi^a\gamma^5)\tau^b\gamma^5\psi_N + \theta_A^b \bar{\psi}_N(-\pi^b + i\tau^b\sigma\gamma^5)\psi_N \ . \end{array}$$

The terms with the σ field cancel out directly and those with the π^a fields vanish upon using $\tau^a\tau^b + \tau^b\tau^a = 2\delta^{ab}$.

The Noether curents associated with these symmetry transformations are given by the standard formula

$$\theta^a j_\mu^a = \sum_{\text{fields}} \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \delta\phi \ .$$

In the present case this gives

$$\begin{array}{rcl} V_\mu^a & = & \bar{\psi}_N \gamma_\mu T^a \psi_N + \epsilon^{abc} \pi^b \partial_\mu \pi^c \ , \\ A_\mu^a & = & \bar{\psi}_N \gamma_\mu \gamma^5 T^a \psi_N + \sigma \partial_\mu \pi^a - \pi^a \partial_\mu \sigma \ . \end{array}$$

To check that these currents are conserved we need the operator equations of motion of the fields ψ_N, π^a and σ. These read

$$\begin{array}{rcl} \gamma^\mu \partial_\mu \psi_N & = & -ig_{\pi N}\,(\sigma + i\pi^a\tau^a\gamma^5)\psi_N \ , \\ \partial_\mu \bar{\psi}_N \gamma^\mu & = & ig_{\pi N}\,\bar{\psi}_N\,(\sigma + i\pi^a\tau^a\gamma^5) \ , \\ \partial^2 \pi^a & = & m^2\pi^a - \lambda(\sigma^2 + \pi^b\pi^b)\pi^a - ig_{\pi N}\,\bar{\psi}_N \tau^a \gamma^5 \psi_N \ , \\ \partial^2 \sigma & = & m^2 \sigma - \lambda(\sigma^2 + \pi^b\pi^b)\sigma - g_{\pi N}\,\bar{\psi}_N \psi_N \ . \end{array}$$

With the help of these equations we have

$$
\begin{aligned}
\partial^\mu V^a_\mu &= \partial_\mu \bar{\psi}_N \gamma^\mu T^a \psi_N + \bar{\psi}_N T^a \gamma^\mu \partial_\mu \psi_N + \epsilon^{abc} \pi^b \partial^2 \pi^c \\
&= ig_{\pi N}\, \bar{\psi}_N (\sigma + i\pi^b \tau^b \gamma^5) T^a \psi_N - ig_{\pi N}\, \bar{\psi}_N T^a (\sigma + i\pi^b \tau^b \gamma^5) \psi_N \\
&\quad + \epsilon^{abc} \pi^b (-ig_{\pi N}\, \bar{\psi}_N \tau^c \gamma^5 \psi_N) \\
&= -\frac{1}{2} g_{\pi N}\, \pi^b\, \bar{\psi}_N \gamma^5 [\tau^b,\ \tau^a] \psi_N - ig_{\pi N}\, \epsilon^{abc} \pi^b \bar{\psi}_N \tau^c \gamma^5 \psi_N = 0\ ,
\end{aligned}
$$

upon using $[\tau^b,\ \tau^a] = -2i\epsilon^{abc}\tau^c$. Similarly,

$$
\begin{aligned}
\partial^\mu A^a_\mu &= \partial_\mu \bar{\psi}_N \gamma^\mu \gamma^5 T^a \psi_N - \bar{\psi}_N T^a \gamma^5 \gamma^\mu \partial_\mu \psi_N + \sigma \partial^2 \pi^a - \pi^a \partial^2 \sigma \\
&= ig_{\pi N}\, \bar{\psi}_N (\sigma + i\pi^b \tau^b \gamma^5) \gamma^5 T^a \psi_N + ig_{\pi N}\, \bar{\psi}_N \gamma^5 T^a (\sigma + i\pi^b \tau^b \gamma^5) \psi_N \\
&\quad - ig_{\pi N}\, \sigma\, \bar{\psi}_N \gamma^5 \tau^a \psi_N + g_{\pi N}\, \pi^a \psi_N \psi_N \\
&= -\frac{1}{2} g_{\pi N}\, \pi^b\, \bar{\psi}_N \left\{\tau^b,\ \tau^a\right\} \psi_N + g_{\pi N}\, \pi^a \psi_N \psi_N = 0\ .
\end{aligned}
$$

Thus both currents are conserved.

Invariance of $\mathcal{L}$ under the $SU(2) \times SU(2)$ transformations can be made manifest by splitting the field ψ_N into the right- and left-chiral parts:

$$
\begin{aligned}
\psi_N &= \mathbf{P}_R \psi_N + \mathbf{P}_L \psi_N = \mathbf{P}_R\, (\psi_N)_R + \mathbf{P}_L\, (\psi_N)_L\ , \\
\bar{\psi}_N &= \bar{\psi}_N \mathbf{P}_L + \bar{\psi}_N \mathbf{P}_R = \overline{(\psi_N)_R}\, \mathbf{P}_L + \overline{(\psi_N)_L}\, \mathbf{P}_R\ ,
\end{aligned}
$$

so that the kinetic term takes the form

$$
\bar{\psi}_N i\, \partial\!\!\!/ \psi_N = \overline{(\psi_N)_R}\, i\, \partial\!\!\!/ (\psi_N)_R + \overline{(\psi_N)_L}\, i\, \partial\!\!\!/ (\psi_N)_L\ ,
$$

and by writing

$$
\sigma + i\pi^a \tau^a \gamma^5 = \Phi\, \mathbf{P}_R + \Phi^\dagger\, \mathbf{P}_L\ ,
$$

where

$$
\Phi = \begin{pmatrix} \sigma + i\pi^3 & i\pi^1 + \pi^2 \\ i\pi^1 - \pi^2 & \sigma - i\pi^3 \end{pmatrix}\ .
$$

One then has

$$
\mathrm{tr}(\Phi^\dagger \Phi) = 2(\sigma^2 + \pi^a \pi^a)\ ,
$$

and the Lagrangian can be written in the form[3]

$$\mathcal{L} = \overline{(\psi_N)_R}\, i\partial\!\!\!/ (\psi_N)_R + \overline{(\psi_N)_L}\, i\partial\!\!\!/ (\psi_N)_L + \frac{1}{4}\mathrm{tr}(\partial_\mu \Phi^\dagger \partial^\mu \Phi)$$
$$-g_{\pi N}\, \overline{(\psi_N)_R}\, \Phi^\dagger\, (\psi_N)_L - g_{\pi N}\, \overline{(\psi_N)_L}\, \Phi\, (\psi_N)_R + \frac{1}{4} m^2 \mathrm{tr}(\Phi^\dagger \Phi) - \frac{\lambda}{16}[\mathrm{tr}(\Phi^\dagger\Phi)]^2 \ .$$

It is now clear that $\mathcal{L}$ is invariant under the transformations

$$\begin{aligned} (\psi_N)_R &\rightarrow e^{-i\theta^a_R T^a} (\psi_N)_R \ , \\ (\psi_N)_L &\rightarrow e^{-i\theta^a_L T^a} (\psi_N)_L \ , \\ \Phi &\rightarrow e^{-i\theta^a_L T^a}\, \Phi\, e^{+i\theta^a_R T^a} \ , \end{aligned}$$

forming the $SU(2)_L \times SU(2)_R$ group. It is straightforward to check that the vector transformations correspond to $\theta^a_R = \theta^a_L = \theta^a_V$ and the axial ones to $\theta^a_R = -\theta^a_L = \theta^a_A$.

It is easy to see that the σ-model Lagrangian is in addition invariant under the following transformations[4]

$$\begin{aligned} (\psi_N)_R &\rightarrow e^{-i\theta_B} (\psi_N)_R \ , \\ (\psi_N)_L &\rightarrow e^{-i\theta_B} (\psi_N)_L \ , \\ \Phi &\rightarrow \Phi \ , \end{aligned}$$

whose Noether current is just $j^\mu_B = \bar{\psi}\gamma^\mu \psi$ (checking that it is conserved is straightforward). The associated charge $B = \int d^3\mathbf{x}\, j^0_B(t,\mathbf{x})$ can be identified with the baryon number. Thus, the full global symmetry group of the σ-model is

$$SU(2)_L \times SU(2)_R \times U(1)_B \ .$$

Since the scalar fields mass squared term in the Lagrangian has the wrong sign, the $SU(2)_L \times SU(2)_R$ symmetry is broken spontaneously down to the vector $SU(2)_V$ symmetry (corresponding to $\theta^a_L = \theta^a_R = \theta^a_V$). One easily finds that the potential

$$V = -\frac{1}{2} m^2 (\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2)^2 \ ,$$

[3] Since $\Phi^\dagger \Phi$ is proportional to the unit matrix, the last term can be written alternatively as $-(\lambda/8)\mathrm{tr}[(\Phi^\dagger\Phi)^2]$.

[4] Superficially, the transformations $(\psi_N)_R \rightarrow e^{-i\theta_A}(\psi_N)_R$, $(\psi_N)_L \rightarrow e^{+i\theta_A}(\psi_N)_L$ and $\Phi \rightarrow e^{2i\theta_A}\Phi$ also leave $\mathcal{L}$ invariant; they are, however, not allowed because they lead to $\sigma' + i\pi^{3\prime} = e^{2i\theta_A}(\sigma + i\pi^3)$ and $\sigma' - i\pi^{3\prime} = e^{2i\theta_A}(\sigma - i\pi^3)$.

has the minimum for[5] $\langle\sigma\rangle = v$ such that $(-m^2 + \lambda v^2)v = 0$. In other words, for this value of v^2 the term linear in σ is absent from the potential after shifting the σ field: $\sigma \to \sigma + v$. The terms without derivatives and quadratic in the fields in the Lagrangian then read

$$\mathcal{L}_{\text{mass}} = -g_{\pi N} v\, \bar{\psi}_N \psi_N - \frac{1}{2}(2\lambda v^2)\sigma^2 - \frac{1}{2}(-m^2 + \lambda v^2)\sigma^2 \ .$$

It follows that the three pion fields π^a are massless while the nucleons and σ are massive.

If a small term $-a\sigma$ is added to the original Lagrangian, $\mathcal{L} \to \mathcal{L} - a\sigma$, the equation of motion of the field σ gets modified to

$$\partial^2\sigma = m^2\sigma - \lambda(\sigma^2 + \pi^b\pi^b)\sigma - g_{\pi N}\, \bar{\psi}_N \psi_N - a \ .$$

It is easy to see that the vector currents V^a_μ are still conserved (the σ equation of motion was not used in showing that $\partial^\mu V^a_\mu = 0$) because the symmetry $SU(2)_L \times SU(2)_R$ breaks down to the vector $SU(2)_V$ symmetry. The axial currents are no longer conserved:

$$\partial^\mu A^a_\mu = a\, \pi^a \ .$$

In addition, the equation which determines in the lowest order the vacuum expectation value v of the σ field gets modified and reads

$$(-m^2 + \lambda v^2)v = -a \ ,$$

so that after shifting the σ field $\sigma \to \sigma + v$ the mass terms in $\mathcal{L}$ read

$$\mathcal{L}_{\text{mass}} = -g_{\pi N} v\, \bar{\psi}_N \psi_N - \frac{1}{2}\left(2\lambda v^2 - \frac{a}{v}\right)\sigma^2 - \frac{1}{2}\left(-\frac{a}{v}\right)(\sigma^2 + \vec{\pi}^2) \ .$$

It is also clear that for $V(\sigma) \to V(\sigma) + a\sigma$ with $a > 0$ it is the minimum with $v < 0$ which is lower than the one for positive v.[6] It follows that the π^a fields have now a

[5] The minimum of the potential $V(\sigma, \pi^a)$ is for $\langle\sigma\rangle^2 + \langle\pi^a\rangle^2 = v^2$. The choice of the vacuum $\langle\sigma\rangle \neq 0$, $\langle\pi^a\rangle = 0$ is justified by assuming that $\mathcal{L}$ has an infinitesimally small term $-a\sigma$ (with $a \to 0$) which breaks the symmetry explicitly and determining the direction of the spontaneous symmetry breaking in agreement with the Dashen's theorem.

[6] For $v < 0$ the nucleon mass $m_N = g_{\pi N} v$ is positive if $g_{\pi N} < 0$ (which is a possible choice). Nothing is wrong, however with $m_N < 0$.

positive mass squared $M_\pi^2 = -a/v > 0$. Furhermore, after shifting the σ field the axial current A_μ^a acquires a term linear in the pion field

$$A_\mu^a = v\,\partial_\mu \pi^a + \dots \ ,$$

so that[7]

$$\langle \Omega | A_\mu^a(x) | \pi^b(\mathbf{p}) \rangle = \langle \Omega_0 | v \partial_\mu \pi_I^a(x) + \dots | (\pi^b(\mathbf{p}))_0 \rangle = -iv\, p_\mu\, e^{-ix\cdot p}\, \delta^{ab} \ ,$$

($\pi_I^a(x)$ is the interaction picture pion field operator) which allows to identify in the lowest order $-v = f_\pi$. Thus, the PCAC assumption is in the σ model realized by the operator relation

$$\partial^\mu A_\mu^a = f_\pi M_\pi^2\, \pi^a \ ,$$

the divergence of the axial currents is just proportional to the canonical pion field.[8]

Solution to Problem 12.
In order to couple the proton (ψ_p) and charged pion ($\pi^\pm$) fields to the electromagnetic field we introduce $\pi^0 \equiv \pi^3$, $\pi^\mp \equiv (\pi^1 \mp i\pi^2)/\sqrt{2}$ and replace in the kinetic terms of pions and nucleons ordinary derivatives by the covariant ones

$$\begin{aligned}
&\frac{1}{2}\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + \bar{\psi}_N i \not{\partial} \psi_N \to \\
&\qquad \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 + \pi^-(\overleftarrow{\partial}_\mu - ieA_\mu)(\overrightarrow{\partial}^\mu + ieA^\mu)\pi^+ + \bar{\psi}_N(i\not{\partial} - e\,Q\not{A})\psi_N \ ,
\end{aligned}$$

with Q the electric charge matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ .$$

This corresponds to "gauging" the $U(1)_{\rm EM}$ subgroup generated by $Q = Q_V^3 + B$ of the the full global $SU(2)_L \times SU(2)_R \times U(1)_B$ symmetry group of the model (see Problem 11).

---

[7] Working in the lowest order we do not need to specify the renormalization scheme we are working in.

[8] The main point of PCAC is the assumption that $\partial^\mu A_\mu^a$ is a good operator for creating and annihilating pions from the vacuum (in the LSZ formula) irrespectively of whether there is any canonical pion field in the theory Lagrangian. This assumption turns out not to be justified when (as in Problem 14) the electromagnetic interactions are taken into account.

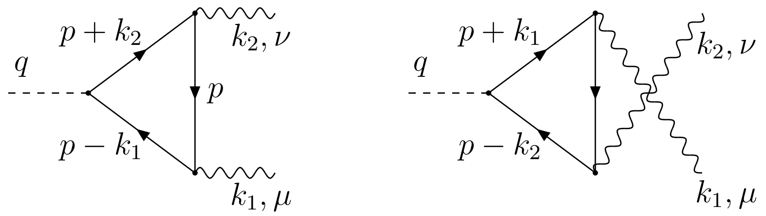


Figure 11: Diagrams in the linear σ-model contributing directly to the $\pi^0 \to \gamma\gamma$ decay amplitude.

Since after spontaneous (due to the negative mass squared term in the potential) and explicit (by adding a term $-a\sigma$ to the symmetric Lagrangian) chiral symmetry breaking neutral pions do not have linear couplings to the charged pions (parity conservation!), the terms relevant for computing the $\pi^0 \to \gamma\gamma$ amplitude at the lowest nontrivial order are

$$\mathcal{L}_{\rm int} = -ig_{\pi N}\bar{\psi}_N\gamma^5\tau^3\pi^0\psi_N - e\,\bar{\psi}_N\gamma^\mu Q\psi_N A_\mu \ .$$

and the diagrams to be computed are shown in figure 11. From these diagrams for the tensor $T^{\mu\nu}(k_1,k_2)$ in the decay amplitude

$$-i\mathcal{M} = i\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\,T^{\mu\nu}(k_1,k_2) = i\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\,\epsilon^{\mu\nu\kappa\rho}k_{1\kappa}k_{2\rho}\,T(q^2=M_\pi^2)\ ,$$

we get

$$iT^{\mu\nu}(k_1,k_2) = (-1)^F e^2 g_{\pi N}{\rm tr}(Q^2\tau^3)\int\frac{d^4p}{(2\pi)^4}\,i\left(\frac{{\rm NUM}_1}{{\rm DEN}_1}+\frac{{\rm NUM}_2}{{\rm DEN}_2}\right)\ ,$$

with

$${\rm NUM}_1 = {\rm tr}\left[\gamma^5(\not{p}_1+m_N)\gamma^\mu(\not{p}+m_N)\gamma^\nu(\not{p}_2+m_N)\right]\ ,$$

where to simplify the calculation of the trace we have used $p_1 \equiv p-k_1$ and $p_2 \equiv p+k_2$ and

$${\rm DEN}_1 = [(p-k_1)^2-m_N^2][p^2-m_N^2][(p+k_2)^2-m_N^2]\ .$$

${\rm NUM}_2$ and ${\rm DEN}_2$ are obtained from ${\rm NUM}_1$ and ${\rm DEN}_1$ by exchanging $(k_1,\mu)\leftrightarrow(k_2,\nu)$. The numerators are easy to work out:

$$\begin{aligned}{\rm NUM}_1 &= m_N{\rm tr}\left(\gamma^5\not{p}_1\gamma^\mu\not{p}\gamma^\nu\right)+m_N{\rm tr}\left(\gamma^5\not{p}_1\gamma^\mu\gamma^\nu\not{p}_2\right)+m_N{\rm tr}\left(\gamma^5\gamma^\mu\not{p}\gamma^\nu\not{p}_2\right)\\ &= -4im_N\epsilon^{\lambda\mu\kappa\nu}p_{1\lambda}p_\kappa - 4im_N\epsilon^{\lambda\mu\nu\kappa}p_{1\lambda}p_{2\kappa} - 4im_N\epsilon^{\mu\lambda\nu\kappa}p_\lambda p_{2\kappa}\ .\end{aligned}$$

Inserting here $p_1 = p - k_1$ and $p_2 = p + k_2$ one finds that most of the terms cancel out and one is left with

$$\mathrm{NUM}_1 = 4im_N\epsilon^{\mu\nu\lambda\kappa}k_{1\lambda}k_{2\kappa} = \mathrm{NUM}_2 \ .$$

Thus, the numerators are independent of the integration variable p so that the integrals over d^4p which seemed to be linearly divergent are actually finite. They can be done in the standard way:

$$\int\frac{d^4p}{(2\pi)^4}\frac{i}{[(p-k_1)^2-m_N^2][p^2-m_N^2][(p+k_2)^2-m_N^2]}$$
$$= \int_0^1 dx\int_0^1 dy\int\frac{d^4p}{(2\pi)^4}\frac{2x\ i}{[(1-x)p^2+xy(p+k_2)^2+x(1-y)(p-k_1)^2-m_N]^3} \ .$$

Using the kinematics $k_i^2 = 0$, $2k_1\cdot k_2 = q^2$ the denominator can be written as

$$[(p-x(1-y)k_1+xyk_2)^2+x^2y(1-y)q^2-m_N^2]^3 \ ,$$

and after shifting the integration variable (this is allowed because the integral is not linearly divergent) and performing the standard Wick rotation one is left with

$$\int\frac{d^4p}{(2\pi)^4}\frac{i}{[p^2-a^2]^3} = \int\frac{d^4_{\mathrm{E}}p}{(2\pi)^4}\frac{1}{[p_{\mathrm{E}}^2+a^2]^3} = \frac{1}{16\pi^2}\ \frac{1}{2a^2} \ ,$$

where $a^2 = m_N^2 - x^2y(1-y)q^2$.

Assembling all the pieces together (taking into account that the two diagrams of figure 11 give equal contributions) we have

$$T^{\mu\nu}(k_1,k_2) = -\frac{e^2g_{\pi N}}{16\pi^2}\,\mathrm{tr}(Q^2\tau^3)\,4m_N\epsilon^{\mu\nu\lambda\kappa}k_{1\lambda}k_{2\kappa}\int_0^1 dx\int_0^1 dy\,\frac{2x}{m_N^2-x^2y(1-y)q^2}$$
$$= -\frac{e^2}{4\pi^2}\,\frac{g_{\pi N}}{m_N}\left(1+\frac{1}{12}\frac{q^2}{m_N^2}+\dots\right)\epsilon^{\mu\nu\lambda\kappa}k_{1\lambda}k_{2\kappa} \ .$$

Thus

$$T(q^2) = -\frac{e^2}{4\pi^2}\frac{g_{\pi N}}{m_N}\left(1+\frac{1}{12}\frac{q^2}{m_N^2}+\dots\right) = \frac{e^2}{4\pi^2 f_\pi}\left(1+\frac{1}{12}\frac{q^2}{m_N^2}+\dots\right) \ ,$$

(in the second step we have used the relation $g_{\pi N}/m_N = 1/v = -1/f_\pi$) i.e. $T(0)\neq 0$ in contradiction with the naive current algebra arguments (see Problem 14).

The $\pi^0 \to \gamma\gamma$ amplitude structure is uniquely determined by the gauge invariance and parity conservation in strong and electromagnetic interactions (see Problem V.29) and can be obtained from the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -h\,\pi^0 \epsilon^{\lambda\mu\kappa\nu} F_{\lambda\mu} F_{\kappa\nu} = 4h\,\pi^0 \epsilon^{\lambda\mu\kappa\nu} \partial_\lambda A_\mu \partial_\kappa A_\nu \ .$$

This gives the amplitude

$$-i\mathcal{M} = -8ih\,\epsilon^{\mu\nu\lambda\kappa} k_{1\lambda} k_{2\kappa}\,\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \ ,$$

(there is a combinatoric factor of 2). It follows that the factor $T(M_\pi^2)$ is related to the effective Langrangian coupling h by

$$-8ih = iT(M_\pi^2) = -i\frac{e^2}{4\pi^2}\frac{g_{\pi N}}{m_N}\left(1 + \frac{1}{12}\frac{M_\pi^2}{m_N^2} + \ldots\right) \ ,$$

that is

$$h = \frac{e^2}{32\pi^2}\frac{g_{\pi N}}{m_N}\left(1 + \frac{1}{12}\frac{M_\pi^2}{m_N^2} + \ldots\right) \ .$$

Squaring the amplitude and summing over polarizations of the photons using the Feynman's trick we get (see Problem V.29) the width

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha_{\text{EM}}^2 g_{\pi N}^2}{64\pi^3}\frac{M_\pi^3}{m_N^2}\left(1 + \frac{1}{12}\frac{M_\pi^2}{m_N^2} + \ldots\right)^2$$

(there is an extra factor of 1/2 because photons are indistinguishable). For $\alpha_{\text{EM}} = 1/137.036$, $g_{\pi N}^2/4\pi = 14.6$, $M_\pi = 135$ MeV and $m_N = 938$ MeV one gets

$$\Gamma(\pi^0 \to \gamma\gamma) = 13.76 \text{ eV}.$$

If instead the relation $g_{\pi N}^2/m_N^2 = 1/f_\pi^2$ (valid in the σ-model in the lowest order) is used with $f_\pi = 93$ MeV, one obtains

$$\Gamma(\pi^0 \to \gamma\gamma) = 7.63 \text{ eV}.$$

The measured π^0 lifetime is $\tau_{\pi^0} = (8.4 \pm 0.6) \times 10^{-17}$ sec and the branching ratio of $\pi^0 \to \gamma\gamma$ is 98.8%. Thus experimentally

$$\Gamma(\pi^0 \to \gamma\gamma) = 0.988 \times \frac{\hbar}{\tau_{\pi^0}} = 0.988 \times \frac{6.582 \times 10^{-22}\ \text{MeV} \cdot \text{sec}}{(8.4 \pm 0.6) \times 10^{-17}\ \text{sec}} = (7.7 \pm 0.5) \text{ eV}.$$

Solution to Problem 13.

1) If each of the terms under the integral were sepatrately integrable, a shift of the integration variable $p^\mu + a^\mu \to p^\mu$ would show that the difference of these integrals is zero. However, as the power counting shows, the individual integrals are linearly divergent. The shift of the integration variable can, therefore, result in a nonzero contribution of the surface term. We therefore write

$$a^\mu f(a^2) = i \int \frac{d^4p}{(2\pi)^4} \left[\frac{p^\mu + a^\mu}{[(p+a)^2 - m^2]^2} - \frac{p^\mu}{[p^2 - m^2]^2} \right] ,$$

or

$$a^2 f(a^2) = i \int \frac{d^4p}{(2\pi)^4} \left[\frac{(p+a)\cdot a}{[(p+a)^2 - m^2]^2} - \frac{p \cdot a}{[p^2 - m^2]^2} \right] ,$$

Next we continue into the Euclidean space $a^0 \to ia^4$ and, simultaneously, rotate the integration contour anticlockwise making the change of variable $p^0 = ip^4$, so that

$$-a^2 f(-a^2) = \int \frac{d_E^4 p}{(2\pi)^4} \left[\frac{(p+a)\cdot a}{[(p+a)^2 + m^2]^2} - \frac{p \cdot a}{[p^2 + m^2]^2} \right] ,$$

where now all the scalar products are Euclidean. Next, expanding the first term in powers of a^μ we have

$$-a^2 f(-a^2) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d_E^4 p}{(2\pi)^4} \frac{\partial^n}{\partial p^{\nu_1} \dots \partial p^{\nu_n}} \frac{a^{\nu_1} \dots a^{\nu_n}\, p \cdot a}{(p^2 + m^2)^2} ,$$

Each of the terms can be viewed as an integral of the divergence of a vector field and can be reduced to an integral of the "flux" of this field through the spherical surface of radius $R \to \infty$. Since the surface of the sphere in 4 dimensions goes as R^3, only the $n = 1$ term will give a nonvanishing result. Hence f is a constant independent of a^2. Thus,

$$a^2 f = -\frac{1}{(2\pi)^4 (R^2 + m^2)^2} \int_R d\sigma_p^\nu \, a^\nu (p \cdot a) ,$$

By 4-dimensional rotational invariance of the integral this must be

$$a^2 f(-a^2) = -\frac{1}{(2\pi)^4 (R^2 + m^2)^2} \frac{a^2}{4} \int_R d\sigma_p^\nu \, p^\nu = -\frac{1}{(2\pi)^4 (R^2 + m^2)^2} \frac{a^2}{4} 2\pi^2 R^4 ,$$

and, taking the limit $R \to \infty$, we get

$$f = -\frac{1}{32\pi^2} \ .$$

2) As in the previous case, the individual integrals are linearly divergent and their difference receives a contribution from the surface term. Again, only the first term in the Taylor expansion in powers of a^μ can contribute (i.e. the result must be linear in a^μ). The same argument shows that the masses m and the momenta k_1 and k_2 can be set to zero. Moreover, since the trace involves γ^5 the result must be a pseudotensor of rank 3. Since the only available pseudotensor is $\epsilon^{\lambda\mu\nu\kappa}$ we write

$$f\epsilon^{\lambda\mu\nu\kappa}a_\kappa = i\int \frac{d^4p}{(2\pi)^4}\,\frac{\partial}{\partial p^\kappa}\,a^\kappa\,\mathrm{tr}\left[\frac{1}{\not p}\gamma^\mu\frac{1}{\not p}\gamma^\nu\frac{1}{\not p}\gamma^\lambda\gamma^5\right] = i\int \frac{d^4p}{(2\pi)^4}\,\frac{\partial}{\partial p^\kappa}\,a^\kappa\,\frac{4i}{(p^2)^2}\epsilon^{\mu\nu\sigma\lambda}\,p_\sigma \ .$$

Contracting both sides with $\epsilon_{\mu\nu\lambda\rho}\,a^\rho$ one gets

$$f\,a^2 = i\int \frac{d^4p}{(2\pi)^4}\,\frac{\partial}{\partial p^\kappa}\,a^\kappa\,\frac{-4i}{(p^2)^2}\,(p\cdot a) \ .$$

Rotating the integration contour in the standard way anticlockwise and changing the integration variable $p^0 = ip^4$ with the simultaneous continuation $a^0 \to ia^4$ gives

$$f\,a^2 = \int \frac{d^4_E p}{(2\pi)^4}\,\frac{\partial}{\partial p^\kappa}\,a^\kappa\,\frac{4i}{(p^2)^2}\,(p\cdot a) = \frac{4i}{(2\pi)^4 R^4}\int_R d\sigma_p^\nu\,a^\nu\,(p\cdot a) \ ,$$

where the scalar products are now Euclidean. By 4-dimensional rotational invariance this must give $f = i/8\pi^2$.

3) The expression can be rewritten in the form

$$i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\frac{1}{\not p - \not k_2 + \not a - m}\gamma^\nu\frac{1}{\not p + \not k_1 + \not a - m}\gamma^\lambda\gamma^5 \right.$$
$$\left. -\frac{1}{\not p - \not k_2 - m}\gamma^\nu\frac{1}{\not p + \not k_1 - m}\gamma^\lambda\gamma^5\right] \ ,$$

with $a^\kappa = (k_2 - k_1)^\kappa$ which makes clear that it would be zero if shifting of the variable $p + a \to p$ was allowed. As it is not (the individual integrals superficially diverge quadratically, however taking the trace kills the p^2 dependence of the numerators leaving only linear divergences) the nonzero result must be due to the surface term contribution. Hence, since the expression must be a pseudotensor of rank 2, we write

$$f\,\epsilon^{\lambda\nu\kappa\rho} k_{1\kappa} k_{2\rho} = i \int \frac{d^4p}{(2\pi)^4}\, a^\kappa \frac{\partial}{\partial p^\kappa}\, \mathrm{tr}\left[\frac{1}{\not p - \not k_2 - m} \gamma^\nu \frac{1}{\not p + \not k_1 - m} \gamma^\lambda \gamma^5 \right] \; .$$

The trace under the integral is

$$-4i \left[\epsilon^{\lambda\nu\kappa\rho}\, p_\kappa k_{2\rho} + \epsilon^{\lambda\nu\kappa\rho}\, p_\kappa k_{1\rho} + \epsilon^{\lambda\nu\kappa\rho}\, k_{1\kappa} k_{2\rho} \right] \; ,$$

and contracting both sides with $\epsilon_{\lambda\nu\sigma\delta} k_1^\sigma k_2^\delta$ we obtain

$$2f\,[(k_1 \cdot k_2)^2 - k_1^2 k_2^2] = 8 \int \frac{d^4p}{(2\pi)^4}\, a^\kappa \frac{\partial}{\partial p^\kappa} \frac{h(p,k_2,k_1)}{[(p-k_2)^2 - m^2][(p+k_1)^2 - m^2]} \; ,$$

with

$$\begin{aligned} h(p,k_1,k_2) \equiv\ & (k_1 \cdot k_2)(p \cdot k_2) - (p \cdot k_1) k_2^2 + (p \cdot k_2) k_1^2 \\ & -(p \cdot k_1)(k_1 \cdot k_2) + (k_1 \cdot k_2)^2 - k_1^2 k_2^2 \; . \end{aligned}$$

At this point one can make the standard Wick rotation of the integration contour followed by the change $p^0 = ip^4$ of the integration variable and simultaneous continuation $k_i^0 \to i k_i^4$, $a^0 \to i a^4$, so that

$$\begin{aligned} f\,[(k_1 \cdot k_2)^2 - k_1^2 k_2^2] &= 4i \int \frac{d_E^4p}{(2\pi)^4}\, a^\kappa \frac{\partial}{\partial p^\kappa} \frac{h(p,k_1,k_2)}{[(p-k_2)^2 + m^2][(p+k_1)^2 - +^2]} \\ &= \frac{4i}{(2\pi)^4 R^4} \int_R d\sigma_p^\nu a^\nu\, h(p,k_1,k_2) \; , \end{aligned}$$

where now the scalar products in $h(p,k_1,k_2)$ and in the left-hand side are Euclidean. For the integral over the sphere the 4-dimensional rotational invariance implies (the last two terms of $h(p,k_1,k_2)$ being independent of p give zero upon integration)

$$\int_R d\sigma_p^\nu a^\nu (p \cdot k_i) = \frac{1}{4}(a \cdot k_i) \int_R d\sigma_p^\nu p^\nu = \frac{1}{4}(a \cdot k_i) 2\pi^2 R^4 \; ,$$

which for $a = k_2 - k_1$ leads to the final result

$$f\,[(k_1 \cdot k_2)^2 - k_1^2 k_2^2] = 4i\, \frac{1}{16\pi^4 R^4}\, \frac{2\pi^2 R^4}{4}\, [2k_1^2 k_2^2 - 2(k_1 \cdot k_2)^2] \; .$$

Hence $f = -i/4\pi^2$.

Solution to Problem 14.
The starting point is the σ-model of Problem 11 with the $U(1)_{\rm EM}$ subgroup of its full $SU(2)_L \times SU(2)_R \times U(1)_B$ global symmetry gauged as in Problem 12. In the model obtained in this way the original global symmetry is broken down to $U(1)_L \times U(1)_R \times U(1)_B$ of which the $U(1)_L \times U(1)_R$ part is generated by T^3_L, T^3_R. The extra terms introduced in Problem 12 into the Lagrangian of Problem 11

$$\begin{aligned}\Delta\mathcal{L} &= -e\bar{\psi}_N Q \not{A}\psi_N - ieA_\mu(\pi^-\partial^\mu\pi^+ - \pi^+\partial^\mu\pi^-) + e^2 A^\mu A_\mu \pi^-\pi^+ \\ &= -e\bar{\psi}_N Q \not{A}\psi_N + eA_\mu(\pi^2\partial^\mu\pi^1 - \pi^2\partial^\mu\pi^1) + \frac{e^2}{2}A^\mu A_\mu(\pi^1\pi^1 + \pi^2\pi^2)\ ,\end{aligned}$$

are invariant only with respect to the vector and axial-vector transformations with $\theta^a_{A,V} \equiv (\theta^2_R - \theta^a_L) = \theta^3_{A,V}\delta^{a3}$ and the transformations corresponding to the baryon number (this is easy to see: T^3 and the unit matrix both commute with the electric charge matrix Q, the transformations with θ^3_V rotate $\pi^\pm$ and π^0 independently while those with θ^3_A rotate only the π^0 and σ fields). Moreover, they do not modify the axial current A^3_μ which in the absence of the explicit breaking term $-a\sigma$ remains conserved.

When the term $-a\sigma$ is added to the Lagrangian one still (naively) derives with the help of the equations of motion the operator relation

$$\partial^\lambda A^3_\lambda = f_\pi M^2_\pi \pi^0\ .$$

Since the $U(1)_A$ symmetry[9] is spontaneously broken, the matrix element of the axial current A^3_λ between the state of a physical (massive) neutral pion and the vacuum is

$$\langle\Omega|A^3_\lambda(y)|\pi^0(\mathbf{q})\rangle = if_\pi\, q_\lambda\, e^{-iq\cdot y}\ ,$$

with $f_\pi \neq 0$ (in fact the above matrix element *defines* f_π) so that with the explicit symmetry breaking

$$\langle\Omega|\partial^\lambda A^3_\lambda(y)|\pi^0(\mathbf{q})\rangle = f_\pi\, M^2_\pi\, e^{-iq\cdot y}\ .$$

In the LSZ formalism the S-matrix element corresponding to the $\pi^0 \to \gamma\gamma$ decay can be represented as

$$\begin{aligned}\langle(\gamma(\mathbf{k}_1)\gamma(\mathbf{k}_2))_-|(\pi^0(\mathbf{q}))_+\rangle &= \mathcal{Z}_A^{-1}\epsilon^*_\mu(\mathbf{k}_1)\epsilon^*_\nu(\mathbf{k}_2) \\ &\int d^4x_1 d^4x_2 d^4y\, e^{ik_1\cdot x_1}\, e^{ik_2\cdot x_2}\, e^{-iq\cdot y}\,(-i\partial^2_{x_1})(-i\partial^2_{x_2})i(\partial^2_y + M^2_\pi) \\ &\qquad \times\langle\Omega_-|T\{\phi_\pi(y)A^\mu(x_1)A^\nu(x_2)\}|\Omega_+\rangle\ ,\end{aligned}$$

[9] Do not confuse this $U(1)_A$ symmetry with the $U(1)_A$ symmetry of the QCD Lagrangian!

where ϕ_π is an operator with the property

$$\langle\Omega|\phi_\pi(y)|\pi^0(\mathbf{q})\rangle = e^{-iq\cdot y} \ .$$

Integrating the y derivative by parts and using the photon field operator equation of motion $\partial^2 A^\mu = e\, J^\mu_{\rm EM}$ this can be rewritten in the form (see Problem VII.6)

$$\langle(\gamma(\mathbf{k}_1)\gamma(\mathbf{k}_2))_-|(\pi^0(\mathbf{q}))_+\rangle = \mathcal{Z}_A^{-1}\epsilon^*_\mu(\mathbf{k}_1)\epsilon^*_\nu(\mathbf{k}_2)\, i(q^2-M^2_\pi)\, e^2\int d^4x_1 d^4x_2 d^4y\, e^{-iq\cdot y}$$
$$\times e^{ik_1\cdot x_1}\, e^{ik_2\cdot x_2}\, \langle\Omega_-|T^*\left\{\phi_\pi(y)J^\mu_{\rm EM}(x_1)J^\nu_{\rm EM}(x_2)\right\}|\Omega_+\rangle \ .$$

To exploit the PCAC hypothesis one takes for ϕ_π the operator $f^{-1}_\pi M^{-2}_\pi \partial^\lambda_y A^3_\lambda(y)$ which is expected to be "good" for continuing the $\pi^0 \to \gamma\gamma$ amplitude of the pion mass shell in the sense that the variation with q^2 of the Green's function obtained in this way will not be unusually rapid but will be dictated by kinematics (i.e. by the distance to the nearest singularity). Taking the ∂^λ_y derivative outside the T^* product *ignoring possible nonzero value of the equal time commutators* of A^3_0 with the currents $J^\nu_{\rm EM}$ and integrating it by parts one arrives at

$$\langle(\gamma(\mathbf{k}_1)\gamma(\mathbf{k}_2))_-|(\pi^0(\mathbf{q}))_+\rangle = i\mathcal{Z}_A^{-1}\epsilon^*_\nu(\mathbf{k}_2)\epsilon^*_\mu(\mathbf{k}_1)\,\frac{q^2-M^2_\pi}{f_\pi M^2_\pi}\, q_\lambda\, ie^2\int d^4x_1 d^4x_2 d^4y\, e^{-iq\cdot y}$$
$$\times e^{ik_2\cdot x_2}\, e^{ik_1\cdot x_1}\, \langle\Omega_-|T^*\left\{A^\lambda_3(y)J^\mu_{\rm EM}(x_1)J^\nu_{\rm EM}(x_2)\right\}|\Omega_+\rangle \ .$$

Taking into account that $\mathcal{Z}_A = 1$ in the lowest order in e, and expressing the $\langle(\gamma(\mathbf{k}_1)\gamma(\mathbf{k}_2))_-|(\pi^0(\mathbf{q}))_+\rangle$ amplitude as in Problem 12 through the tensor $T^{\mu\nu}(k_1,k_2)$ we get

$$T^{\mu\nu}(k_1,k_2) = \frac{q^2-M^2_\pi}{f_\pi M^2_\pi}\, q_\lambda\, T^{\lambda\mu\nu}(k_1,k_2) \ ,$$

with

$$T^{\lambda\mu\nu}(k_1,k_2) = ie^2\int d^4x_1 d^4x_2\, e^{ik_1\cdot x_1}\, e^{ik_2\cdot x_2}\, \langle\Omega_-|T^*\left\{A^\lambda_3(0)J^\mu_{\rm EM}(x_1)J^\nu_{\rm EM}(x_2)\right\}|\Omega_+\rangle \ .$$

The most general form of $T_{\lambda\mu\nu}(k_1,k_2)$ consistent for $k^2_i = 0$ with gauge invariance, parity conservation and symmetric with respect to $(k_1,\mu) \leftrightarrow (k_2,\nu)$ reads[10]

$$T_{\lambda\mu\nu}(k_1,k_2) = \frac{1}{2}q^2\epsilon_{\lambda\mu\nu\rho}(k_1-k_2)^\rho\, F_3(q^2)$$

[10]To get this form just use the most general parametrization of the (off-shell) trangle amplitude $\tilde{\Gamma}^5_{\lambda\nu\mu}$ in terms of 8 invariant formfactors $A_{1,2}$ and B_{1-6}. Set $B_i \to F_i$ and express the functions A_1 and A_2 in terms of F_i, consistently with the gauge invariance; for $k^2_i = 0$ this gives $A_1 = (k_1\cdot k_2)F_3$ and $A_2 = (k_1\cdot k_2)F_2$; for $k^2_i = 0$ Bose symmetry implies $F_4 = -F_1$, $F_2 = -F_3$ and $F_6 = +F_5$.

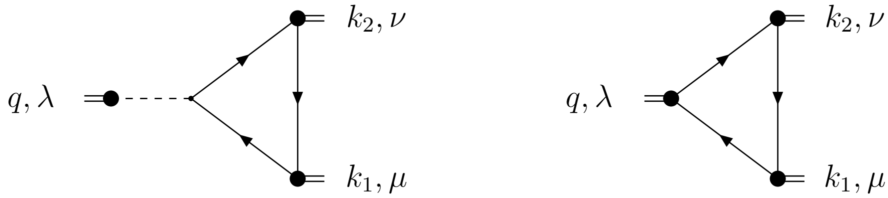


Figure 12: Two types of contributions to the Green's function $\langle\Omega|T[A_3^\lambda(y)J^\mu_{\rm EM}(x_1)J^\nu_{\rm EM}(x_2)]|\Omega\rangle$ in the linear σ-model. The double lines with the big dots represent insertions of the currents. Crossed diagrams are not shown.

$$
\begin{aligned}
&+(\epsilon_{\lambda\nu\kappa\rho}k_{1\mu} - \epsilon_{\lambda\mu\kappa\rho}k_{2\nu})\, k_1^\kappa k_2^\rho\, F_1(q^2)\\
&+(\epsilon_{\lambda\mu\kappa\rho}k_{1\nu} - \epsilon_{\lambda\nu\kappa\rho}k_{2\mu})\, k_1^\kappa k_2^\rho\, F_3(q^2)\\
&+q^\lambda\, \epsilon_{\mu\nu\kappa\rho}k_1^\kappa k_2^\rho\, F_5(q^2)\ .
\end{aligned}
$$

(where $q = k_1 + k_2$ and $q^2 = 2k_1 \cdot k_2$) for which

$$
q^\lambda\, T_{\lambda\mu\nu}(k_1, k_2) = q^2 \epsilon_{\mu\nu\kappa\rho}\, k_1^\kappa k_2^\rho\, [F_1(q^2) - F_3(q^2)]\ ,
$$

so that the formfactor $T(q^2)$ defined in Problem 12 which determines the $\pi^0 \to \gamma\gamma$ is given by

$$
T(q^2) = \frac{q^2 - M_\pi^2}{f_\pi M_\pi^2}\, q^2\, [F_5(q^2) - F_3(q^2)]\ .
$$

If the PCAC assumption is right, the formfactors $F_5(q^2)$ and $F_3(q^2)$ cannot have a pole at $q^2 = 0$ and, in consequence, $T(0) = 0$ (the Sutherland-Veltman theorem).

These formal considerations can be checked explicitly in the σ-model of Problems 11 and 12 in which PCAC is realized as the operator relation and the currents in in question are explicitly given in terms of the canonical Lagrangian fields. The relevant terms in the currents are[11]

$$
\begin{aligned}
A_\lambda^3 &= \bar\psi_N T^3 \gamma_\lambda \gamma^5 \psi_N - f_\pi \partial_\lambda \pi^0 + \dots\ ,\\
J^\mu_{\rm EM} &= \bar\psi_N Q \gamma^\mu \psi_N\ .
\end{aligned}
$$

Correspondingly to the two terms in A_λ^3 there are two types of contributions to $T_{\lambda\mu\nu}$, shown in figure 12:

$$
T_{\lambda\mu\nu} = T^{(1)}_{\lambda\mu\nu} + T^{(2)}_{\lambda\mu\nu}\ .
$$

[11]For the sake of simplicity we consider below a "reduced" σ-model without the $\pi^\pm$ fields. With these fields included there would be additional graphs contributing to $\langle A_3^\lambda J^\nu_{\rm EM} J^\mu_{\rm EM}\rangle$ but they would not change the final conclusions.

The term $T^{(1)}_{\lambda\mu\nu}$ corresponding to the first diagram of figure 12 can easily be written down[12] in terms of the tensor $T_{\mu\nu}$:

$$T^{(1)}_{\lambda\mu\nu} = -i\left(-if_\pi\, q_\lambda\, \frac{i}{q^2 - M^2_\pi}\, iT_{\mu\nu}\right) = f_\pi\, q_\lambda\, \frac{1}{q^2 - M^2_\pi}\, T_{\mu\nu} \ .$$

Since the amplitude $T^{(2)}_{\lambda\mu\nu}$ does not have a pole at $q^2 = M^2_\pi$, it is clear that the LSZ prescription written in terms of the correlator of three currents correctly reproduces the *on-shell* amplitude of the $\pi^0 \to \gamma\gamma$ decay. Furthermore, if the PCAC assumption that the divergence of the axial current is a good pion interpolating field (as suggested in the σ-model by the naive operator equality $\partial^\lambda A^3_\lambda = f_\pi M^2_\pi \pi^0$) and the assumption about vanishing the equal time commutator $[A^3_0(\mathbf{y}, t),\ J^\mu_{\rm EM}(\mathbf{x}, t)]$ are true, the same form of the LSZ prescription should also correctly give the *off-shell* $\pi^0 \to \gamma\gamma$ amplitude in range of values of q^2 near $q^2 = M^2_\pi$ including $q^2 = 0$ (because, at the hadronic scale $M^2_\pi \approx 0$). One should therefore get $T(0) = 0$, in agreement with the Sutherland-Veltman argument. On the other hand, the $\pi^0 \to \gamma\gamma$ amplitude comuputed in the σ-model directly (Problem 12), i.e. by applying the LSZ prescription to the *canonical* pion field, gives $T(0) \neq 0$. Since the argument that $q^\lambda T_{\lambda\mu\nu} = 0$ at $q^2 = 0$ seems flawless (it is only the absence of a pole at $q^2 = 0$ which can be disputed, but there does not exist any kinematical reason for its existence), it is the arguments that allowed to express the *off-shell* amplitude in this form that must fail. We will now explicitly verify that $q^\lambda T_{\lambda\mu\nu} = 0$ at $q^2 = 0$ in the σ-model and will relate this to the anomalous Ward identity.

If the formula derived above were to correctly reproduce the $\pi^0 \to \gamma\gamma$ amplitude the following Ward identity

$$f_\pi\, T_{\mu\nu} = -q^\lambda\, T^{(2)}_{\lambda\mu\nu} \ .$$

should hold. The amplitude $T^{(2)}_{\lambda\mu\nu}$ is given by the second diagram of figure 12 and a similar one with $(k_1, \mu) \leftrightarrow (k_2, \nu)$. In the momentum space they give

$$T^{(2)}_{\lambda\mu\nu} = (-)^F ie^2 \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, {\rm tr}\left[\frac{i}{\not p - \not k_1 - m}\gamma^\mu \frac{i}{\not p\ - m}\gamma^\nu \frac{i}{\not p + \not k_2 - m}\gamma^\lambda\gamma^5\right] + [(k_1, \mu) \leftrightarrow (k_2, \nu)] \ ,$$

[12]Following the origin of different factors may be a little bit tricky here: the first $-i$ comes from the definition of $T_{\lambda\mu\nu}$ and the fact that the amplitude $iT_{\mu\nu}$ of Problem 12 differs from the the one corresponding to the triangle in the left diagram of figure 12 by $(-ie)^2$ (the Feynman rule for the electromagnetic current insertion does not have $-ie$ which is associated with the usual photon vertex). $i/(q^2 - M^2_\pi)$ is the propagator for the dashed line. Finally the derivative in the current A^3_λ acts on the y argument of the propagator $i\Delta^F(z - y)$, where z is the position of the vertex connecting the dashed line with the triangle, hence $-f_\pi\partial^\lambda$ produces $-if_\pi q^\lambda$.

(ie^2 comes from the definition of $T_{\lambda\mu\nu}$ and the extra 1/2 comes from $\mathrm{tr}(T^3Q^2)$). As follows from Problem 13 the integral in this expression is linearly divergent and the result depends on the way the momenta are ascribed to the internal lines. Hence, to the result obtained with the momenta ascribed as above one must add the term

$$\frac{e^2}{8\pi^2}\,\epsilon_{\lambda\mu\nu\kappa}\,a^\kappa\ ,$$

where a^κ is an arbitrary vector which in this problem should be a linear combination of k_1^κ and k_2^κ (no other vectors are available). After symmetrizing this with respect to $(k_1,\mu)\leftrightarrow(k_2,\nu)$ the arbitrary term which can be added takes the form

$$\frac{e^2}{8\pi^2}\,\epsilon_{\lambda\mu\nu\kappa}\,b\,(k_1-k_2)^\kappa\ ,$$

with some constant b to be fixed by the electromagnetic current conservation condition, which must be imposed on the amplitude $T_{\lambda\nu\mu}$.

The amplitude $T_{\lambda\mu\nu}$ involving conserved electromagnetic currents must satisfy

$$k_1^\mu\,T_{\lambda\mu\nu}=k^\mu\,T^{(2)}_{\lambda\mu\nu}=0\ ,$$
$$k_2^\nu\,T_{\lambda\mu\nu}=k^\nu\,T^{(2)}_{\lambda\mu\nu}=0\ ,$$

(the amplitude $T^{\mu\nu}$ in $T^{(1)}_{\lambda\mu\nu}$ satisfies these conditions by itself). Writing $\not k_1=(\not p-m_N)-(\not p-\not k_1-m_N)$ in the expression arising from the second diagram of figure 12 and $\not k_1=(\not p+\not k_1-m_N)+(\not p-m_N)$ in the crossed one we have

$$\begin{aligned}k_1^\mu T^{(2)}_{\lambda\mu\nu}=&\frac{e^2}{2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\frac{i}{\not p-\not k_1-m}\gamma_\nu\frac{i}{\not p+\not k_2-m}\gamma_\lambda\gamma^5-\frac{i}{\not p-m}\gamma_\nu\frac{i}{\not p+\not k_2-m}\gamma_\lambda\gamma^5\right]\\&+\frac{e^2}{2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\frac{i}{\not p-\not k_2-m}\gamma_\nu\frac{i}{\not p-m}\gamma_\lambda\gamma^5-\frac{i}{\not p-\not k_2-m}\gamma_\nu\frac{i}{\not p+\not k_1-m}\gamma_\lambda\gamma^5\right]\\&+\frac{e^2}{8\pi^2}\,\epsilon_{\lambda\mu\nu\rho}\,b\,k_1^\mu\,(k_1-k_2)^\rho\ .\end{aligned}$$

Of the four terms under the integrals the two middle ones must give zero, the corresponding results being pseudotensors of rank 2 depending on a single vector k_2. Thus,

$$\begin{aligned}k_1^\mu T^{(2)}_{\lambda\mu\nu}=ie^2\,\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Bigg[&\frac{1}{\not p-\not k_1-m}\gamma_\nu\frac{1}{\not p+\not k_2-m}\gamma_\lambda\gamma^5\\&-\frac{1}{\not p-\not k_2-m}\gamma_\nu\frac{1}{\not p+\not k_1-m}\gamma_\lambda\gamma^5\Bigg]+\frac{e^2}{8\pi^2}\epsilon_{\lambda\nu\kappa\rho}\,b\,k_1^\kappa k_2^\rho\ .\end{aligned}$$

Using the result for the third integral in Problem 13 this is

$$k_1^\mu T^{(2)}_{\lambda\mu\nu} = ie^2 \left(\frac{-i}{8\pi^2}\,\epsilon_{\lambda\nu\kappa\rho}\,k_1^\kappa k_2^\rho\right) + \frac{e^2}{8\pi^2}\epsilon_{\lambda\nu\kappa\rho}\,b\,k_1^\kappa k_2^\rho = \frac{e^2}{8\pi^2}\,(1+b)\,\epsilon_{\lambda\nu\kappa\rho}\;\;k_1^\kappa k_2^\rho.$$

Similar result is also obtained for $k_2^\nu T^{(2)}_{\lambda\mu\nu}$ because of the symmetry $(k_1,\mu)\leftrightarrow(k_2,\nu)$. Thus the requirement of the electromagnetic current conservation selects $b=-1$. With the value of b fixed by the electromagnetic current conservation we can now examine the contraction $k_2^\nu T^{(2)}_{\lambda\mu\nu}$. To this end we use the decompositions

$$\begin{aligned}\not{q} = \not{k}_1 + \not{k}_2 &= (\not{p} + \not{k}_2 - m) - (\not{p} - \not{k}_2 + m) + 2m\ ,\\ \not{q} = \not{k}_1 + \not{k}_2 &= (\not{p} + \not{k}_1 - m) - (\not{p} - \not{k}_2 + m) + 2m\ .\end{aligned}$$

This gives

$$\begin{aligned}q^\lambda T^{(2)}_{\lambda\mu\nu} = &-\frac{e^2}{2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\frac{1}{\not{p}-\not{k}_1-m}\gamma_\mu\frac{1}{\not{p}-m}\gamma_\nu\gamma^5 + \gamma_\mu\frac{1}{\not{p}-m}\gamma_\nu\frac{1}{\not{p}+\not{k}_2-m}\gamma^5\right]\\ &-\frac{e^2}{2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\frac{1}{\not{p}-\not{k}_2-m}\gamma_\nu\frac{1}{\not{p}-m}\gamma_\mu\gamma^5 + \gamma_\nu\frac{1}{\not{p}-m}\gamma_\mu\frac{1}{\not{p}+\not{k}_1-m}\gamma^5\right]\\ &-me^2\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\frac{1}{\not{p}-\not{k}_1-m}\gamma_\mu\frac{1}{\not{p}-m}\gamma_\nu\frac{1}{\not{p}+\not{k}_2-m}\gamma^5\right]\\ &-me^2\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\frac{1}{\not{p}-\not{k}_2-m}\gamma_\nu\frac{1}{\not{p}-m}\gamma_\mu\frac{1}{\not{p}+\not{k}_1-m}\gamma^5\right]\\ &-\frac{e^2}{8\pi^2}\,\epsilon_{\lambda\mu\nu\rho}\,q^\lambda\,(k_1-k_2)^\rho\ .\end{aligned}$$

The integrals in the first two lines must vanish (each of them would have to give a rank 2 pseudotensor depending on a single four-vector) and the integrals in the next two lines combine to give

$$q^\lambda T^{(2)}_{\lambda\mu\nu} = \frac{m}{g_{\pi N}}T_{\mu\nu} + \frac{e^2}{4\pi^2}\,\epsilon_{\mu\nu\kappa\rho}\,k_1^\kappa k_2^\rho\ .$$

Inserting here the explicit form of the tensor $T_{\mu\nu}$ computed in Problem 12

$$q^\lambda T^{(2)}_{\lambda\mu\nu} = \frac{m}{g_{\pi N}}\left\{-\frac{e^2}{4\pi^2}\frac{g_{\pi N}}{m}\left(1+\frac{q^2}{12m^2}+\dots\right)\epsilon_{\mu\nu\kappa\rho}\,k_1^\kappa k_2^\rho\right\} + \frac{e^2}{4\pi^2}\,\epsilon_{\mu\nu\kappa\rho}\,k_1^\kappa k_2^\rho\ ,$$

we see that indeed, the Sutherland-Veltman theorem is satisfied, that is $q^\lambda T_{\lambda\nu\mu}=0$ for $q^2=0$ ($q^\lambda T^{(1)}_{\lambda\mu\nu}=0$ for $q^2=0$ separately). Furthermore, since $m/g_{\pi N}=v=-f_\pi$

we see that the result differs from the one needed to reproduce the off-shell $\pi^0 \to \gamma\gamma$ amplitude by the last term. By the direct computation we therefore discover that

$$q^\lambda T_{\lambda\mu\nu} = q^\lambda (T^{(1)}_{\lambda\mu\nu} + T^{(2)}_{\lambda\mu\nu} =) = \frac{f_\pi q^2}{q^2 - M_\pi^2} T_{\mu\nu} - f_\pi T_{\mu\nu} + \frac{e^2}{4\pi^2}\,\epsilon_{\mu\nu\kappa\rho}\, k_1^\kappa k_2^\rho \ .$$

or

$$T_{\mu\nu} = \frac{q^2 - M_\pi^2}{f_\pi M_\pi^2}\left(q^\lambda T_{\lambda\mu\nu} - \frac{e^2}{4\pi^2}\,\epsilon_{\mu\nu\kappa\rho}\, k_1^\kappa k_2^\rho\right) \ ,$$

which for $q^2 = M_\pi^2$ reduces to $T_{\mu\nu} = T_{\mu\nu}$ but for $q^2 = 0$ gives

$$T(0) = \frac{e^2}{4\pi^2 f_\pi} \ ,$$

which is precisely the result obtained by direct calculation of Problem 12. It is clear that the term $q^\lambda T_{\lambda\mu\nu}$ arises from $\langle T(A_3^\lambda J^\nu_{\rm EM} J^\mu_{\rm EM})\rangle$. The second term must be due to the noncanonical values of the commutators $\delta(y^0 - x_1^0)[A_3^0(y),\ J^\mu_{\rm EM}(x_1)]$ and $\delta(y^0 - x_2^0)[A_3^0(y),\ J^\nu_{\rm EM}(x_2)]$. Indeed, in the adopted LSZ approach the modifcation of the PCAC relation to

$$\partial^\lambda A^3_\lambda = f_\pi M_\pi^2 \pi^0 - \frac{e^2}{32\pi^2}\,\epsilon_{\kappa\mu\rho\nu} f^{\kappa\mu} f^{\rho\nu} \ ,$$

would only give a correction of order e^4.

The modification of the PCAC relation and the necessary consistency of the coefficient of the $f\tilde{f}$ term with the noncanonical value of the equal-time commutator of A_3^0 with $J^\nu_{\rm EM}$ becomes evident in a different approach. The $\pi^0 \to \gamma\gamma$ amplitude can alternatively be written in the form

$$\langle(\gamma(\mathbf{k}_2)\gamma(\mathbf{k}_1))_-|(\pi^0(\mathbf{q}))_+\rangle = (-i)(q^2 - M_\pi^2)\int d^4y\, e^{-iq\cdot y}\,\langle(\gamma(\mathbf{k}_2)\gamma(\mathbf{k}_1))_-|\phi_\pi(y)|\Omega_+\rangle \ ,$$

Using here the modified PCAC relation gives ($\tilde{f}^{\kappa\mu} = -\epsilon^{\kappa\mu\rho\nu} f_{\rho\nu}$)

$$\begin{aligned}\langle(\gamma(\mathbf{k}_2)\gamma(\mathbf{k}_1))_-|(\pi^0(\mathbf{q}))_+\rangle &= \frac{q^2 - M_\pi^2}{i f_\pi M_\pi^2}\int d^4y\, e^{-iq\cdot y} \\ &\times\left\{\partial^\lambda\langle(\gamma(\mathbf{k}_2)\gamma(\mathbf{k}_1))_-|A^3_\lambda(y)|\Omega_+\rangle - \frac{e^2}{32\pi^2}\langle(\gamma(\mathbf{k}_2)\gamma(\mathbf{k}_1))_-|f\tilde{f}(y)|\Omega_+\rangle\right\} \ ,\end{aligned}$$

The first term gives just the amplitude which must satisfy the Sutherland-Veltman theorem, i.e. vanishes at $q^2 = 0$ (but it is this term that has a pole at $q^2 = M_\pi^2$ and produces the on-shell amplitude of the $\pi^0 \to \gamma\gamma$ decay). The second term at $q^2 = 0$ gives

$$\frac{ie^2}{32\pi^2 f_\pi} \times 2(i)^2 \times 4\epsilon_{\kappa\mu\rho\nu}\, k_1^\kappa k_2^\rho\, \epsilon^{*\mu}(\mathbf{k}_1)\epsilon^{*\nu}(\mathbf{k}_2) \ ,$$

i.e. $T(0) = e^2/4\pi^2 f_\pi$ as previously.

Solution to Problem 15.
The chiral Lagrangian consists in principle of infinitely many terms. They are classified according to the number of derivatives and the power of chiral symmetry breaking mass factors. Each of the terms of its $\mathcal{O}(p^4)$ part (denoted $\mathcal{L}^{(4)}_{\rm chiral}$) built out of the unitary, dimensionless matrix $U(x)$ must involve either four derivatives, or two derivatives and M^2, or just M^4 and no derivatives (since Lorentz indices must be contracted, the number of derivatives in each term must be even). Since each term must be of dimension 4, no explicit f factors can appear. For the usual exponential parametrization $U^\dagger = \exp\left(\frac{i}{f}\pi^a\tau^a\right)$

$$U^\dagger = \cos\frac{|\vec{\pi}|}{f} + i\frac{\pi^a\tau^a}{|\vec{\pi}|}\sin\frac{|\vec{\pi}|}{f} \ ,$$

so that $UU + U^\dagger U^\dagger = 1$. Hence the only possible Hermitian term with M^4 and four U's is $[\mathrm{tr}(U + U^\dagger)]^2 \propto \mathrm{tr}[(U + U^\dagger)^2]$. Similarly, the only possible term with M^2, two derivatives and four U's is $\mathrm{tr}[(U + U^\dagger)\partial_\mu U\partial^\mu U^\dagger] \propto [\mathrm{tr}(U + U^\dagger)]\mathrm{tr}(\partial_\mu U\partial^\mu U^\dagger)$. **(niedokończone)**

Thus the most general form of the $\mathcal{O}(p^4)$ chiral Lagrangian (in the limit of *exact* isospin $SU(2)_V$ symmetry) reads

$$\begin{aligned}\mathcal{L}^{(4)}_{\rm chiral} = \frac{1}{4}\alpha_1 \left[\mathrm{tr}(\partial_\mu U\partial^\mu U^\dagger)\right]^2 + \frac{1}{4}\alpha_2\,\mathrm{tr}(\partial_\mu U\partial^\nu U^\dagger)\,\mathrm{tr}(\partial^\mu U\partial_\nu U^\dagger)\\ +\frac{1}{32}\alpha_3 M^4 \left[\mathrm{tr}(U + U^\dagger)\right]^2 + \frac{1}{8}\alpha_4 M^2\,\mathrm{tr}((U + U^\dagger)\partial_\mu U\partial^\mu U^\dagger) \ .\end{aligned}$$

The lowest order term $\mathcal{L}^{(2)}_{\rm chiral}$ expanded in fields gives rise to the usual kinetic term, the quartic interaction

$$\mathcal{L}^{(2)}_{\rm int4} = \frac{1}{6f^2}\left[(\vec{\pi}\partial_\mu\vec{\pi})(\vec{\pi}\partial^\mu\vec{\pi}) - \vec{\pi}^2(\partial_\mu\vec{\pi}\partial^\mu\vec{\pi})\right] + \frac{1}{24}\frac{M^2}{f^2}\vec{\pi}^4,$$

and to infinitely many other terms with more and more π fields. The Feynman rule for the 4-pion vertex arising from $\mathcal{L}^{(2)}_{\rm int4}$ is (see figure 2 Set 6)

$$
\begin{aligned}
f^{(2)}_{abcd}(k_1,k_2,k_3,k_4) = & \frac{1}{3f^2}[2(k_1\cdot k_2)+2(k_3\cdot k_4)-(k_1+k_2)\cdot(k_3+k_4)+M^2]\,\delta^{ab}\delta^{cd} \\
& +\frac{1}{3f^2}[2(k_1\cdot k_3)+2(k_2\cdot k_4)-(k_1+k_3)\cdot(k_2+k_4)+M^2]\,\delta^{ac}\delta^{bd} \\
& +\frac{1}{3f^2}[2(k_1\cdot k_4)+2(k_2\cdot k_3)-(k_1+k_4)\cdot(k_2+k_3)+M^2]\,\delta^{ad}\delta^{bc} \ .
\end{aligned}
$$

To expand $\mathcal{L}^{(4)}_{\rm chiral}$ in powers of the pion field it is helpful to notice that for the 2×2 matrix U

$$
U+U^\dagger = 2\cos\frac{|\vec\pi|}{f} \ ,
$$

times the unit 2×2 matrix, so that

$$
\begin{aligned}
\mathcal{L}^{(4)} = & \frac{1}{4f^2}\alpha_1(\partial_\mu\vec\pi\partial^\mu\vec\pi)^2+\frac{1}{4f^2}\alpha_2(\partial_\mu\vec\pi\partial_\nu\vec\pi)(\partial^\mu\vec\pi\partial^\nu\vec\pi)+\dots \\
& +\frac{1}{2}\alpha_3 M^4\cos^2\frac{|\vec\pi|}{f}+\alpha_4\frac{M^2}{f^2}\frac{f^2}{4}{\rm tr}\left(\partial_\mu U\partial^\mu U^\dagger\right)\cos\frac{|\vec\pi|}{f} \ ,
\end{aligned}
$$

where ellipses stand for other terms proportional to α_1 and α_2. This gives rise to (the last term is readily expanded by using the expansion of $\mathcal{L}^{(2)}$):

$$
\begin{aligned}
\mathcal{L}^{(4)}_{\rm kin} = & \frac{1}{2}\alpha_4\frac{M^2}{f^2}\,(\partial_\mu\vec\pi\partial^\mu\vec\pi)-\frac{1}{2}\alpha_3\frac{M^4}{f^2}\,\vec\pi^2 \\
\mathcal{L}^{(4)}_{\rm int4} = & \frac{\alpha_1}{f^4}\,(\partial_\mu\vec\pi\partial^\mu\vec\pi)(\partial_\nu\vec\pi\partial^\nu\vec\pi)+\frac{\alpha_2}{f^4}\,(\partial_\mu\vec\pi\partial_\nu\vec\pi)(\partial_\nu\vec\pi\partial^\nu\vec\pi) \\
& +\alpha_3\frac{M^4}{6f^4}\,\vec\pi^4+\alpha_4\frac{M^2}{12f^4}\left[2(\vec\pi\partial_\mu\vec\pi)(\vec\pi\partial^\mu\vec\pi)-5\vec\pi^2\partial_\mu\vec\pi\partial^\mu\vec\pi\right] \ .
\end{aligned}
$$

According to the general scheme of the perturbation expansion appropriate for chiral effective Lagrangians, to the pion self energy contribute, in the first nontrivial order, the one loop diagram with the vertex arising from $\mathcal{L}^{(2)}_{\rm int4}$ as well as the tree level diagrams obtained from $\mathcal{L}^{(4)}_{\rm kin}$. Together they give

$$
-i\Sigma_{ab}(p^2) = \frac{1}{2}\sum_c\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-M^2}\ if^{(2)}_{abcc}(p,-p,q,-q)+i\delta^{ab}\left(\alpha_4\frac{M^2}{f^2}p^2-\alpha_3\frac{M^4}{f^2}\right) \ ,
$$

and, hence, modify the pion mass and cause departure of its $\mathcal{Z}$ wave function factor from unity. Using the explicit form of $f^{(2)}_{abcc}(p,-p,q,-q)$ we get

$$\Sigma_{ab}(p^2) = -\frac{1}{6f^2}\delta^{ab}\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-M^2}[-4p^2-4q^4+5M^2] - \delta^{ab}\left(\alpha_4\frac{M^2}{f^2}p^2 - \alpha_3\frac{M^4}{f^2}\right)$$
$$= \left(\frac{1}{16\pi^2}\frac{1}{6f^2}\left(4p^2-M^2\right)a(M) - \alpha_4\frac{M^2}{f^2}p^2 + \alpha_3\frac{M^4}{f^2}\right)\delta^{ab} \equiv \delta^{ab}\Sigma(p^2)\ .$$

We read off that the physical pion mass is

$$M_\pi^2 = M^2 + \Sigma(M^2) = M^2 + \frac{1}{16\pi^2}\frac{M^2}{2f^2}a(M) - \alpha_4\frac{M^4}{f^2} + \alpha_3\frac{M^4}{f^2}\ ,$$

and the $\mathcal{Z}$ factor is given by

$$\mathcal{Z} = \frac{1}{1-\Sigma'} \approx 1 + \Sigma' = 1 + \frac{1}{16\pi^2}\frac{2}{3f^2}a(M) - \alpha_4\frac{M^2}{f^2}\ .$$

The physical pion decay constant f_π is defined by the order G_F amplitude of the $\pi^-(\mathbf{q}) \to \mu^-(\mathbf{p})\bar{\nu}_\mu(\mathbf{k})$ decay:

$$-i\mathcal{M}(\pi^- \to \mu^-\bar{\nu}_\mu) = -i\frac{G_F\cos\theta_C}{\sqrt{2}}\bar{u}(\mathbf{p})\gamma^\lambda(1-\gamma^5)v(\mathbf{k}) \times (-i\sqrt{2}f_\pi q_\lambda)\ .$$

obtained from the effective Lagrangian of the low eneregy weak and strong interactions:

$$\mathcal{L} = \mathcal{L}_{\rm chiral} + \mathcal{L}^{\rm lept}_{\rm kin} - \frac{G_F}{\sqrt{2}}J^\lambda J^\dagger_\lambda\ ,$$

where (see Problem 9, Set 5) $J_\lambda = J^{\rm lept}_\lambda + V_\lambda - A_\lambda$ where the weak vector V_λ and axial A_λ currents are constructed out of the Noether currents of the $SU(2)\times SU(2)$ symmetry of $\mathcal{L}_{\rm chiral}$:

$$V_\lambda - A_\lambda = \cos\theta_C[(V^1-iV^2)_\lambda - (A^1-iA^2)_\lambda]\ ,$$

with $V^a_\lambda = (j^a_R)_\lambda + (j^a_L)_\lambda$, $A^a_\lambda = (j^a_R)_\lambda - (j^a_L)_\lambda$.

The Noether currents $(j^a_R)_\lambda$ and $(j^a_L)_\lambda$ (conserved in the limit of $M=0$) associated with the $SU(2)\times SU(2)$ symmetry of $\mathcal{L}^{(2)}_{\rm chiral}$ following from $\mathcal{L}^{(2)}$ have been derived in Problem 15, Set 3. $\mathcal{L}^{(4)}$ modifies these currents. The modification is not entirely obvious: for example, applying the Noether theorem directly to the $SU(2)_L \times SU(2)_R$ symmetric part of $\mathcal{L}^{(2)} + \mathcal{L}^{(4)}$ (i.e. to $\mathcal{L}^{(2)} + \mathcal{L}^{(4)}$ with M set to zero) would erroneously give the current independent of α_4. In fact, to determine the currents for $M \neq 0$ one has to appeal to the Gasser-Leutwyler procedure, that is, one has to gauge the full effective Lagrangian, so that it becomes invariant with respect to *local* transformations $U \to V_L(x)UV_R^\dagger(x)$, $\chi \to V_L(x)\chi V_R^\dagger(x)$. This requires introducing the gauge fields $l_\mu = \frac{1}{2}\tau^a l_\mu^a$ and $r_\mu = \frac{1}{2}\tau^a r_\mu^a$ and replacing the ordinary derivatives $\partial_\mu U$ by the covariant ones

$$D_\mu U = \partial_\mu U + i l_\mu U - i U r_\mu \ ;$$

The Noether currents are then given by the variations

$$(j_L^a)^\mu = \frac{\partial \mathcal{L}}{\partial l_\mu^a}\ , \qquad (j_R^a)^\mu = \frac{\partial \mathcal{L}}{\partial r_\mu^a}\ .$$

computed for $l_\mu^a = r_\mu^a = 0$, $\chi = M^2$. This gives the parts of the currents arising from $\mathcal{L}^{(2)}$ as obtained in Problem 15, Set 3. From $\mathcal{L}^{(4)}$ the only part of the currents linear in the π^a fields arise from the term proportional to α_4. The other terms arising from $\mathcal{L}^{(4)}$ contain higher powers of π^a and at the tree level do not contribute to the matrix element we are interested in. To the matrix element $\langle \Omega_{\rm hadr}|(J_\lambda^{\rm hadr})^\dagger|\pi^-(\mathbf{q})\rangle$ contributes only the axial weak current $(A^1 - iA^2)_\lambda$

$$\begin{aligned}(A^1 + iA^2)_\lambda &= (j_R^1 - j_L^1)_\lambda + i(j_R^2 - j_L^2)_\lambda \\ &= -\sqrt{2}f\left\{\partial_\lambda\pi^- - \frac{2}{3f^2}\left[\pi^-(\vec\pi\partial_\lambda\vec\pi) - \vec\pi^2\partial_\lambda\pi^-\right] + \alpha_4\frac{M^2}{f^2}\partial_\lambda\pi^- + \dots\right\}\ ,\end{aligned}$$

(other contributions from $\mathcal{L}^{(4)}$ involve more pion fields and do not contribute in this order to the pion decay matrix element.

Using this result and the Feynman rule for the four-pion vertex generated by $\mathcal{L}^{(2)}_{\rm int4}$ the corrected matrix element $\langle \Omega_{\rm hadr}|(J_\lambda^{\rm hadr})^\dagger|\pi^-(\mathbf{q})\rangle = \langle \Omega_{\rm hadr}|A_\lambda^1 + iA_\lambda^2|\pi^-(\mathbf{q})\rangle$ is

$$\langle \Omega_{\rm hadr}|A_\lambda^1 + iA_\lambda^2|\pi^-(\mathbf{q})\rangle = -\mathcal{Z}^{1/2}\sqrt{2}f(-iq_\lambda)\left\{1 - \frac{1}{16\pi^2}\frac{4}{3f^2}a(M) + \alpha_4\frac{M^2}{f^2}\right\}\ .$$

With the result for $\mathcal{Z}$ we finally have

$$\begin{aligned} M_\pi^2 &= M^2\left(1 + \frac{1}{16\pi^2}\frac{1}{2f_\pi^2}a(M) - \alpha_4\frac{M^2}{f^2} + \alpha_3\frac{M^4}{f^2}\right)\ , \\ f_\pi &= f\left(1 - \frac{1}{16\pi^2}\frac{1}{f^2}a(M) + \frac{1}{2}\alpha_4\frac{M^2}{f^2}\right)\ . \end{aligned}$$

The requirement that $M_\pi^2 \equiv M_{\rm phys}^2$ and $f_\pi \equiv f_\pi^{\rm phys}$ are finite fixes the infinite parts of α_3 and α_4:

$$\begin{aligned}
\alpha_3 &= \frac{1}{16\pi^2}\,\frac{3}{2}\eta_{\rm div} + \hat{\alpha}_3 \ , \\
\alpha_4 &= \frac{1}{16\pi^2}\,2\eta_{\rm div} + \hat{\alpha}_4 \ ,
\end{aligned}$$

Since both Lagrangian parameters f and M are also free, to determine the finite parts $\hat{\alpha}_3$ and $\hat{\alpha}_4$ additional observables are needed. Notice, that the $\mathcal{Z}$ factor remains infinite.

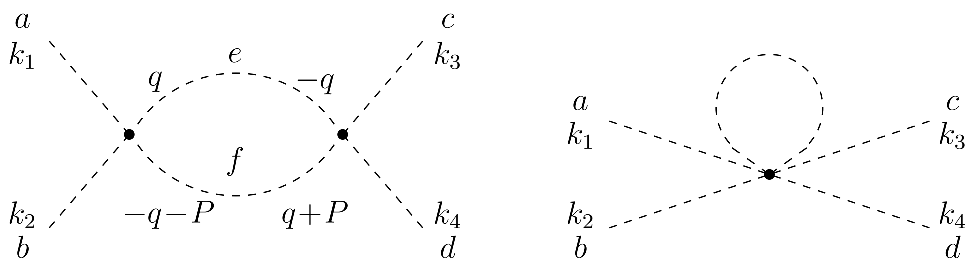


Figure 13: 1-loop diagrams contributing to the pion scattering amplitude at order $\mathcal{O}(p^4)$. To the first diagram giving the factor $\mathcal{A}^{(2;1)}_{(A_s)}$ there are two crossed diagrams giving $\mathcal{A}^{(2;1)}_{(A_t)}$ and $\mathcal{A}^{(2;1)}_{(A_u)}$. All momenta are directed towards the vertices.

Solution to Problem 15.
The general binary scattering amplitude $\mathcal{M}_{ab;cd}(k_1, k_2, k_3, k_4)$ takes the form

$$\mathcal{M}_{ab;cd}(k_1, k_2, k_3, k_4) = \mathcal{A}(s, t, u)\delta^{ab}\delta^{cd} + \mathcal{A}(t, s, u)\delta^{ac}\delta^{bd} + \mathcal{A}(u, t, s)\delta^{ad}\delta^{bc} .$$

The Mandelstam variables s, t and u satisfying the relation $s + t + u = 4M_\pi^2$ are defined by $s = (k_1 + k_2)^2$, $t = (k_1 + k_3)^2$ and $u = (k_1 + k_4)^2$. The function $\mathcal{A}(s, t, u)$ is in the chiral perturbative expansion determined successively as

$$\mathcal{A}(s, t, u) = \mathcal{A}^{(2;0)} + \mathcal{A}^{(2;1)} + \mathcal{A}^{(4;0)} + \dots$$

where the second number in the brackets is the number of loops and the first one indicates part of the Lagrangian from which the vertices are taken (this notation would have to be modified in higher loops). The lowest order pion scattering amplitude has been computed in Problem 11, Set 6. The result, which follows immediately from the four-pion vertex function $f^{(2)}_{abcd}(k_1, k_2, -k_3, -k_4)$, generated by $\mathcal{L}^{(2)}_{\text{int}4}$, has to be modified, however, because the mass shell condition is $k_i^2 = M_\pi^2$ (M_π is the physical pion mass) but to $\mathcal{O}(p^4)$ accuracy $M_\pi^2 \neq M^2$ (see Problem 14). Thus we have

$$\mathcal{A}^{(2;0)}(s, t, u) = -\frac{1}{3f^2}(3s - 4M_\pi^2 + M^2) = -\frac{1}{f^2}(s - M^2) + \frac{4}{3f^2}(M_\pi^2 - M^2) .$$

The one loop contribution $\mathcal{A}^{(2,1)}$ arises from the two graphs shown in figure 13 and from the $\mathcal{Z}^{1/2}$ factors, one per each external line. The contribution $\mathcal{M}^{(2;1)}_{(A_s)}$ of the left diagram of figure 13 to the $\pi\pi$ scattering amplitude $\mathcal{M}^{(2;1)}$ is

$$-i\mathcal{M}^{(2;1)}_{(A_s)} = \frac{(i)^2}{2}\int \frac{d^4q}{(2\pi)^4} \frac{(i)^2}{[q^2 - M^2][(q + P)^2 - M^2]}$$
$$\times \sum_{e,f} f^{(2)}_{abef}(k_1, k_2, q, -q - P)\, f^{(2)}_{efcd}(-q, q + P, k_3, k_4) ,$$

where $P = k_1 + k_2 = -(k_3 + k_4)$ and $1/2$ is the usual combinatoric factor. The two vertex factors $f^{(2)}_{abef}(k_1, k_2, q, -q-P)$ and $f^{(2)}_{efcd}(-q, q+P, k_3, k_4)$ simplify for on-shell[13] momenta k_i to:

$$
\begin{aligned}
f^{(2)}_{abef}(k_1, k_2, q, -q-P) = & \frac{1}{3f^2}[2s - M^2 - 2q^2 - 2q \cdot P]\, \delta_{ab}\delta_{ef} \\
& + \frac{1}{3f^2}[q^2 + q \cdot P + 3(k_1 - k_2) \cdot q - 2k_1 \cdot k_2]\, \delta_{ae}\delta_{bf} \\
& + \frac{1}{3f^2}[q^2 + q \cdot P - 3(k_1 - k_2) \cdot q - 2k_1 \cdot k_2]\, \delta^{af}\delta^{be} \ ,
\end{aligned}
$$

$$
\begin{aligned}
f^{(2)}_{efcd}(-q, q+P, k_3, k_4) = & \frac{1}{3f_\pi^2}[2s - M^2 - 2q^2 - 2q \cdot P]\, \delta_{ef}\delta_{cd} \\
& + \frac{1}{3f_\pi^2}[q^2 + q \cdot P - 3(k_3 - k_4) \cdot q - 2k_3 \cdot k_4]\, \delta^{ec}\delta^{fd} \\
& + \frac{1}{3f_\pi^2}[q^2 + q \cdot P + 3(k_3 - k_4) \cdot q - 2k_3 \cdot k_4]\, \delta^{de}\delta^{fc} \ ,
\end{aligned}
$$

where we have replaced P^2 by the Mandelstam variable s.

Multiplying these expressions we get three different structures: $\delta^{ab}\delta^{cd}$, $\delta^{ac}\delta^{bd}$ and $\delta^{ad}\delta^{bc}$. The first one arises from multiplying the first term in $f^{(2)}_{abef}(k_1, k_2, q, -q-P)$ by the last two in $f^{(2)}_{efcd}(-q, q+P, k_3, k_4)$ and the first one in $f^{(2)}_{efcd}(-q, q+P, k_3, k_4)$ by the last two in $f^{(2)}_{abef}(k_1, k_2, q, -q-P)$ and reads

$$
\begin{aligned}
& \frac{1}{9f^4}\delta^{ab}\delta^{cd}[2s - M^2 - 2q^2 - 2q \cdot P][2s + 5M^2 - 2q^2 - 2q \cdot P] \\
& = \frac{1}{9f^4}\delta^{ab}\delta^{cd}[3s - 3M^2 - (q^2 - M^2) - ((q+P)^2 - M^2)] \\
& \qquad \times [3s + 3M^2 - (q^2 - M_\pi^2) - ((q+P)^2 - M^2)] \ .
\end{aligned}
$$

Thus the $\delta^{ab}\delta^{cd}$ term in $\mathcal{M}^{(2;1)}_{(A_s)}$ is given by

$$
\begin{aligned}
& \frac{1}{18f^4}\delta^{ab}\delta^{cd}\int \frac{d^4q}{(2\pi)^4}\frac{(i)^2}{[q^2 - M^2][(q+P)^2 - M^2]} \\
& \qquad \Big[9(s^4 - M^4) - 6s(q^2 - M^2) - 6s((q+P)^2 - M^2)
\end{aligned}
$$

[13] To the order $\mathcal{O}(p^4)$ in the one loop diagram one can identify M_π with M.

$$
+(q^2-M^2)^2+((q+P)^2-M^2)^2+2(q^2-M^2)((q+P)^2-M^2)\Big]
$$
$$
=\frac{1}{16\pi^2}\frac{1}{18f^4}\delta^{ab}\delta^{cd}\left[9(s^4-M^4)\,b_0(s,M,M)-10s\,a(M)\right]\ .
$$

We have used the fact that in the dimensional regularization $\int d^dq=0$ and

$$
i\int\frac{d^4q}{(2\pi)^4}\left[\frac{(q+P)^2-M^2}{q^2-M^2}+\frac{q^2-M^2}{(q+P)^2-M^2}\right]=i\int\frac{d^4q}{(2\pi)^4}\frac{2P^2}{q^2-M^2}=\frac{1}{16\pi^2}\,2s\,a(M)\ ,
$$

as follows from shifting the integration variable $q+P\to q$ in the second term.

The structure $\delta^{ac}\delta^{bd}$ comes from mutiplying the second term in $f^{(2)}_{abef}(k_1,k_2,q,-q-P)$ by the second one in $f^{(2)}_{efcd}(-q,q+P,k_3,k_4)$ and the third term in $f^{(2)}_{abef}(k_1,k_2,q,-q-P)$ by the third one in $f^{(2)}_{efcd}(-q,q+P,k_3,k_4)$. The multiplication gives

$$
\frac{1}{9f^4}\delta_{ac}\delta_{bd}[-18(k_1-k_2)\cdot q\,(k_3-k_4)\cdot q+2(q\cdot P+q^2)^2
$$
$$
-4(k_1\cdot k_2)(q\cdot P+q^2)-4(k_3\cdot k_4)(q\cdot P+q^2)+8(k_1\cdot k_2)(k_3\cdot k_4)]\ ,
$$

Using the decomposition

$$
q^2+q\cdot P=\frac{1}{2}(q^2-M^2)+\frac{1}{2}((q+P)^2-M^2)+M^2-\frac{1}{2}P^2\ ,
$$

and the integral[14]

$$
\int\frac{d^4q}{(2\pi)^4}\frac{iq^\mu q^\nu}{(q^2-M^2)((q+P)^2-M^2)}=\frac{g^{\mu\nu}}{16\pi^2}\left\{\frac{1}{6}a(M)-\frac{1}{12}(P^2-4M^2)\,b_0(s,M,M)\right.
$$
$$
\left.+\frac{1}{18}P^2-\frac{1}{3}M^2\right\}+P^\mu P^\nu\{\ldots\}\ ,
$$

we get for the $\delta^{ac}\delta^{bd}$ term in $\mathcal{M}^{(2;1)}_{(A_s)}$ the expression

$$
\frac{1}{16\pi^2}\frac{1}{18f^4}\delta^{ac}\delta^{bd}\left\{(t-u)\left[\frac{3}{2}(s-4M^2)\,b_0(s,M,M)-3a(M)-s+6M^2\right]\right.
$$
$$
\left.+\frac{9}{2}(s-2M^2)^2\,b_0(s,M,M)-(5s-12M^2)\,a(M)\right\}\ .
$$

[14] The terms proportional to $P^\mu P^\nu=-(k_1+k_2)^\mu(k_3+k_4)^\nu$ do not contribute to the amplitude because they are contracted with $(k_1-k_2)_\mu$ or $(k_3-k_3)_\nu$ and vanish on-shell.

We have used the fact that $(k_1 - k_2) \cdot (k_3 - k_4) = t - u$.

Finally, the structure $\delta^{ad}\delta^{bc}$ in $\mathcal{M}^{(2;1)}_{(A_s)}$ results from multiplying the second term in $f^{(2)}_{abef}(k_1, k_2, q, -q-P)$ by the third one in $f^{(2)}_{efcd}(-q, q+P, k_3, k_4)$ and the third term in $f^{(2)}_{abef}(k_1, k_2, q, -q-P)$ by the second one in $f^{(2)}_{efcd}(-q, q+P, k_3, k_4)$. This is very similar to the calculation above and gives for the $\delta^{ad}\delta^{bc}$ term in $\mathcal{M}^{(2;1)}_{(A_s)}$:

$$\frac{1}{16\pi^2}\frac{1}{18f^4}\delta^{ac}\delta^{bd}\left\{(u-t)\left[\frac{3}{2}(s-4M^2)\, b_0(s,M,M) - 3a(M) - s + 6M^2\right]\right.$$
$$\left. + \frac{9}{2}(s-2M^2)^2\, b_0(s,M,M) - (5s - 12M^2)\, a(M)\right\} .$$

The contribution $\mathcal{M}^{(2;1)}_{(A_t)}$ of the t-channel diagram (in which k_1 and k_3 are in the same vertex) crossed with respect to the one shown in figure 13 can be obtained by making in $\mathcal{M}^{(2;1)}_{(A_s)}$ the substitutions: $b \leftrightarrow c$, $k_2 \leftrightarrow k_3$, $s \leftrightarrow t$, $u \leftrightarrow u$. Similarly, the contribution $\mathcal{M}^{(2;1)}_{(A_u)}$ of the u-channel diagram (in which k_1 and k_4 are in the same vertex) can be obtained by making in $\mathcal{M}^{(2;1)}_{(A_s)}$ the substitutions: $b \leftrightarrow d$, $k_2 \leftrightarrow k_4$, $s \leftrightarrow u$, $t \leftrightarrow t$. Adding the contributions $\mathcal{M}^{(2;1)}_{(A_s)}$, $\mathcal{M}^{(2;1)}_{(A_t)}$ and $\mathcal{M}^{(2;1)}_{(A_u)}$ get for the function $\mathcal{A}^{(2;1)}_{(A)}$ the expression

$$\mathcal{A}^{(2;1)}_{(A)}(s,t,u) = \frac{1}{16\pi^2}\frac{1}{18f^4}\left\{9(s^2 - M^4)\, b_0(s,M,M) - 10s\, a(M)\right.$$
$$+ (s-u)\left[\frac{3}{2}(t-4M^2)\, b_0(t,M,M) - 3a(M) - t + 6M^2\right]$$
$$+ \frac{9}{2}(t-2M^2)^2\, b_0(t,M,M) - (5t-12M^2)\, a(M)$$
$$+ (s-t)\left[\frac{3}{2}(u-4M^2)\, b_0(u,M,M) - 3a(M) - u + 6M^2\right]$$
$$\left. + \frac{9}{2}(u-2M^2)^2\, b_0(u,M,M) - (5u-12M^2)\, a(M)\right\} .$$

Terms of the sixth order in π in $\mathcal{L}^{(2)}$ read

$$\mathcal{L}^{(2)}_{\text{int6}} = \frac{1}{45f^4}\left[\vec{\pi}^4\,(\partial_\mu\vec{\pi}\partial^\mu\vec{\pi}) - \vec{\pi}^2\,(\vec{\pi}\partial_\mu\vec{\pi})(\vec{\pi}\partial^\mu\vec{\pi})\right] - \frac{1}{720}\frac{M^2}{f^4}\,\vec{\pi}^6 .$$

They give rise to the six-pion vertex (needed in the right diagram of figure 13) with the Feynman rule $if^{(2)}_{abcdef}(k_1,k_2,k_3,k_4,k_5,k_6)$. The vertex function $if^{(2)}_{abcdef}$ can be conveniently split into three parts corresponding to the three terms in $\mathcal{L}^{(2)}_{\rm int6}$. Correspondingly, the one loop contribution to the $\pi\pi$ scattering amplitude given by

$$-i\mathcal{M}^{(2;1)}_{(B)} = \frac{1}{2}\sum_e \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-M^2}\; if^{(2)}_{abcdee}(k_1,k_2,k_3,k_4,q,-q) \; .$$

can also be split into three contributions $\mathcal{M}^{(2;1)}_{(B_1)}$, $\mathcal{M}^{(2;1)}_{(B_2)}$ and $\mathcal{M}^{(2;1)}_{(B_3)}$.

The last part in $if^{(2)}_{abcdef}$ is[15]

$$\begin{aligned} if^{(2)}_{abcdef}(k_1,k_2,k_3,k_4,k_5,k_6) = -\frac{i}{15}\frac{M^2}{f^4}\Big[&\delta^{ef}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right)\\ &+\delta^{ed}\left(\delta^{ab}\delta^{cf}+\delta^{ac}\delta^{bf}+\delta^{af}\delta^{bc}\right)\\ &+\delta^{ec}\left(\delta^{ab}\delta^{fd}+\delta^{af}\delta^{bd}+\delta^{ad}\delta^{bf}\right)\\ &+\delta^{eb}\left(\delta^{af}\delta^{cd}+\delta^{ac}\delta^{fd}+\delta^{ad}\delta^{fc}\right)\\ &+\delta^{ea}\left(\delta^{fb}\delta^{cd}+\delta^{fc}\delta^{bd}+\delta^{fd}\delta^{bc}\right)\Big]\end{aligned}$$

Contracting two indices gives

$$\sum_e if^{(2)}_{abcdee} = -i\frac{7}{15}\frac{M^2}{f^4}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right) \; ,$$

and the corresponding contribution $\mathcal{M}^{(2;1)}_{(B_3)}$ is

$$\mathcal{M}^{(2;1)}_{(B_3)} = \frac{7}{30}\frac{M^2}{f^4}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right)\, a(M) \; .$$

The part of $if^{(2)}_{abcdef}(k_1,k_2,k_3,k_4,k_5,k_6)$ resulting from the first term in $\mathcal{L}^{(2)}_{\rm int6}$ is

$$\begin{aligned} if^{(2)}_{abcdef}(k_1,k_2,k_3,k_4,k_5,k_6) = -i\frac{16}{45f^4}\Big[&(k_5\cdot k_6)\delta^{ef}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right)\\ &+(k_5\cdot k_4)\delta^{ed}\left(\delta^{ab}\delta^{cf}+\delta^{ac}\delta^{bf}+\delta^{af}\delta^{bc}\right)\\ &+(k_5\cdot k_3)\delta^{ec}\left(\delta^{ab}\delta^{fd}+\delta^{af}\delta^{bd}+\delta^{ad}\delta^{bf}\right)\\ &+(k_5\cdot k_2)\delta^{eb}\left(\delta^{af}\delta^{cd}+\delta^{ac}\delta^{fd}+\delta^{ad}\delta^{fc}\right)\end{aligned}$$

[15] There are 15 different triplets of pairs that can be formed out of 6 indices; if all indices assume the same value the numerical coefficient in $if^{(3)}_{aaaaaa}$ should be $6!/720 = 1$.

$$
\begin{aligned}
&+(k_5\cdot k_1)\delta^{ea}\left(\delta^{fb}\delta^{cd}+\delta^{fc}\delta^{bd}+\delta^{fd}\delta^{bc}\right)\\
&+(k_6\cdot k_4)\delta^{fd}\left(\delta^{ab}\delta^{ce}+\delta^{ac}\delta^{be}+\delta^{ae}\delta^{bc}\right)\\
&+(k_6\cdot k_3)\delta^{fc}\left(\delta^{ab}\delta^{ed}+\delta^{ae}\delta^{bd}+\delta^{ad}\delta^{be}\right)\\
&+(k_6\cdot k_2)\delta^{fb}\left(\delta^{ae}\delta^{cd}+\delta^{ac}\delta^{ed}+\delta^{ad}\delta^{ec}\right)\\
&+(k_6\cdot k_1)\delta^{fa}\left(\delta^{eb}\delta^{cd}+\delta^{ec}\delta^{bd}+\delta^{ed}\delta^{bc}\right)\\
&+(k_4\cdot k_3)\delta^{dc}\left(\delta^{ab}\delta^{ef}+\delta^{ae}\delta^{bf}+\delta^{af}\delta^{be}\right)\\
&+(k_4\cdot k_2)\delta^{db}\left(\delta^{ae}\delta^{cf}+\delta^{ac}\delta^{ef}+\delta^{af}\delta^{ec}\right)\\
&+(k_4\cdot k_1)\delta^{da}\left(\delta^{eb}\delta^{cf}+\delta^{ec}\delta^{bf}+\delta^{ef}\delta^{bc}\right)\\
&+(k_3\cdot k_2)\delta^{cb}\left(\delta^{ae}\delta^{df}+\delta^{ad}\delta^{ef}+\delta^{af}\delta^{ed}\right)\\
&+(k_3\cdot k_1)\delta^{ca}\left(\delta^{eb}\delta^{df}+\delta^{ed}\delta^{bf}+\delta^{ef}\delta^{bd}\right)\\
&+(k_2\cdot k_1)\delta^{ba}\left(\delta^{ec}\delta^{df}+\delta^{ed}\delta^{cf}+\delta^{ef}\delta^{cd}\right)\Big]
\end{aligned}
$$

With indices contracted as is required, this vertex reads

$$
\begin{aligned}
\sum_e if^{(2)}_{abcdee}(k_1,k_2,k_3,k_4,q,-q) = -i\frac{16}{45f^4}[&-3q^2\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right)\\
&+5(k_2\cdot k_1+k_4\cdot k_3)\delta^{ab}\delta^{cd}\\
&+5(k_3\cdot k_1+k_4\cdot k_2)\delta^{db}\delta^{ca}\\
&+5(k_4\cdot k_1+k_3\cdot k_2)\delta^{ad}\delta^{bc}]\ .
\end{aligned}
$$

All terms with $q\cdot k_i$ has canceled out. The corresponding contribution to $\mathcal{M}^{(2;1)}_{(B_1)}$ is

$$
\begin{aligned}
\mathcal{M}^{(2;1)}_{(B_1)} = \frac{8}{45f^4}\Big[&-3M^2\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right)\\
&+5(s-2M^2)\delta^{ab}\delta^{cd}+5(t-2M^2)\delta^{db}\delta^{ca}+5(u-2M^2)\delta^{ad}\delta^{bc}\Big]\ a(M)\ .
\end{aligned}
$$

The middle term of $\mathcal{L}^{(2)}_{\text{int6}}$ gives the following term in $if^{(2)}_{abcdef}(k_1,k_2,k_3,k_4,k_5,k_6)$

$$
\begin{aligned}
if^{(2)}_{abcdef}(k_1,k_2,k_3,k_4,k_5,k_6) = i\frac{4}{45f^4}\Big\{\delta^{fe}\Big[&(k_1\cdot k_2+k_3\cdot k_4)(\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})\\
&+(k_1\cdot k_3+k_2\cdot k_4)(\delta^{ab}\delta^{cd}+\delta^{ad}\delta^{bc})\\
&+(k_1\cdot k_4+k_2\cdot k_3)(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\delta^{fd}\Big[(k_1\cdot k_2+k_3\cdot k_5)(\delta^{ac}\delta^{be}+\delta^{ae}\delta^{bc})\\
&\quad+(k_1\cdot k_3+k_2\cdot k_5)(\delta^{ab}\delta^{ce}+\delta^{ae}\delta^{bc})\\
&\quad+(k_1\cdot k_5+k_2\cdot k_3)(\delta^{ab}\delta^{ce}+\delta^{ac}\delta^{be})\Big]\\
&+\delta^{fc}\Big[(k_1\cdot k_2+k_4\cdot k_5)(\delta^{ad}\delta^{be}+\delta^{ae}\delta^{bd})\\
&\quad+(k_1\cdot k_4+k_2\cdot k_5)(\delta^{ab}\delta^{de}+\delta^{ae}\delta^{bd})\\
&\quad+(k_1\cdot k_5+k_2\cdot k_4)(\delta^{ab}\delta^{de}+\delta^{ad}\delta^{be})\Big]\\
&+\delta^{fb}\Big[(k_1\cdot k_3+k_4\cdot k_5)(\delta^{ad}\delta^{ce}+\delta^{ae}\delta^{cd})\\
&\quad+(k_1\cdot k_4+k_3\cdot k_5)(\delta^{ac}\delta^{de}+\delta^{ae}\delta^{cd})\\
&\quad+(k_1\cdot k_5+k_3\cdot k_4)(\delta^{ac}\delta^{de}+\delta^{ad}\delta^{ce})\Big]\\
&+\delta^{fa}\Big[(k_2\cdot k_3+k_4\cdot k_5)(\delta^{bd}\delta^{ce}+\delta^{be}\delta^{cd})\\
&\quad+(k_2\cdot k_4+k_3\cdot k_5)(\delta^{bc}\delta^{de}+\delta^{be}\delta^{cd})\\
&\quad+(k_2\cdot k_5+k_3\cdot k_4)(\delta^{bc}\delta^{de}+\delta^{bd}\delta^{ce})\Big]\\
&+\delta^{ed}\Big[(k_1\cdot k_2+k_3\cdot k_6)(\delta^{ac}\delta^{bf}+\delta^{af}\delta^{bc})\\
&\quad+(k_1\cdot k_3+k_2\cdot k_6)(\delta^{ab}\delta^{cf}+\delta^{af}\delta^{bc})\\
&\quad+(k_1\cdot k_6+k_2\cdot k_3)(\delta^{ab}\delta^{cf}+\delta^{ac}\delta^{bf})\Big]\\
&+\delta^{ec}\Big[(k_1\cdot k_2+k_4\cdot k_6)(\delta^{ad}\delta^{bf}+\delta^{af}\delta^{bd})\\
&\quad+(k_1\cdot k_4+k_2\cdot k_6)(\delta^{ab}\delta^{df}+\delta^{af}\delta^{bd})\\
&\quad+(k_1\cdot k_6+k_2\cdot k_4)(\delta^{ab}\delta^{df}+\delta^{ad}\delta^{bf})\Big]\\
&+\delta^{eb}\Big[(k_1\cdot k_3+k_4\cdot k_6)(\delta^{ad}\delta^{cf}+\delta^{af}\delta^{cd})\\
&\quad+(k_1\cdot k_4+k_3\cdot k_6)(\delta^{ac}\delta^{df}+\delta^{af}\delta^{cd})\\
&\quad+(k_1\cdot k_6+k_3\cdot k_4)(\delta^{ac}\delta^{df}+\delta^{ad}\delta^{cf})\Big]\\
&+\delta^{ea}\Big[(k_2\cdot k_3+k_4\cdot k_6)(\delta^{bd}\delta^{cf}+\delta^{bf}\delta^{cd})\\
&\quad+(k_2\cdot k_4+k_3\cdot k_6)(\delta^{bc}\delta^{df}+\delta^{bf}\delta^{cd})\\
&\quad+(k_2\cdot k_6+k_3\cdot k_4)(\delta^{bc}\delta^{df}+\delta^{bd}\delta^{cf})\Big]\\
&+\delta^{dc}\Big[(k_1\cdot k_2+k_5\cdot k_6)(\delta^{ae}\delta^{bf}+\delta^{af}\delta^{be})\\
&\quad+(k_1\cdot k_5+k_2\cdot k_6)(\delta^{ab}\delta^{ef}+\delta^{af}\delta^{be})\\
&\quad+(k_1\cdot k_6+k_2\cdot k_5)(\delta^{ab}\delta^{ef}+\delta^{ae}\delta^{bf})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\delta^{db}\Big[(k_1\cdot k_3+k_5\cdot k_6)(\delta^{ae}\delta^{cf}+\delta^{af}\delta^{ce})\\
&\quad+(k_1\cdot k_5+k_3\cdot k_6)(\delta^{ac}\delta^{ef}+\delta^{af}\delta^{ce})\\
&\quad+(k_1\cdot k_6+k_3\cdot k_5)(\delta^{ac}\delta^{ef}+\delta^{ae}\delta^{cf})\Big]\\
&+\delta^{da}\Big[(k_2\cdot k_3+k_5\cdot k_6)(\delta^{be}\delta^{cf}+\delta^{bf}\delta^{ce})\\
&\quad+(k_2\cdot k_5+k_3\cdot k_6)(\delta^{bc}\delta^{ef}+\delta^{bf}\delta^{ce})\\
&\quad+(k_2\cdot k_6+k_3\cdot k_5)(\delta^{bc}\delta^{ef}+\delta^{be}\delta^{cf})\Big]\\
&+\delta^{cb}\Big[(k_1\cdot k_4+k_5\cdot k_6)(\delta^{ae}\delta^{df}+\delta^{af}\delta^{de})\\
&\quad+(k_1\cdot k_5+k_4\cdot k_6)(\delta^{ad}\delta^{ef}+\delta^{af}\delta^{de})\\
&\quad+(k_1\cdot k_6+k_4\cdot k_5)(\delta^{ad}\delta^{ef}+\delta^{ae}\delta^{df})\Big]\\
&+\delta^{ca}\Big[(k_2\cdot k_4+k_5\cdot k_6)(\delta^{be}\delta^{df}+\delta^{bf}\delta^{de})\\
&\quad+(k_2\cdot k_5+k_4\cdot k_6)(\delta^{bd}\delta^{ef}+\delta^{bf}\delta^{de})\\
&\quad+(k_2\cdot k_6+k_4\cdot k_5)(\delta^{bd}\delta^{ef}+\delta^{be}\delta^{df})\Big]\\
&+\delta^{ba}\Big[(k_3\cdot k_4+k_5\cdot k_6)(\delta^{ce}\delta^{df}+\delta^{cf}\delta^{de})\\
&\quad+(k_3\cdot k_5+k_4\cdot k_6)(\delta^{cd}\delta^{ef}+\delta^{cf}\delta^{de})\\
&\quad+(k_3\cdot k_6+k_4\cdot k_5)(\delta^{cd}\delta^{ef}+\delta^{ce}\delta^{df})\Big]\Big\}
\end{aligned}
$$

$\sum_e if^{(2)}_{abcdee}(k_1,k_2,k_3,k_4,q,-q)$ has many terms. Dropping terms linear in q as they will not contribute to $\mathcal{M}^{(2;1)}_{(B_2)}$ it reads

$$
\begin{aligned}
\sum_e if^{(2)}_{abcdee}(k_1,k_2,k_3,k_4,q,-q) = i\frac{4}{45f^4}\Big\{&7(k_1+k_2)\cdot(k_3+k_4)\delta^{ab}\delta^{cd}\\
&+7(k_1+k_3)\cdot(k_2+k_4)\delta^{ac}\delta^{bd}\\
&+7(k_1+k_4)\cdot(k_2+k_3)\delta^{ad}\delta^{bc}\\
&+2(k_1\cdot k_2+k_3\cdot k_4-2q^2)\delta^{ab}\delta^{cd}\\
&+2(k_1\cdot k_3+k_2\cdot k_4-2q^2)\delta^{ac}\delta^{bd}\\
&+2(k_1\cdot k_4+k_2\cdot k_3-2q^2)\delta^{ad}\delta^{bc}\Big\}\ .
\end{aligned}
$$

so that taking the external momenta on shell we get

$$
\begin{aligned}
\sum_e if^{(2)}_{abcdee}(k_1,k_2,k_3,k_4,q,-q) = -i\frac{4}{45f^4}\Big\{&[5s+4M^2+4q^2]\delta^{ab}\delta^{cd}\\
&+[5t+4M^2+4q^2]\delta^{ac}\delta^{bd}
\end{aligned}
$$

$$+[5u + 4M^2 + 4q^2]\delta^{ad}\delta^{bc} \} \ .$$

With this vertex the corresponding contribution $\mathcal{M}^{(2;1)}_{(B_2)}$ is

$$\mathcal{M}^{(2;1)}_{(B_2)} = \frac{2}{45f^4}\left[(5s + 8M^2)\delta^{ab}\delta^{cd} + (5t + 8M^2)\delta^{ac}\delta^{bd} + (5u + 8M^2)\delta^{ad}\delta^{bc}\right] a(M) \ .$$

Finally, we consider the tree level contribution of $\mathcal{L}^{(4)}$. The relevant interaction term derived in Problem 14 reads

$$\begin{aligned}\mathcal{L}^{(4)}_{\text{int}4} &= \alpha_3\frac{M^4}{6f^4}\vec{\pi}^4 + \alpha_4\frac{M^2}{12f^4}\left[2(\vec{\pi}\partial_\mu\vec{\pi})(\vec{\pi}\partial^\mu\vec{\pi}) - 5\vec{\pi}^2(\partial_\mu\vec{\pi}\partial^\mu\vec{\pi})\right] \\ &+\frac{1}{4f^2}\alpha_1(\partial_\mu\vec{\pi}\partial^\mu\vec{\pi})^2 + \frac{1}{4f^2}\alpha_2(\partial_\mu\vec{\pi}\partial_\nu\vec{\pi})(\partial^\mu\vec{\pi}\partial^\nu\vec{\pi}) + \ldots\end{aligned}$$

It adds to the vertex function $f^{(2)}_{abcd}(k_1, k_2, k_3, k_4)$ the following higher order term

$$\begin{aligned}f^{(4)}_{abcd}(k_1, k_2, k_3, k_4) = &\frac{1}{f_\pi^4}\delta^{ab}\delta^{cd}\,[8\alpha_1\,(k_1\cdot k_2)(k_3\cdot k_4) \\ &\qquad +4\alpha_2\,(k_1\cdot k_3)(k_2\cdot k_4) + 4\alpha_2\,(k_1\cdot k_4)(k_2\cdot k_3)] \\ &+\frac{1}{f_\pi^4}\delta^{ac}\delta^{bd}\,[8\alpha_1\,(k_1\cdot k_3)(k_2\cdot k_4) \\ &\qquad +4\alpha_2\,(k_1\cdot k_2)(k_3\cdot k_4) + 4\alpha_2\,(k_1\cdot k_4)(k_2\cdot k_3)] \\ &+\frac{1}{f_\pi^4}\delta^{ad}\delta^{bc}\,[8\alpha_1\,(k_1\cdot k_4)(k_2\cdot k_3) \\ &\qquad +4\alpha_2\,(k_1\cdot k_2)(k_3\cdot k_4) + 4\alpha_2\,(k_1\cdot k_3)(k_2\cdot k_4)] \\ &+\alpha_3\frac{4M_\pi^4}{3f_\pi^4}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) \\ &+\alpha_4\frac{M_\pi^2}{3f_\pi^4}\Big\{[5(k_1\cdot k_2) + 5(k_3\cdot k_4) - (k_1+k_2)\cdot(k_3+k_4)]\,\delta^{ab}\delta^{cd} \\ &\qquad +[5(k_1\cdot k_3) + 5(k_2\cdot k_4) - (k_1+k_3)\cdot(k_2+k_4)]\,\delta^{ac}\delta^{bd} \\ &\qquad +[5(k_1\cdot k_4) + 5(k_2\cdot k_3) - (k_1+k_4)\cdot(k_2+k_3)]\,\delta^{ad}\delta^{bc}\ \Big\} \ .\end{aligned}$$

We can now summarize all the contributions to $\mathcal{A}(s, t, u)$. After rearranging terms the contribution $\mathcal{A}^{(2;1)}_{(A)}(s, t, u)$ reads

$$\mathcal{A}^{(2;1)}_{(A)}(s, t, u) = \frac{1}{16\pi^2}\frac{1}{18f^4}\Big\{-(14s - 16M^2)a(M) + 9(s^2 - M^4)\,b_0(s, M, M)$$

$$+(3t^2-3tu-6M^2t+12M^2u-6M^4)b_0(t,M,M)$$
$$+(3u^2-3tu-6M^2u+12M^2t-6M^4)b_0(u,M,M)$$
$$+t^2+u^2+4tu+12M^2s-10M^2(t+u)\Big\} \ .$$

Collecting terms arising from the second diagram in figure 13 we have

$$\mathcal{A}^{(2;1)}_{(B)}(s,t,u) = \frac{1}{16\pi^2}\left\{\frac{1}{45f^4}(50s-88M^2)\,a(M) + \frac{7}{30f^4}M^2\,a(M)\right\} \ .$$

Using the result from the preceding Problem, the contribution of the $\mathcal{Z}$ factors amounts to

$$\mathcal{A}^{(2;1)}_{(\mathcal{Z})}(s,t,u) = -\frac{1}{16\pi^2}\frac{4}{3f^4}(s-M^2)\,a(M) + 2\alpha_4\frac{M^2}{f^4}(s-M^2) \ .$$

Finally, the tree level contribution of vertices from $\mathcal{L}^{(4)}$ is obtained by setting in $f^{(4)}_{abcd}(k_1,k_2,k_3,k_4)$ the momenta on-shell and reads

$$\begin{aligned}\mathcal{A}^{(4;0)}(s,t,u) = {} & -\frac{1}{2f^4}[4\alpha_1(s-2M^2)^2+\alpha_2(s^2+(t-u)^2)] \\ & -\alpha_3\frac{4}{3}\frac{M^4}{f^4} - \alpha_4\frac{M^4}{f^4}\left(2s-\frac{10}{3}M^2\right) \ .\end{aligned}$$

We first study cancelation of the divergences. Since $[b_0(s,M,M)]_{\rm div} = \eta_{\rm div}$ and $[a(M)]_{\rm div} = M^2\eta_{\rm div}$, upon using judiciously the relation $s+t+u = 4M_\pi^2 \approx 4M^2$, we find

$$\begin{aligned}
\mathcal{A}^{(2;1)}_{(A)} &= \frac{1}{16\pi^2}\frac{1}{18f^4}[12(t^2+u^2+tu)+52M^2s-125M^4]\eta_{\rm div} + \text{finite} \ , \\
\mathcal{A}^{(2;1)}_{(B)} &= \frac{1}{16\pi^2}\frac{1}{90f^4}[100M^2s-155M^4]\eta_{\rm div} + \text{finite} \ , \\
\mathcal{A}^{(2;1)}_{(\mathcal{Z})} &= \frac{1}{16\pi^2}\frac{4}{3f^4}(M^4-M^2s)\eta_{\rm div} + 2\alpha_4\frac{M^2}{f^4}(s-M^2) + \text{finite} \ , \\
\mathcal{A}^{(4;0)} &= -\frac{1}{f^4}\left[(2\alpha_1+\alpha_2)(t^2+u^2)+4\alpha_1 tu\right] \\
&\quad -\frac{1}{f^4}\left[2\alpha_1(4M^2s-12M^4)+\alpha_2(4M^2s-8M^4)\right] \\
&\quad -\alpha_3\frac{4}{3}\frac{M^4}{f^4} - \alpha_4\frac{M^4}{f^4}\left(2s-\frac{10}{3}M^2\right) \ .
\end{aligned}$$

Cancelation of the divergent terms of order s^2 (i.e. $\propto t^2$, $\propto u^2$ and $\propto tu$) fixes infinite parts of α_1 and α_2:

$$\alpha_1 = \frac{1}{16\pi^2}\frac{1}{6}\eta_{\rm div} + \hat{\alpha}_1 \ ,$$
$$\alpha_2 = \frac{1}{16\pi^2}\frac{1}{3}\eta_{\rm div} + \hat{\alpha}_2 \ .$$

The infinite parts of α_3 and α_4 have been fixed (see Problem 14) by M_π and f_π and the cancelation of the remaining divergences in the scattering amplitude is a nontrivial check of the consistency of the method. Collecting divergent terms proportional to $M^2 s$ we get

$$\frac{1}{16\pi^2}\left\{\left(\frac{52}{18}+\frac{50}{45}-\frac{4}{3}\right)\eta_{\rm div} + 2\alpha_4 - 8\alpha_1 - 4\alpha_2 - 2\alpha_4\right\}\frac{M^2 s}{f^4} \ ,$$

and it is easy to see that this sum is finite. Finally, the divergent terms proportional to M^4 are:

$$\frac{1}{16\pi^2}\left\{\left(-\frac{125}{18}-\frac{155}{90}+\frac{4}{3}\right)\eta_{\rm div} - 2\alpha_4 + 24\alpha_1 + 8\alpha_2 - \frac{4}{3}\alpha_3 + \frac{10}{3}\alpha_4\right\}\frac{M^4}{f^4} \ ,$$

which as also easily seen to be finite with infinite parts of α_3 and α_4 fixed be f_π and M_π (see Problem 14).

Using the relation between M^2 and M_π^2 found in Problem 14 the tree level contribution $\mathcal{A}^{(2;0)}(s,t,u)$ takes the form

$$\mathcal{A}^{(2;0)}(s,t,u) = -\frac{1}{f^2}(s-M^2) + \frac{1}{16\pi^2}\frac{2M^2}{3f^4}\,a(M) + \frac{4M^4}{3f^4}(\alpha_3-\alpha_4) \ ,$$

Combining then all terms the final form of the A factor is

$$\begin{aligned}
\mathcal{A}(s,t,u) = &-\frac{1}{f^2}(s-M^2) + \frac{1}{16\pi^2}\frac{1}{f^4}\left(-s+\frac{7}{6}\right)a(M) \\
&+\frac{1}{16\pi^2}\frac{1}{18f^4}\Big\{9(s^2-M^4)\,b_0(s,M,M) \\
&\qquad +(3t^2-3tu-6M^2t+12M^2u-6M^4)b_0(t,M,M) \\
&\qquad +(3u^2-3tu-6M^2u+12M^2t-6M^4)b_0(u,M,M) \\
&\qquad +t^2+u^2+4tu+22M^2s-40M^4\Big\} \\
&-\frac{1}{2f^4}[4\alpha_1(s-2M^2)^2+\alpha_2(s^2+(t-u)^2)] \ .
\end{aligned}$$

Thus, when the amplitude is expressed in terms of f^2 and M^2 (and not in terms of the physical quantities f_π^2 and M_π^2) it is independent of $\hat{\alpha}_3$ and $\hat{\alpha}_4$ which in turn affect the relations between the Lagrangian parameters f^2, M^2 and the physical quantities f_π^2 and M_π^2.

Since the finite parts $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are yet to be determined, one can simplify the formulae by taking

$$\eta_{\rm div} = \frac{1}{\epsilon} - 1 - \gamma_{\rm E} + \ln\frac{1}{4\pi} \ ,$$

so that all terms with the function $a(M)$ drop out if one choses $\mu = M$. As a result, the finite constants $\hat{\alpha}_i$ should be then understood to be normalized at $\mu = M$. For the $\pi\pi$ scattering amplitude we then have

$$\begin{aligned}\mathcal{A}(s,t,u) = &-\frac{1}{f^2}(s-M^2)\\ &+\frac{1}{16\pi^2}\frac{1}{18f^4}\Big\{9(s^2-M^4)\,\tilde{b}_0(s,M,M)\\ &\qquad +(3t^2-3tu-6M^2t+12M^2u-6M^4)\tilde{b}_0(t,M,M)\\ &\qquad +(3u^2-3tu-6M^2u+12M^2t-6M^4)\tilde{b}_0(u,M,M)\\ &\qquad +13t^2+13u^2+16tu+88M^2s-181M^4\Big\}\\ &-\frac{1}{2f^4}[4\hat{\alpha}_1(s-2M^2)^2+\hat{\alpha}_2(s^2+(t-u)^2)] \ ,\end{aligned}$$

where the function $\tilde{b}_0(s,M,M)$ is given by

$$\tilde{b}_0(s,M,M) = \int_0^1 dx\ln\left(1+\frac{x(x-1)s-i0}{M^2}\right) \ .$$

To check the unitarity we recall that (Problem 6, set 6) that the isospin amplitudes are given by

$$\begin{aligned}\mathcal{M}^{I=0} &= 3\mathcal{A}(s,t,u)+\mathcal{A}(t,s,u)+\mathcal{A}(u,t,s) \ ,\\ \mathcal{M}^{I=1} &= \mathcal{A}(t,s,u)-\mathcal{A}(u,t,s) \ ,\\ \mathcal{M}^{I=2} &= \mathcal{A}(t,s,u)+\mathcal{A}(u,t,s) \ .\end{aligned}$$

So that in the lowest order

$$\mathcal{M}^{I=0} = -\frac{2s-M^2}{f^2} \ , \qquad \mathcal{M}^{I=1} = \frac{u-t}{f^2} \ , \qquad \mathcal{M}^{I=2} = \frac{2M^2-s}{f^2} \ ,$$

and the lowest order isospin partial amplitudes read

$$\mathcal{T}_{J=0}^{I=0} = -\frac{2s - M^2}{16\pi f^2} \ , \qquad \mathcal{T}_{J=1}^{I=1} = -\frac{1}{3}\frac{(s - 4M^2)}{16\pi f^2} \ , \qquad \mathcal{T}_{J=0}^{I=2} = \frac{2M^2 - s}{16\pi f^2} \ .$$

For $M \neq 0$ finding analytically the complete one-loop isospin partial amplitudes is not possible. However, for the unitarity condition one needs only their imaginary parts. In the physical domain, $s > 0$, $t < 0$, $u < 0$ only the function $b_0(s)$ develops an imaginary part given by

$$\operatorname{Im} b_0(s, M, M) = -\pi \sqrt{1 - \frac{4M^2}{s}}\, \Theta(s - 4M^2) \ .$$

Thus, for $s > 4M^2$ we get

$$\begin{aligned}
\operatorname{Im} \mathcal{M}^{I=0} &= \frac{1}{16\pi^2}\frac{1}{18f^4}\Big\{27(s^2 - M^4) + [3s^2 - 3su + 12M^2u - 6M^2s - 6M^4] \\
&\qquad +[3s^2 - 3st + 12M^2t - 6M^2s - 6M^4]\Big\} \operatorname{Im} b_0(s, M, M) \\
&= -\frac{1}{16\pi}\frac{1}{18f^4}(36s^2 - 36M^2s + 9M^4)\sqrt{1 - \frac{4M^2}{s}} \ ,
\end{aligned}$$

and it is clear that at one loop only $\mathcal{T}_{J=0}^{I=0}$ develops an imaginary part

$$\operatorname{Im} \mathcal{T}_{J=0}^{I=0} = -\frac{1}{(16\pi)^2 2f^4}(2s - M^2)^2 \sqrt{1 - \frac{4M^2}{s}} \ ,$$

so that the unitarity relation

$$\left[-\frac{2s - M^2}{16\pi f^2}\right]^2 = -\frac{2}{\sqrt{1 - 4M^2/s}} \operatorname{Im} \mathcal{T}_{J=0}^{I=0} \ ,$$

is satisfied. Similarly,

$$\begin{aligned}
\operatorname{Im} \mathcal{M}^{I=0} &= \frac{1}{16\pi^2}\frac{1}{18f^4}\Big\{[3s^2 - 3su + 12M^2u - 6M^2s - 6M^4] \\
&\qquad [3s^2 - 3st + 12M^2t - 6M^2s - 6M^4]\Big\} \operatorname{Im} b_0(s, M, M) \\
&= -\frac{1}{16\pi}\frac{1}{6f^4}(s - 4M^2)(t - u)\sqrt{1 - \frac{4M^2}{s}} \ ,
\end{aligned}$$

so that with $t-u=(s-4M^2)\cos\theta$ and

$$\frac{1}{32\pi}\int_0^\pi d\theta\,\sin\theta\,P_1(\cos\theta)\cos\theta=\frac{1}{16\pi}\frac{1}{3}\ ,$$

one easily finds

$$\mathrm{Im}\,\mathcal{T}^{I=1}_{J=1}=-\frac{1}{(16\pi)^2 18 f^4}\,(s-4M^2)^2\sqrt{1-\frac{4M^2}{s}}\ ,$$

which again agrees with the unitarity relation. Finally, for $\mathcal{T}^{I=2}$ we find

$$\mathrm{Im}\,\mathcal{T}^{I=2}_{J=0}=-\frac{1}{(16\pi)^2 2 f^4}\,(s-2M^2)^2\sqrt{1-\frac{4M^2}{s}}\ ,$$

as it should be.