Problems in Quantum Field Theory of Fundamental Interactions. V

Problem V.1
A particle A can decay in several different ways: $A \to B_1 C_1 \ldots$, $A \to B_2 C_2 \ldots$, etc. The respective decay widths are Γ_1, Γ_2, ... In a typical experiment each individual particle A comes from the accelerator and is brought to rest in a block of material. The moment of putting it to rest is taken for $t = 0$. Then it decays after some time which is measured. What is the time distribution dN_1/dt of the registered decays $A \to B_1 C_1 \ldots$, assuming the products of the other decays ($A \to B_2 C_2 \ldots$ and others) are not registered or, as was the case in the ground breaking experiment with the θ^+/τ^+ particles,[1] are incorrectly attributed to decays of another particle A' ($A' \neq A$ is assumed)? Can one determine from it the lifetime τ_A of the particle A? And the branching fraction $\mathrm{Br}(A \to B_1 C_1 \ldots)$?

Problem V.2
In the LAB frame a particle of mass M moves with velocity V along the z-axis and decays (in flight) into two other particles of masses m_1 and m_2. Find the correlation of the LAB energy of the particle 1 with the angle $\vartheta_1^{\mathrm{LAB}}$ its LAB momentum forms with the z_{LAB}-axis. Show that if V is sufficiently large ($V < 1$, of course), the angle $\vartheta_1^{\mathrm{LAB}}$ cannot exceed a certain value (determine it). What is the energy distribution of the particles of mass m_1 measured in the Laboratory system, if the angular distribution (the distribution of the variable $\cos\vartheta^{\mathrm{CM}}$) of the produced particles in the CM system follows the $(1 + \cos\vartheta^{\mathrm{CM}})^2$ law?

Problem V.3
What are the possible angles ψ_{LAB} in the LAB system between the momenta of two particles which are produced in the decay of a massive particle having in the LAB system the velocity V? Write down the explicit formula for $\tan\psi_{\mathrm{LAB}}$ in the case of two identical daughter particles.

Problem V.4
A neutral pion π^0 decays in flight into two photons (see also Problems I.38 and V.34). What is the angular distribution (w.r.t. the direction of the parent pion) of the photons in the Laboratory system in which π^0 has velocity V? Find also the Laboratory frame distribution of the angles between the two photons.

[1] In the historic experiment both θ^+ and τ^+ were in fact the positively charged kaon and the observed decay channels were $K^+(\theta^+) \to \pi^0\pi^+$ and $K^+(\tau^+) \to \pi^0\pi^0\pi^+$, $\pi^-\pi^+\pi^+$. Since decays of the same spinless particle into two and three pions would necessarily mean violation of parity which was believed to be a good symmetry, it was taken for granted that two and three pion final states must correspond to decays of two different particles.

Problem V.5
Calculate the volume Φ_2 of the two-particle phase space

$$\Phi_2 = \int d\Gamma_{\mathbf{p}_1} d\Gamma_{\mathbf{p}_2} \, (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q) \, ,$$

where $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, assuming that - as in the case of computing the decay width of a particle of mass M into two other particles of masses m_1 and m_2 in the decaying particle rest frame - $q^\mu = (M, \mathbf{0})$. Recall that $d\Gamma_{\mathbf{p}} = d^3\mathbf{p}/(2\pi)^3 2E_{\mathbf{p}}$.

Problem V.6
Calculate the volume of the phase space of three particles

$$\Phi_3 = \int d\Gamma_{\mathbf{p}_1} d\Gamma_{\mathbf{p}_2} d\Gamma_{\mathbf{p}_3} \, (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - q) \, ,$$

where $p_i^2 = m_i^2$. Keep the masses m_i nonzero as long as possible and set them to zero only at the end.
Hint: using the delta function integrate over $d\Gamma_{\mathbf{p}_3}$ and then express the remaining (one-dimensional) delta function in the form $\delta(\cos\theta - f(E_1, E_2))$ where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. Reduce the remaining integral to a one over $dE_1 dE_2 d(\cos\theta)$ and seek its geometrical interpretation. Then set $m_i = 0$ and evaluate the integral.

Problem V.7
Calculate the width of the decay of a fermion $f^{(a)}$ of mass m_a into another fermion $f^{(b)}$ of mass m_b and a spinless particle (S or P) of mass M (assuming that $m_a > M + m_b$). Assume the interaction of the form

$$\mathcal{H}_{\rm int}(x) = h \, \varphi(x) \, \bar{\psi}_{(b)}(x) \Gamma \psi_{(a)}(x) + \text{H.c.},$$

and consider two cases: $\Gamma = I$ (S) and $\Gamma = -i\gamma^5$ (P). In both cases write down the amplitudes of the decays into $f^{(b)}$ with definite spin projection onto the z-axis and the amplitudes of the decays into $f^{(b)}$ with definite helicities, assuming that the spin projection onto the z-axis of the initial fermion $f^{(a)}$ was $+1/2$. What is the relative orbital angular momentum of the two final state particles? Compute also the helicity amplitudes corresponding to a definite total angular momentum. Express the width $\Gamma(f^{(a)} \to f^{(b)} S(P))$ through these helicity amplitudes (and also through the amplitudes corresponding to definite orbital angular momentum).

Problem V.8
Compute the differential rate $d\Gamma/d\cos\theta$ of the decay $f^{(a)} \to f^{(b)} S$ (taking for θ the angle between the z-axis and the three-momentum of the final fermion $f^{(b)}$) induced by the same interaction as in Problem V.7 with $\Gamma = I$, assuming that the decaying fermion $f^{(a)}$ has the spin projection $+1/2$ ($-1/2$) onto the z-axis and summing over possible spin projections of the final fermion $f^{(b)}$. Do this in two ways: first by using the explicit form

of the spinor corresponding to $f^{(a)}$ in the given spin projection and next by summing over all $f^{(a)}$ spin projections σ_a after having written its spinor in the form

$$u(\mathbf{q}, \sigma_a) = \Sigma_q(s_q)\, u(\mathbf{q}, \sigma_a)\,.$$

The spin projection operator

$$\Sigma_q(s_q) = \frac{1}{2}\left(1 + \gamma^5 \not{s}_q\right),$$

with $s_q^\mu = (L_q)^\mu{}_\nu s^\nu_{\mathrm{rest}}$ projects onto the spinor corresponding to the fermion state $|\mathbf{q}, \sigma\rangle$ obtained with the help of the standard Lorentz transformation $U(L_q)$ from the rest frame eigenstate $|\mathbf{0}, \sigma\rangle$ of the operator $\mathbf{s}_{\mathrm{rest}}\cdot\mathbf{J}$.

Using the spin projector technique compute also the analogous differential rates of the decays of $f^{(a)}$ polarized along the z-axis into $f^{(b)}$ in a concrete spin state: with definite spin projection onto the z-axis ($\sigma_b = \pm\frac{1}{2}$) and with definite helicity ($\lambda_b = \pm\frac{1}{2}$).

Problem V.9
Consider the decay of a (fully) polarized fermion $f^{(a)}$ of mass m_a into another fermion $f^{(b)}$ of mass m_b and a spinless particle of mass M (for $m_a > M + m_b$). Assume the interaction of the form

$$\mathcal{H}_{\mathrm{int}}(x) = h\,\varphi(x)\,\bar{\psi}_{(b)}(x)(1 - i\lambda\gamma^5)\psi_{(a)}(x) + \mathrm{H.c.}\,,$$

with complex λ. (This can be a model of the $\Lambda^0 \to p\,\pi^-$ decay induced by the weak interactions). Combining the results of Problem V.7 compute rates of the decays into final states with definite spin projection and definite helicity of $f^{(b)}$ and in both cases discuss possible experimental signals of parity nonconservation in this decay. Recover the same rates using the spin projectors of Problem V.8.

Problem V.10
Taking for the interaction

$$\mathcal{H}_{\mathrm{int}}(x) = h\,\partial_\mu\varphi(x)\,\bar{\psi}_{(b)}(x)\gamma^\mu\Gamma\psi_{(a)}(x) + \mathrm{H.c.}\,,$$

where $\Gamma = I$ or $\Gamma = -i\gamma^5$. Compute the same decay rates as in the Problem V.7.

Problem V.11
Calculate the decay width of a spinless particle of mass M into a fermion-antifermion (of masses $m < \frac{1}{2}M$) pair: a) with definite spin projections onto the z-axis, b) with definite helicities. Assume the interaction of the form

$$\mathcal{H}_{\mathrm{int}}(x) = h\,\bar{\psi}(x)\Gamma\psi(x)\varphi(x)\,,$$

and consider two cases: $\Gamma = I$ and $\Gamma = -i\gamma^5$. Explain the different forms of the widths in these two cases by appealing to the orbital momentum of the final fermion-antifermion pair and parity conservation. Check that the total angular momentum of the final state particles is $j = 0$.

Problem V.12
Using the spin projection technique (Problem V.8) calculate the rates of the decays of a spinless particle into into a fermion-antifermion pair: with definite spin projections onto the z-axis and with definite helicities assuming the interaction of the form

$$\mathcal{H}_{\rm int}(x) = h\,\bar{\psi}(x)(1 - i\lambda\gamma^5)\psi(x)\phi(x) + {\rm H.c.}$$

where λ can be complex. Discuss the possible signals of parity and CP nonconservation. Notice that for complex λ necessarily $\phi \neq \phi^\dagger$, i.e. the spinless particle and its antiparticle must be different in this case.

Problem V.13
The Hamiltonian of the effective theory of weak interactions at low energies has the form[2]

$$\mathcal{H}_{\rm weak} = \frac{G_F}{\sqrt{2}}\,J^\dagger_\lambda J^\lambda\,, \qquad \text{where} \qquad J^\lambda = J^\lambda_{\rm lept} + J^\lambda_{\rm hadr}\,,$$

$$J^\lambda_{\rm lept} = \bar{\psi}_{(e)}\gamma^\lambda(1-\gamma^5)\psi_{(\nu_e)} + \bar{\psi}_{(\mu)}\gamma^\lambda(1-\gamma^5)\psi_{(\nu_\mu)} + \bar{\psi}_{(\tau)}\gamma^\lambda(1-\gamma^5)\psi_{(\nu_\tau)}\,.$$

Parametrize the most general form of the matrix element $\langle\Omega_{\rm hadr}|J^{\lambda\dagger}_{\rm hadr}(x)|\pi^-(\mathbf{q})\rangle$ by one real constant f_π - the so-called pion decay constant - and calculate in the lowest order in G_F (treating neutrino as massless) the ratio of the decay widths $R_\pi = \Gamma(\pi^\pm \to e^\pm\nu_e)/\Gamma(\pi^\pm \to \mu^\pm\nu_\mu)$ and compare it with the corresponding ratio of the phase spaces available in these two decays and with the experimental data. Compare also with the data the value of $R_K = \Gamma(K^\pm \to e^\pm\nu_e)/\Gamma(K^\pm \to \mu^\pm\nu_\mu)$ obtained in the similar way. What are the possible sources of the small discrepancies (in both cases)? Using the value $G_F = 1.16639 \times 10^{-5}$ GeV^{-2} and the π^- lifetime $\tau_{\pi^-} = 2.6 \times 10^{-8}$ sec. determine the value of f_π. Correct the result for the Cabibbo angle. Use also the appropriate experimental data to determine f_K.

Problem V.14
Assume $\mathcal{H}_{\rm weak}$ takes the form of a product of two scalar currents:

$$\mathcal{H}_{\rm weak} = \frac{G_F}{\sqrt{2}}\,J^\dagger J\,,$$

where $J = \bar{\psi}_{(e)}(1-\gamma^5)\psi_{(\nu_e)} + \bar{\psi}_{(\mu)}(1-\gamma^5)\psi_{(\nu_\mu)} + J_{\rm hadr}$ (instead of the product of two vector currents considered in Problem V.13), and parametrizing appropriately the hadronic matrix element $\langle\Omega_{\rm hadr}|J^\dagger_{\rm hadr}(x)|\pi^-(\mathbf{q})\rangle$ compute the same ratio of the decay widths as in Problem V.13. Which of the two results matches the data?

Problem V.15
Analyse the decay $\pi^- \to \ell^-\bar{\nu}_\ell$ mediated by the $V - A$ weak interactions of Problem V.13 from the point of view of the relative angular momentum of the final fermions. Compute the amplitudes of the π^- decay into ℓ^- and $\bar{\nu}_\ell$ with definite helicities. Discuss the signal of parity violation.

[2] In fact this is only the charged current part of the full effective Hamiltonian of weak interactions that is derived from the Standard Model.

Problem V.16
The τ lepton is heavy enough to decay into hadrons. Using the term

$$\mathcal{H}_{\rm weak} = \frac{G_F}{\sqrt{2}}\,[\bar\psi_{(\nu_\tau)}\gamma_\lambda(1-\gamma^5)\psi_{(\tau)}]J^\lambda_{\rm hadr}\,,$$

of the weak interaction Hamiltonian, the values of pion and Kaon decay constants f_π and f_K obtained in Problem V.13, and the τ lifetime $\tau_\tau = 2.906\times10^{-13}$ sec. compute the branching fractions $\mathrm{Br}(\tau^-\to\nu_\tau\pi^-)$ and $\mathrm{Br}(\tau^-\to\nu_\tau K^-)$ and compare the predictions with the data.

Problem V.17
Analyse the decay of a fully polarized τ^- mediated by the $V-A$ weak interactions into $\pi^-\nu_\tau$ from the point of view of the relative angular momentum of the final particles. Assume that the τ^- spin projection onto the z-axis is $+\frac{1}{2}$. Write down also the amplitudes of the decay of a polarized τ^- into neutrinos of definite helicities.

Problem V.18
What would be the energy distribution of electrons in the decay $\mu^-\to e^-\bar\nu_e\nu_\mu$ if the leptonic weak current $J^\lambda_{\rm lept}$ entering the weak interaction Hamiltonian of problem V.13 had the form

$$J^\lambda_{\rm lept} = \bar\psi_{(e)}\gamma^\lambda(1-\tilde\lambda\gamma^5)\psi_{(e)} + \bar\psi_{(\mu)}\gamma^\lambda(1-\lambda\gamma^5)\psi_{(\nu_\mu)} + \dots\,,$$

with arbitrary complex λ and $\tilde\lambda$ parametrizing possible departures from the pure $V-A$ structure?

Problem V.19
Determine the angular distribution of electrons produced in the decay $\mu^-\to e^-\bar\nu_e\nu_\mu$ of a fully polarized muon.

Problem V.20
Find the energy distribution of electrons produced in the decay $\mu^-\to e^-\bar\nu_e\nu_\mu$ if the weak interactions are mediated by massive spin 1 bosons as in the Standard Model. The relevant term in the Hamiltonian is

$$\mathcal{H}_{\rm weak} = \frac{g_2}{2\sqrt{2}}\,W^+_\lambda\bar\psi_{(\nu_\mu)}\gamma^\lambda(1-\gamma^5)\psi_{(\mu)} + \frac{g_2}{2\sqrt{2}}\,W^-_\lambda\bar\psi_{(e)}\gamma^\lambda(1-\gamma^5)\psi_{(\nu_e)} + \dots\,,$$

where $W^+_\lambda = (W^-_\lambda)^\dagger$ is the (free) field operator if the spin 1 boson and g_2 is the coupling constant. For simplicity assume that electron and neutrinos are massless.

Problem V.21
The most general (consistent with invariance of the strong interactions with respect to parity and time reversal transformations) form of the matrix element of the hadronic weak current (appearing in the effective Hamiltonian of weak interactions, given in Problem V.13) $J^\mu_{\rm hadr} = V^\mu - A^\mu$ relevant for the free neutron decay is (see Problem II.32)

$$\langle p(\mathbf{k}_p)|J^{\mu\dagger}_{\rm hadr}(x)|n(\mathbf{q})\rangle = e^{-ix\cdot Q}\,\bar u(\mathbf{k}_p)\,[f^\mu_V(Q^2) - f^\mu_A(Q^2)\,\gamma^5]\,u(\mathbf{q})\,.$$

Here $Q \equiv q - p$ and $f^{\mu}_{V,A} = \gamma^{\mu} g_{V,A}(Q^2) + i\sigma^{\mu\kappa} Q_{\kappa} r_{V,A}(Q^2) + Q^{\mu} h_{V,A}(Q^2)$ with a priori arbitrary functions $g_{V,A}$, $r_{V,A}$ and $h_{V,A}$ of the Lorentz invariant Q^2. Arguing that in computing the amplitude of the free neutron decay $n \to p\, e^- \bar{\nu}_e$ the terms other than $g_V(0)$ and $g_A(0)$ can be neglected, and using the fact that $g_V(0) \approx 1$ (in agreement with the CVC hypothesis of Feynman and Gell-Mann) determine the differential energy distribution of the electrons produced in this decay. Use the neutron lifetime to find the value of $g_A(0)$. Prove (analytically) that only $|g_A(0)|$ can be obtained in this way.

Problem V.22

Find the angular distribution of electrons produced in the decay $n \to p\, e^- \nu_e$ of a fully polarized neutron. Derive the analytic expression for $d\Gamma/d(\cos\theta) dE_e$ and integrate it numerically to find the coefficient a in the formula

$$\frac{1}{\Gamma}\frac{d\Gamma}{d(\cos\theta)} = \frac{1}{2}(1 + a\cos\theta)\,,$$

for both possible signs of $g_A(0)$ and the value of $|g_A(0)|$ obtained in Problem V.21.

Problem V.23

Using the effective Hamiltonian of the weak interactions given in Problem V.13 compute the partial width $\Gamma(\pi^- \to \pi^0 e^- \bar{\nu}_e)$. Use the parametrization

$$\langle \pi^0(\mathbf{p})| J^{\lambda\dagger}_{\rm hadr}(x) |\pi^-(\mathbf{q})\rangle = e^{-ix\cdot Q}[(q+p)^{\lambda} f_+(Q^2) + (q-p)^{\lambda} f_-(Q^2)],$$

with two formfactors (see Problem II.32) $f_+(Q^2)$ and $f_-(Q^2)$, where $Q = q - p$, of the the relevant hadronic matrix element. Argue then that the formfactor $f_-(Q^2)$ does not contribute in the limit $m_e \to 0$ (therefore can be neglected) and that $f_+(Q^2)$ can be approximated by $f_+(0)$.

Next, write the hadronic current as the difference $J^{\lambda}_{\rm hadr} = V^{\lambda} - A^{\lambda}$ of the vector and axial vector currents and argue that the matrix element $\langle \pi^0(\mathbf{p})|A^{\dagger}_{\lambda}(x)|\pi^-(\mathbf{q})\rangle$ vanishes. Making the assumption that the strangeness conserving part of the vector current $V^{\Delta S=0}_{\lambda}$ is the combination $V^1_{\lambda} - iV^2_{\lambda}$ of the conserved Noether currents V^a_{λ}, $a = 1, 2, 3$, associated with the isospin symmetry of the strong interactions, justify neglecting $f_-(Q^2)$ in a different way and predict the value of $f_+(0)$. Check if the prediction agrees with the experimental data. Correct the prediction taking into account the Cabibbo angle θ_C by writing $V^{\Delta S=0}_{\lambda} = \cos\theta_C(V^1_{\lambda} - iV^2_{\lambda})$.

Finally, assuming that the strangeness changing part of V_{λ} has the form $V^{\Delta S=-1}_{\lambda} = \sin\theta_C(V^4_{\lambda} - iV^5_{\lambda})$, where V^a_{λ}, $a = 1, \dots, 8$, are now the Noether currents of the (approximate) *Eightfold Way* $SU(3)$ symmetry, find (still approximating $f_+(Q^2)$ by $f_+(0)$ and neglecting $f_-(Q^2)$) the rate of the decay $K^- \to \pi^0 e^- \bar{\nu}_e$ and compare the prediction with the data. Plot the doubly differential rate $d\Gamma(K^- \to \pi^0 e^- \bar{\nu}_e)/dE_{\pi^0} dE_e$ as a function of the electron energy E_e for representative values of the π^0 energy[3] E_{π^0}. Check that in-

[3]The shape of the electron energy distribution for fixed π^0 energy does not depend on the value of $f_+(Q^2)$ which enters as a multiplicative factor and is therefore a good test of the theory being insensitive to the approximation $f_+(Q^2) \approx f_+(0)$.

tegrating numerically $d\Gamma/dE_{\pi^0}dE_e$ over the appropriate domain, one recovers the decay width $\Gamma(K^- \to \pi^0 e^- \bar{\nu}_e)$.

Use the same approximations to compute also the rate of the decay $K^- \to \pi^0 \mu^- \bar{\nu}_\mu$ and compare the prediction with the data.

Predict also in this way (neglecting small effects of CP violation) the rates of the K_L and K_S decays into $\pi^\pm \ell^\mp \nu_\ell$ as well as into $K^\pm e^\mp \nu_e$ and check how well these predictions compare with the data.

Problem V.24

The Hamiltonian of the electromagnetic interactions has the form $\mathcal{H}^{\rm EM}_{\rm int} = eJ^\lambda_{\rm EM} A_\lambda$, where A_μ is the photon field operator and the electromagnetic current is

$$J^\lambda_{\rm EM} = -\bar{\psi}_{(e)}\gamma^\lambda\psi_{(e)} - \bar{\psi}_{(\mu)}\gamma^\lambda\psi_{(\mu)} - \bar{\psi}_{(\tau)}\gamma^\lambda\psi_{(\tau)} + J^\lambda_{\rm EM\ hadr}\,,$$

(the minus signs arise because $Q_e = Q_\mu = Q_\tau = -1$). Parametrize the hadronic matrix element $\langle\Omega_{\rm hadr}|J^\lambda_{\rm EM\ hadr}(x)|\rho^0(\mathbf{q},\sigma)\rangle$ respecting the electromagnetic current conservation and compute (in the lowest order in e) the width of the $\rho^0 \to \ell^-\ell^+$ decay (for $\ell = e$, μ, or τ). Use the experimental data for $\Gamma(\rho^0 \to e^+e^-)$ to fix the single constant in the parametrization of the hadronic matrix element.

Next, using the relation

$$Q_{\rm EM} = I_3 + \frac{1}{2}B\,,$$

(where $Q_{\rm EM}$ is the electromagnetic charge, I_3 the third isospin component and B - the baryon number operator) which holds true for all nonstrange ($S = 0$) hadrons, relate the matrix elements $\langle\Omega_{\rm hadr}|J^\lambda_{\rm EM\ hadr}|\rho^0(\mathbf{q},\sigma)\rangle$ and $\langle\Omega_{\rm hadr}|(J^\lambda_{\rm hadr})^\dagger|\rho^-(\mathbf{q},\sigma)\rangle$ and predict the width $\Gamma(\rho^- \to \ell^-\bar{\nu}_\ell)$ for $\ell = e$ or μ. Why there are no data to which this result could be compared? Predict also the rate of the decay $\tau^- \to \rho^-\nu_\tau$ and compare with the data.

Problem V.25

Write down a renormalizable (i.e. a dimension $[\mathrm{M}]^4$ operator) interaction of a massive vector (spin 1) boson with two scalars (i.e. two spinless particles). Consider all possibilities: *i)* a neutral vector boson and a scalar and its anti-scalar, *ii)* a neutral vector boson and two different neutral scalars, *iii)* a neutral vector boson and two different charged scalars (of the same charge) *iv)* a charged vector boson and two different scalars (at least one scalar must be charged).

Calculate the decay widths (assuming appropriate mass hierarchies): *A)* of a neutral vector boson into scalar and its anti-scalar, *B)* of a charged vector boson into two different scalars *C)* of a scalar into a vector boson and another scalar

Problem V.26

Using the appropriate interactions derived in Problem V.25 compute the differential decay rates: A) of a polarized massive vector boson into two scalars having nonequal masses (consider all possible projections of the initial particle spin onto the z-axis), B) of a scalar into another scalar and a massive vector boson of helicity $\lambda = 0$ and $\lambda = \pm 1$. Analyze these decays from the point of view of the orbital angular momentum of the final state pair and from the point ov view of the angular momentum conservation, using the partial wave expansion of the final state.

Problem V.27
Compute the decay width of a neutral massive spin 1 particle into two photons induced by the interaction

$$\mathcal{H}_{\mathrm{int}} = \kappa\left(\partial_\mu V_\nu + \partial_\nu V_\mu\right) f^{\mu\lambda} f_\lambda^{\ \nu}\,,$$

where V_μ is the field operator of the spin 1 particle and $f^{\mu\lambda} = \partial^\mu A^\lambda - \partial^\lambda A^\mu$ is the field strength tensor operator of the photon. Does the result agree with the general Landau-Yang theorem (Problem I.38)?

Problem V.28
In the Standard Theory the interaction of the $W^\pm$ massive charged vector bosons with leptons has the form

$$\mathcal{L}_{\mathrm{int}} = -\frac{g_2}{2\sqrt{2}}\, W^-_\mu \bar{\psi}_{(\ell)}\gamma^\mu(1-\gamma^5)\psi_{(\nu_\ell)} - \frac{g_2}{2\sqrt{2}}\, W^+_\mu \bar{\psi}_{(\nu_\ell)}\gamma^\mu(1-\gamma^5)\psi_{(\ell)}\,,$$

where $g_2 = e/s_W$ is the coupling constant ($e > 0$ is the electric charge and $s_W \approx \sqrt{0.23}$ - sine of the Weinbeg angle). Compute in the lowest order in g_2 the differential and total decay widths of W^- into the lepton-antineutrino pair for the spin projection of W^- onto the z-axis equal 0, 1 and -1. Assume that the spins of the final fermions are not measured. Explain vanishing of the differential decay rates for some specific ℓ^- emission angles. Discuss the possible signals of parity violation.

Problem V.29
Analyze the decays of a W^- with spin projection 0 and $+1$ onto the z-axis into the lepton-antineutrino pair with definite spin projections from the point of view of the orbital angular momentum. Compute also the rates of polarized W^- decays into ℓ^- and $\bar{\nu}_\ell$ with definite helicities.

Problem V.30
The Standard Theory predicts the interaction of the massive charged vector bosons $W^\pm$ with the top (t) and bottom (b) quarks in the form

$$\mathcal{L}_{\mathrm{int}} = -\frac{g_2}{2\sqrt{2}}\, V^*_{tb}\, W^-_\mu \bar{\psi}_{(b)}\gamma^\mu(1-\gamma^5)\psi_{(t)} - \frac{g_2}{2\sqrt{2}}\, V_{tb}\, W^+_\mu \bar{\psi}_{(t)}\gamma^\mu(1-\gamma^5)\psi_{(b)}\,,$$

where g_2 is the same coupling constant as in Problem V.28 and V_{tb} modifies the coupling compared to the interaction of $W^\pm$ with leptons. Compute (in the lowest order in g_2) the differential and total decay widths of the fully polarized top quark into W^+ and b. Assume that the spins of the final state particles are not measured.

Problem V.31
Compute the rates of decay of polarized top quark into W^+ and b with definite helicities. Decompose the amplitudes of the top quark decay into W^+ and b with definite spin projections onto the z-axis into the amplitudes corresponding to definite orbital angular momentum and spin of the final W^+b pair.

Problem V.32
The interaction

$$\mathcal{H}_{\rm int} = \bar{\psi}_s \sigma^{\mu\nu}(a_L \mathbf{P}_L + a_R \mathbf{P}_R)\psi_b f_{\mu\nu} + \text{H.c.}\,,$$

in which $f_{\mu\nu}$ is the photon field strength operator $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu,\ \gamma^\nu]$ effectively describes (for $a_L \approx 0$) the flavour changing b-quark decay into photon and s-quark.[4] What is the physical dimension of the couplings $a_{L,R}$? Write the "H.c." part of the interaction explicitly. Compute the $b \to s\gamma$ decay width assuming that the initial b quark is unpolarized (average over possible projections of its spin) and the spin and polarization of the final s quark and the photon are not measured (sum over projections of the s quark spin and photon polarizations). To sum over the photon polarizations use the Feynman prescription

$$\sum_\sigma \epsilon_\mu(\mathbf{k},\sigma)\,\epsilon^*_\nu(\mathbf{k},\sigma) \to -g_{\mu\nu}\,.$$

Problem V.33
Using the interaction Hamiltonian

$$\mathcal{H}_{\rm int} = \kappa\,\bar{\psi}_s \sigma^{\mu\nu}\mathbf{P}_R \psi_b f_{\mu\nu} + \text{H.c.}\,,$$

compute the width of the decays $b \to s\gamma_L$ and $b \to s\gamma_R$ where γ_L and γ_R are the left- and right-polarized photons (i.e. of helicity -1 and $+1$, respectively) assuming that the initial b is unpolarized and the spin of the final s is not measured. Check that the full width $\Gamma(b \to s\gamma) = \Gamma(b \to s\gamma_L) + \Gamma(b \to s\gamma_R)$ coincides with the one obtained in Problem V.32 for $a_L = 0$ and $a_R = \kappa$.

Problem V.34
Write down the most general, compatible with the gauge invariance and parity conservation in electromagnetic interactions, form of the decay amplitude of a neutral spinless particle into two photons. Consider the cases of a scalar and of a pseudoscalar (in the latter case this is the amplitude of $\pi^0 \to \gamma\gamma$ decay). Construct the corresponding effective interaction Hamiltonian (Lagrangian). In both cases compute the decay widths.

Problem V.35
Using the effective Hamiltonian (Lagrangian) describing the π^0 (having negative intrinsic parity) decay into two photons derived in Problem V.34 to generate the Feynman rule

[4] In the standard theory of electroweak interactions this interaction is generated by loops of W bosons and u, c and t quarks.

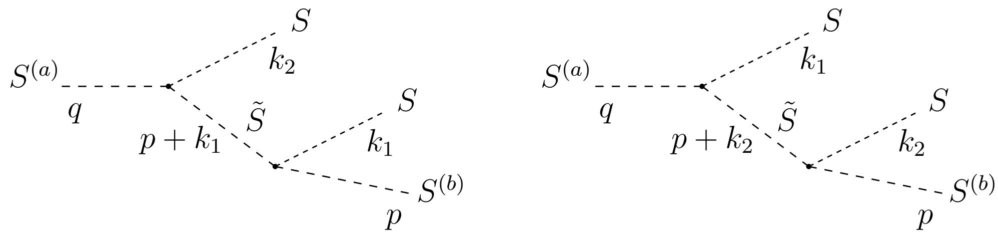

Figure 1: Decay of a spinless particle $S^{(a)}$ into three other spinless particles $S^{(b)}SS$ of which two are identical, with the intermediate state of a massive spinless particle.

also for off-shell photons compute the branching ratio of the decay $\pi^0 \to \gamma\, e^- e^+$ assuming that the branching fraction $\mathrm{BR}(\pi^0 \to e^- e^+ e^- e^+)$ is negligible. Compare the result with the data.

Problem V.36
Using the effective Hamiltonian (Lagrangian) describing the decay into two photons of a neutral mass M spin 0 particle S^0 of positive intrinsic parity (derived in Problem V.34) to generate the Feynman rule also for off-shell photons compute the branching ratio of the decay $S^0 \to \gamma\, e^- e^+$ assuming that only decays $S^0 \to 2\gamma$, $S^0 \to \gamma\, e^- e^+$ and, perhaps, $S^0 \to e^- e^+ e^- e^+$ but with a negligible branching fraction are kinematically allowed. Compare the numerical value of $\mathrm{BR}(S^0 \to \gamma\, e^- e^+)$ with $\mathrm{BR}(\pi^0 \to \gamma\, e^- e^+)$ for $M = M_{\pi^0}$.

Problem V.37
Using the electromagnetic interaction Hamiltonian $\mathcal{H}^{\mathrm{EM}}_{\mathrm{int}} = e J^{\lambda}_{\mathrm{EM}} A_{\lambda}$, where

$$J^{\lambda}_{\mathrm{EM}} = -\bar{\psi}_{(e)}\gamma^{\lambda}\psi_{(e)} - \bar{\psi}_{(\mu)}\gamma^{\lambda}\psi_{(\mu)} - \bar{\psi}_{(\tau)}\gamma^{\lambda}\psi_{(\tau)} + J^{\lambda}_{\mathrm{EM\ hadr}}\,,$$

compute the rate of the decay $\Sigma^0 \to \Lambda^0\gamma$. To this end argue that the most general form of the matrix element

$$\langle \Lambda^0(\mathbf{p})|J^{\lambda}_{\mathrm{EM\ hadr}}(x)|\Sigma^0(\mathbf{q})\rangle\,,$$

can, in the limit of exact $SU(3)$ *Eightfold Way* symmetry (in which limit Λ^0 and Σ^0 belong to the same octet), be approximated by a single magnetic dipole term and the formfactor $F_2(Q^2)$, $Q = q - p$, multiplying this term can be approximated by its value at $Q^2 = 0$ which is called the magnetic dipole transition moment $\mu^{\Lambda\Sigma}$. From the Σ^0 lifetime $\tau_{\Sigma^0} = 7.4 \times 10^{-20}$ sec. find the numerical value of $F_2(0)$ and compare it with the one given by PDG. Using the same approximations compute also the branching fraction of the decay $\Sigma^0 \to \Lambda^0 e^- e^+$. Compare the result with the number quoted by PDG. Try to apply the similar approach to the decay $\Sigma^+ \to p\gamma$. Why is the numerical value of the constant parametrizing the hadronic matrix element $\langle p(\mathbf{p})|J^{\lambda}_{\mathrm{EM\ hadr}}(x)|\Sigma^+(\mathbf{q})\rangle$ much smaller in this case? Can the relevant hadronic matrix element be parametrized by a single constant only?

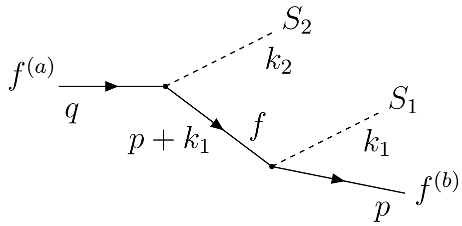

Figure 2: Decay of a spin $\frac{1}{2}$ fermion $f^{(a)}$ into another spin $\frac{1}{2}$ fermion $f^{(b)}$ and two (distinct) spinless particles S_2 and S_1 with the intermediate state of a massive spin $\frac{1}{2}$ fermon f.

Problem V.38
By appealing to the "$\Delta I = \frac{1}{2}$" rule explain the ratio of the lifetimes of the Ξ^- and Ξ^0 baryons: $\tau_{\Xi^-} = 1.639 \times 10^{-10}$ sec., $\tau_{\Xi^0} = 2.90 \times 10^{-10}$ sec. Ξ^- decays with the branching fraction of 99.887% into $\Lambda^0\pi^-$ and Ξ^0 into $\Lambda^0\pi^0$ with branching fraction of 99.525%.

Predict also the ratios of the branching fractions $\text{Br}(\Lambda^0 \to p\,\pi^-)/\text{Br}(\Lambda^0 \to n\,\pi^0)$ and $\text{Br}(K_S \to \pi^+\pi^-)/\text{Br}(K_S \to 2\pi^0)$ and compare these predictions with the data.

Problem V.39
Consider the decay of a spinless particle $S^{(a)}$ of mass M into two spinless particles: massless S and massive $\tilde{S}$ (of mass $m < M$) which decays into another massless particle S and yet another massless one, $S^{(b)}$ (distinct from S), see figure 1. Assuming that the relevant interactions are given by

$$\mathcal{L}_{\text{int}} = -g\,\varphi_{(a)}\varphi\tilde{\varphi} - h\,\varphi_{(b)}\varphi\tilde{\varphi}\,,$$

and including in the $\tilde{S}$ propagator (by means of the subsitution $m^2 \to m^2 - im\Gamma_{\text{tot}}$) the total $\tilde{S}$ width Γ_{tot} show that for $\Gamma_{\text{tot}} \ll m$ (i.e. if the particle $\tilde{S}$ is a narrow resonance) the decay rate can be approximated by $\Gamma(S^{(a)} \to \tilde{S}S) \times \text{Br}(\tilde{S} \to S^{(b)}S)$. Compute the decay rate $\Gamma(S^{(a)} \to S^{(b)}SS)$ numerically and study the role of the Bose-Einstein correlation (i.e. of the interference of the two diagrans shown in figure 1) as a function of the ratio m/Γ_{tot}.

Problem V.40
Consider the decay of a spin $\frac{1}{2}$ fermion $f^{(a)}$ of mass M into another spin $\frac{1}{2}$ fermion $f^{(b)}$ and two (distinct) spinless particles S_2 and S_1 (see Figure 2) due to the interaction

$$\mathcal{H}_{\text{int}} = \bar{\psi}_{f^{(b)}}h(1 - \lambda\gamma^5)\psi_f S_1 + \bar{\psi}_f(c_L\mathbf{P}_L + c_R\mathbf{P}_R)\psi_{f^{(a)}}S_2 + \text{H.c.}\,,$$

taking for simplicity all the final state particles to be massless. Include the total width Γ_{tot} of the intermediate fermion f replacing m^2 by $m^2 - im\Gamma_{\text{tot}}$ in the denominator of its propagator and analyse the problem of applying the narrow width approximation, that is the problem of approximating $\Gamma(f^{(a)} \to f^{(b)}S_1S_2)$ by the appropriate characteristics of the two consecutive two-body decays: first of $f^{(a)}$ and then of f, assuming that the mass m of the fermion f is smaller than the mass of the decaying fermion M and $\Gamma_{\text{tot}} \ll m$ (f is a narrow resonance). In particular, show that for $\lambda = 0$, when the spin of $f^{(b)}$ is not measured, the width of the decay of unpolarized $f^{(a)}$ into $f^{(b)}S_1S_2$ can be approximated by

$$\Gamma(f^{(a)} \to f^{(b)}S_1S_2) \approx \Gamma(f^{(a)} \to fS_2) \times \mathrm{Br}(f \to f^{(b)}S_1)\,.$$

Study also the quality of this approximation by comparing this formula with the exact decay width $\Gamma(f^{(a)} \to f^{(b)}S_1S_2)$ computed numerically.

Problem V.41 (Numerical exercise.)
Write a Monte Carlo numerical programme computing the distribution of the invariant mass $(p + k_1)^2$ of the particles $f^{(b)}$ and S_1 originating from the decay $f^{(a)} \to f^{(b)}S_1S_2$ considered in Problem V.40.

Appendix. The Monte Carlo method

A very convenient method of numerical evaluation of complicated (multidimensional) integrals is the Monte Carlo method. Suppose we want to evaluate the integral

$$I_\Delta = \int_\Delta d^n\mathbf{x}\, f(\mathbf{x})\,,$$

where Δ is some n-dimensional domain of (known) volume V_Δ. The estimate of I_Δ is provided by

$$I_\Delta^{\mathrm{est}} = \frac{1}{N}\sum_{i=1}^{N} V_\Delta\, f(\mathbf{x}_i) \equiv \langle V_\Delta\, f\rangle\,,$$

where $\mathbf{x}_i$ are N uniformly generated random points belonging to Δ. The error of the estimate is given by

$$\left|I_\Delta - I_\Delta^{\mathrm{est}}\right| \sim \frac{1}{\sqrt{N}}\,\sigma_N(\Delta)\,,$$

where

$$\sigma_N^2(\Delta) = \langle V_\Delta^2\, f^2\rangle - \langle V_\Delta\, f\rangle^2\,.$$

Thus the error of the estimate decreases always like $1/\sqrt{N}$, independently of the number of dimensions.

This method is particularily well suited for evaluation of integrals over complicated domains (usually determined by some conditions that are hard to solve). To illustrate this point suppose we need to find

$$I_{\Delta'} = \int_{\Delta'} d^n\mathbf{x}\, f(\mathbf{x})\,,$$

where Δ' is a domain whose boundaries are determined by some conditions $h_a(\mathbf{x}) = 0$, $a = 1, \ldots, m$. If these conditions are complicated the volume $V_{\Delta'}$ of Δ' may be not easy to find, so that it is impossible to use directly the original Monte Carlo formula given above. However if we chose a larger domain Δ of known volume V_Δ and such that $\Delta' \subset \Delta$, the estimate of $I_{\Delta'}$ can be obtained by generating uniformly random points in the whole domain Δ: it is simply given by

$$I_{\Delta'}^{\mathrm{est}} = \frac{1}{N}\sum_{i=1}^{N} V_\Delta\, f(\mathbf{x}_i)\,\Theta(\mathbf{x}_i) \equiv \langle V_\Delta\, f\,\Theta\rangle\,,$$

with

$$\Theta(\mathbf{x}_i) = \begin{cases} 1 & \text{if} \quad \mathbf{x}_i \in \Delta' \\ 0 & \text{if} \quad \mathbf{x}_i \notin \Delta' \end{cases}\,,$$

(if the boundaries of Δ' are determined by a set of conditions $h_a(\mathbf{x}) = 0$, it is straightforward to reject points $\mathbf{x}_i \notin \Delta'$). Indeed, the formula can be rewritten as

$$I_{\Delta'}^{\text{est}} = \frac{1}{N'} \sum_{i=1}^{N'} \left(\frac{N'}{N} V_\Delta \right) f(\mathbf{x}_i)\, \Theta(\mathbf{x}_i)\,,$$

where N' is the number of generated points which belong to Δ' and $(N'/N)V_\Delta$ is just the estimate of the volume $V_{\Delta'}$ (and can be replaced by it, if it is known). The error of the estimate is then given by

$$\left| I_{\Delta'} - I_{\Delta'}^{\text{est}} \right| \sim \frac{1}{\sqrt{N'}}\, \sigma_N(\Delta')\,,$$

with

$$\sigma_N^2(\Delta') = \langle V_\Delta^2\, f^2\, \Theta \rangle - \langle V_\Delta\, f\, \Theta \rangle^2\,.$$