Quantum Field Theory of Fundamental Interactions. Problems, set 14.

Problem 1. Consider a $U(1)$ gauge theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + |(\partial_\mu + igA_\mu)\phi|^2 - m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4$$

with $m^2 < 0$ so that the gauge symmetry is spontaneously broken. Work in the Feynman gauge with $\xi = 1$ so that the Goldstone boson has the same mass as the vector boson V (whose mass arises from gauge symmetry breaking). Assume that parameters of the model are such that the physical scalar particle h^0 (the Higgs boson) is much heavier than the vector boson. Compute the decay widths

$$\Gamma(h^0 \to V_T V_T) \qquad \Gamma(h^0 \to V_T V_L) \qquad \text{and} \quad \Gamma(h^0 \to V_L V_L)$$

where V_T and V_L denote transversally and longitudinally polarized vector bosons (final state $V_T V_T$ means that one has to sum over the two longitudinal polarizations for each vector boson). Show that for $M_h \gg M_V$ one has

$$\Gamma(h^0 \to V_T V_T) \ll \Gamma(h^0 \to V_L V_L) ,$$

and that

$$\Gamma(h^0 \to V_L V_L) = \Gamma(h^0 \to GG) + \mathcal{O}\left(\frac{M_V^2}{M_h^2}\right) ,$$

(the equivalence theorem).

Problem 2. In the theory of Problem 1 compute in the lowest order the amplitude of the elastic scattering of two longitudinally polarized vector bosons $V_L(k_1)V_L(k_2) \to V_L(k_3)V_L(k_4)$ and demonstrate the validity of the equavalence theorem.

Problem 3. In the same theory as in the preceding Problem compute the in the lowest order approximation the $h^0 \to VV$ decay width applying the Landau-Cutcosk'y rule to the relevanyt diagrams contributing at the one-loop to the Higgs boson self energy. Use the Feynman gauge with $\xi = 1$. Note that in the R_ξ gauge the ghosts do couple to the higgs boson.

Problem 4. In the Standard Model compute the lowest order contributions to the amplitudes of the decays $h^0 \to Z^0 Z^0$ and $h^0 \to W^+W^-$ (for $M_h > 2M_Z^2$ these

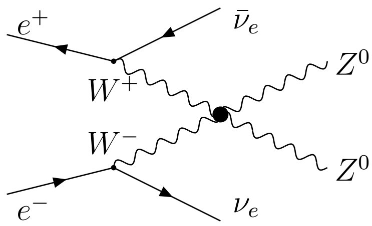

Figure 1: Scattering of spin 1 bosons in e^+e^- collision.

are the dominant channels of the Higgs boson decay) and the corresponding decay widths. Check the equivalence theorem.

Problem 5. In a theory with $U(1)$ gauge symmetry broken spontaneously compute the differential and total cross sections for production of two massive vector bosons ($h^0h^0 \to V_TV_T$, V_TV_L and V_LV_L). Check the validity of the equivalence theorem.

Problem 6. Combining the equivalence theorem with the effective W approximation (see Problem x set y) find the total cross section for the process $e^+e^- \to \nu_e\bar{\nu}_eW_L^+W_L^- \to \nu_e\bar{\nu}_eZ_L^0Z_L^0$ (see figure 1) and the invariant mass distribution $d\sigma/dq^2$ of the final Z^0Z^0 pair for $\sqrt{s} \gg M_W^2, M_Z^2$ assuming that no light Higgs boson with a mass below ~ 1 TeV exists.

Problem 6. Standard Model is a gauge theory with the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ (the group factor subscripts denote c color, L left and Y hypercharge, resectively). There are 3 identical (as far as their quantum numbers are concerned) generations of (two-component) Weyl fermion fields:

$$\ell = \begin{pmatrix} \nu \\ e \end{pmatrix}, \quad e^c, \quad q = \begin{pmatrix} u \\ d \end{pmatrix}, \quad u^c, \quad d^c$$

(here the superscript c is not color; it's just a way to distinguish some fields from the other fields) with quantum numbers of the gauge group: $(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$, $(\mathbf{1},\mathbf{1})_1$, $(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$, $(\overline{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}$, $(\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}$, respectively (subscripts give the $U(1)$ charges and in the formula above we have explicitly displayed the content of $SU(2)$ doublets). Check that all possible type of anomalies cancel out, that is, that the algebraic factor

$$\mathrm{tr}\left(T^A\left\{T^B,\ T^C\right\}\right)$$

where T^A, etc are the generators of the gauge group, vanishes for all A, B and C.

Problem 7. Show that if the gauge group representation R formed by left-chiral fermionic fields of a theory is real, i.e. can be decomposed into a direct sum $R' \oplus \overline{R}'$ where R' is another representation, then there is no anomaly in the $J^L_\mu J^L_\nu J^L_\lambda$ triangle. Show the same for the case of pseudoreal representation. Pseudoreal representation R cannot be written as $R = R' \oplus \overline{R}'$ but is unitarily equivalent to $\overline{R}$, that is, there exist a convoluting matrix S such that $S^{-1}T^aS = -T^{a*}$ where T^a and $-T^{a*}$ are the generators of R and $\overline{R}$, respectively. (In this case there can be however so called global anomalies, but this is beyond the scope of our lecture.)

Problem 8. Consider the amplitude of the top quark decay $t \to b\ell\nu_\ell$. Show that the amplitude is independent of the gauge parameter ξ when the exchange of the unphysical Goldstone boson $G^\pm$ is taken into account in addition to the $W^\pm$ exchange.

Problem 9. Show that the amplitude of $e^+e^- \to f\bar{f}$ (where f can be any fermion but to eliminate the t-channel diagrams assume that $f \neq e$) computed at the tree level in the SM is gauge independent when the contribution of G^0 is taken into account.

Problem 10. Using the general formula for the beta function

$$\beta_g(g) = \frac{1}{16\pi^2}\left[-\frac{11}{3}C_2(G) + \frac{2}{3}S_2(F) + \frac{1}{3}S_2(S)\right]g^3$$

where $C_2(G) = N$ for $SU(N)$ (and 0 for $U(1)$) gauge group and $S_2(F)$ and $S_2(S)$ are the values of the Casimir operator ($\mathrm{tr}(T^a_R T^b_R) = S_2(R)\delta^{ab}$) on (Weyl) fermion and (complex) scalars representations, respectively find the beta functions of the three Standard Model gauge couplings g_s, g_2 and g_y (in the SM there is one complex scalar doublet with quantum numbers $(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$). Next, assuming that there is a scale M_X at which

$$g_s(M) = g_2(M) = \sqrt{\frac{5}{3}}g_y(M)$$

(such a relation is predicted by many Grand Unification models) predict the value of the sinus^2 of the Weinberg angle ($\sin^2\theta_W$) defined by the Standard Model relations

$$g_2 = \frac{e}{\sin\theta_W}, \quad g_y = \frac{e}{\cos\theta_W}$$

Take $e = \sqrt{4\pi\alpha_{\mathrm{EM}}}$ with $\alpha_{\mathrm{EM}} = \hat{\alpha}(M_Z) = 1/127.9$ and $g_s = \sqrt{4\pi\alpha_s}$ with $\alpha_s(M_Z) = 0.118$ where $M_Z = 91$ GeV. What is the unification scale M_X?

Problem 11. Consider the $\nu\bar{\nu} \to W_L^+ W_L^-$ (the subscript L means longitudinal polarization) scattering amplitude in the SM. Show that the leading term in the scattering amplitude (growing with $\sqrt{s}$ - the CMS energy) of the diagram with e exchanged in the t channel is canceled by the Z^0 boson exchange in the s-channel.

Problem 12. Consider $SU(2) \times U(1)_Y$ breaking by several different $SU(2)$ Higgs multiplets Φ_n. They can be characterized by the eigenvalue of the Casimir operator $\mathbf{T}^2$ of $SU(2)$: $\mathbf{T}^2 \Phi_n = t_n(t_n+1)\Phi_n$. Assuming that their vacuum expectation values $\langle \Phi_n \rangle = v_n$ all break $SU(2) \times U(1)$ down to the same $U(1)$ subgroup generated by $Q = T^3 + Y$ (where Y is the generaor of $U(1)_Y$) show that at the tree level the ρ parameter is given by

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = \frac{\sum_n [t_n(t_n+1) - t_{3n}^2] v_n^2}{2\sum_n t_{3n}^2 v_n^2}$$

where t_{3n} is the defined by

$$\langle \Phi_n \rangle T_3 \langle \Phi_n \rangle = t_{3n} v_n^2$$

and $\cos^2\theta_W = g_2^2/(g_2^2 + g_y^2)$, g_2 and g_y being the gauge coupling constants of $SU(2)$ and $U(1)_Y$, respectively.

Problem 13. In the SM Lagrangian there is no $h^0 A_\mu A^\mu$ vertex (neutral particles do not couple directly to photons). Yet the decay $h^0 \to \gamma\gamma$ is induced by loop corrections. Draw all one-loop SM diagrams contributing to this process. Compute explicitly the contribution of the t quark in the limit of light Higgs boson: $m_t/M_h \to \infty$. Is the Appelquist - Carrazone decoupling theorem satisfied?

Problem 14. Suppose there are charged Higgs bosons $H^\pm$ which couple to the neutral Higgs boson h^0 through the term $\mathcal{L}_{\text{int}} = -\kappa H^+ H^- h^0$. Compute the contribution of the $H^\pm$ loop to the amplitude of the decay $h^0 \to \gamma\gamma$. How does this contribution behave in the limit $M_{H^\pm}/M_h \to \infty$?

Problem 15. Suppose the electroweak symmetry $SU(2) \times U(1)$ is broken simultaneously by two Higgs boson doublets H_u and H_d having the quantum numbers $(\mathbf{2}, \frac{1}{2})$ and $(\mathbf{2}, -\frac{1}{2})$, respectively. What scalar particles does such a theory predict? Write down the most general couplings of these Higgs two doublets to quarks. Show that after diagonalization of the quark mass matrices there appear scalar flavour changing neutral currents, i.e. couplings of the form $\propto h^0 \bar{\psi}_c \psi_u$ etc.

Problem 16. Find the one loop renormalization group equations for the Yukawa couplings of the Standard Model.

Problem 17. Show that the Standard Model currents of the baryon B and lepton L numbers are anomalous. Because of this B and L are not conserved if nonperturbative effects are taken into account (Violation of B and L is very tiny yet it is crucial for some baryogenesis scenarios). Check that $B - L$ is not anomalous (that is, the Standard Model predicts strict conservation of $B - L$).