Quantum Field Theory of Fundamental Interactions. Problems set VI.

Problem VI.1

Consider the scattering process $A+B \to C+D$. Show that in the center of mass system (CMS) the factor $F = 4\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}$ can be written as

$$F = 4|\mathbf{k}_i|\sqrt{s}\,,$$

where $\mathbf{k}_i = \mathbf{k}_A = -\mathbf{k}_B$ and

$$s = (k_A + k_B)^2 = (E_A + E_B)^2\,,$$

whereas the final state phase space factor $dQ = (2\pi)^4\delta^{(4)}(k_A + k_B - p_C - p_D)d\Gamma_{\mathbf{p}_C} d\Gamma_{\mathbf{p}_D}$ in the expression $d\sigma = (1/F)\sum|\mathcal{A}|^2 dQ$ can be integrated to give

$$dQ = \frac{|\mathbf{p}_f|}{16\pi^2\sqrt{s}}\, d\Omega_{\mathbf{p}_f}\,,$$

where $\mathbf{p}_f = \mathbf{p}_C = -\mathbf{p}_D$ and $d\Omega_f = d\phi_C d\theta_C \sin\theta_C$ (ϕ_C and θ_C specify the direction of $\mathbf{p}_C$ with respect to $\mathbf{k}_A$), so that the differential cross section reads

$$d\sigma(\theta,\phi) = \frac{1}{64\pi^2 s}\,\frac{|\mathbf{p}_f|}{|\mathbf{k}_i|}\,|\mathcal{A}|^2\, d\Omega_{\mathbf{p}_f}\,.$$

Express $|\mathbf{k}_i|$ and $|\mathbf{p}_f|$ in terms of s and the particle masses.

Problem VI.2

The $pp \to \pi^+ D$ cross section (D stands for Deuterium of mass $M_D = 1874.98$ MeV) measured in the Hydrogen fixed target experiment with the proton kinetic energy[1] $T_p =$ 340 MeV is $\sigma(pp \to \pi^+ D) = 0.18$ mb. In turn, the cross section $\sigma(\pi^+ D \to pp)$ measured in the Deuterium fixed target experiment with $T_\pi = 25$ MeV is about 3 mb. By appealing to the T-invariance of the strong interactions show that these result imply that pion is a spinless particle.

Problem VI.3

Using the weak interaction Hamiltonian

$$\mathcal{H}_{\rm weak} = \frac{G_F}{\sqrt{2}}\, J^\dagger_\lambda J^\lambda \qquad \text{where} \qquad J^\lambda = J^\lambda_{\rm lept} + J^\lambda_{\rm hadr}\,,$$

$$J^\lambda_{\rm lept} = \bar\psi_{(e)}\gamma^\lambda(1-\gamma^5)\psi_{(\nu_e)} + \bar\psi_{(\mu)}\gamma^\lambda(1-\gamma^5)\psi_{(\nu_\mu)} + \bar\psi_{(\tau)}\gamma^\lambda(1-\gamma^5)\psi_{(\nu_\tau)}\,,$$

compute the differential (with respect to the final charged lepton direction) and the total cross sections of the processes $\nu_\mu e^- \to \nu_e \mu^-$ and $\bar\nu_e e^- \to \bar\nu_\ell \ell^-$. Perform calculations both

[1] By kinetic energy one means $T_p \equiv E_p - m_p c^2 = \sqrt{\mathbf{k}_p^2 c^2 + m_p^2 c^4} - m_p c^2$.

in the CMS and in the Laboratory system (electron initially at rest). Give the total CMS cross sections in barns, ($1b = 10^{-28}\ m^2$) for $\sqrt{s} = 10$ MeV and 100 GeV. What is the minimal (threshold) energy of ν_μ capable to initiate the process $\nu_\mu e^- \to \nu_e \mu^-$ in the Laboratory system? Explain the angular dependence of these differential cross sections in the limit in which lepton masses can be neglected by appealing to angular momentum conservation. What is the cross section for the process $\bar{\nu}_\mu e^- \to \bar{\nu}_e \mu^-$?

Problem VI.4
Using the Hamiltonian given in Problem VI.3 find the partial wave amplitudes $\mathcal{T}^{(j)}_{\lambda_\ell \lambda_{\nu_e}, \lambda_{\nu_\ell} \lambda_e}(s)$ of the process $\nu_\ell e^- \to \nu_e \ell^-$ and determine the energy at which the lowest order (in G_F) elastic scattering amplitude fails to satisfy the unitarity bound. Ignore the possible existence of the neutral currents interaction.

Problem VI.5
Assume that the (charged currents) weak interactions are mediated by the spin 1 particles $W^\pm$ of mass $M_W \gg m_e$, so that the Hamiltonian of weak leptonic processes is

$$\mathcal{H}_{\rm weak} = \frac{g_2}{2\sqrt{2}} J^\lambda W^-_\lambda + \frac{g_2}{2\sqrt{2}} (J^\lambda)^\dagger W^+_\lambda \,.$$

Find the partial wave amplitudes of the process $\nu_\ell e^- \to \nu_e \ell^-$ and reconsider the determination of the unitarity bound.

Problem VI.6
Consider a field theory of four real scalar fields π^a, $a = 1, 2, 3$ and η with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^{3} \left(\partial_\mu \pi^a \partial^\mu \pi^a - M_\pi^2\, \pi^a \pi^a \right) + \frac{1}{2} \partial_\mu \eta \partial^\mu \eta - \frac{1}{2} M_\eta^2 \eta^2$$
$$- \frac{\kappa}{2} \left(\eta^2 + \sum_{a=1}^{3} \pi^a \pi^a \right) \eta - \frac{\lambda}{4} \left(\eta^2 + \sum_a \pi^a \pi^a \right)^2 .$$

Find in the lowest order amplitudes of the processes $\pi^+\pi^- \to \pi^0\pi^0$, $\pi^+\pi^+ \to \pi^+\pi^+$, $\pi^+\pi^0 \to \pi^+\pi^0$ etc. where the one particle states of π^+, π^- and π^0 are the common eigenstates of H_0, $\hat{\mathbf{T}}^2 \equiv (\hat{T}^1)^2 + (\hat{T}^2)^2 + (\hat{T}^3)^2$ (the total isospin) and $\hat{T}^3$ (the isospin third component) operators found in Problem III.10. Construct the S-matrix elements in the isospin basis

$$S_{I', I'_3; I, I_3} = \langle I', I'_3, \mathbf{p}_1, \mathbf{p}_2 | T \exp \left(i \int d^4x\, \mathcal{L}_I \right) | I, I_3, \mathbf{k}_1, \mathbf{k}_2 \rangle \,,$$

where $|I, I_3, \mathbf{k}_1, \mathbf{k}_2\rangle$ are the two-particle eigenstates of H_0, $\hat{\mathbf{T}}^2$ and of $\hat{T}^3$. Check by direct calculation that $S_{I', I'_3; I, I_3} = S^I \delta_{I', I} \delta_{I'_3, I_3}$ that is, that the amplitudes do not depend on I_3. Express the amplitudes $\mathcal{A}$ of all possible $\pi\pi$ scatterings in terms of the isospin amplitudes $\mathcal{A}^I$.

By considering transitions between all possible pairs of two-particle states (including the η particle) show that in the limit $\sqrt{s} \gg M_\eta > M_\pi$, where $s = (k_1 + k_2)^2$, there are only three independent nonzero amplitudes which correspond to diagonal transitions within three different representations of the $SO(4)$ group realizable on two-particle states of spinless particles.

Problem VI.7
Realistic interactions of low energy pions (in the limit of vanishing their masses) are described (to a good approximation) by the Lagrangian density

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{tr}\left(\partial^\mu U \partial_\mu U^{-1}\right) + \dots ,$$

(the ellipses stand for terms with more derivatives) where $U^{-1} = \exp\left(i\tau^a \pi^a / f_\pi\right)$ with τ^a the three Pauli matrices and $f_\pi \approx 93$ MeV called the pion decay constant (its value is determined in Problem VI.9). Using this Lagrangian find in the lowest order the amplitudes of all possible binary scatterings ($\pi^+\pi^+ \to \pi^+\pi^+$, $\pi^+\pi^- \to \pi^+\pi^-$, $\pi^+\pi^- \to \pi^0\pi^0$, etc.) and show as in the preceding Problem that

$$\mathcal{A}(I, I_3, \mathbf{k}_1, \mathbf{k}_2 \to I', I'_3, \mathbf{p}_1, \mathbf{p}_2) = \delta_{II'}\, \delta_{I_3 I'_3} \mathcal{A}^I(\mathbf{k_1}, \mathbf{k_2} \to \mathbf{p_1}, \mathbf{p_2}) \,.$$

Find the isospin amplitudes $\mathcal{A}^I \equiv \mathcal{A}(I, I_3, \mathbf{k}_1, \mathbf{k}_2 \to I, I_3, \mathbf{p}_1, \mathbf{p}_2)$.

Problem VI.8
The (fictitious) Hamiltonian of the three π mesons of masses M_π interacting with a neutral spinless particle η of mass M_η has the form

$$\mathcal{H}_{\mathrm{int}}(x) = \frac{\kappa}{2}\left(\eta^2 + \sum_{a=1}^{3} \pi^a \pi^a\right)\eta + \frac{\lambda}{4}\left(\eta^2 + \sum_{a=1}^{3} \pi^a \pi^a\right)^2 .$$

Find the partial amplitudes $\mathcal{T}^{(l)}(s)$ of the elastic $\pi^+\pi^-$ scattering defined by the expansion of the scattering amplitude $\mathcal{A}$

$$\mathcal{A}(s, \cos\theta) = 16\pi \sum_{l=0}^{\infty} (2l+1) \mathcal{T}^{(l)}(s) P_l(\cos\theta) \,,$$

where $P_l(x)$ are the Legendre polynomials. Express the differential and total cross sections in the CMS system through the amplitudes $\mathcal{T}^{(l)}(s)$. What constraints on the coupling constants λ and κ follow from the (asymptotic) unitarity bounds

$$N\left|\mathcal{T}^{(l)}(s)\right| < 1 \,, \qquad N\left|\mathrm{Re}\,\mathcal{T}^{(l)}(s)\right| < \frac{1}{2} \ ?$$

($N = 1$ for different particles and $N = \frac{1}{2}$ for identical final state particles). Optimize the constraint on λ by considering amplitudes of all possible binary reactions (including also those involving the η particle) for $\sqrt{s} \gg M_\eta$, M_π.

Observe, that the partial wave amplitude $\mathcal{T}^{(l=0)}(s)$ of the elastic $\pi^+\pi^-$ scattering computed in the lowest order has a simple pole at $s = M_\eta^2$. Check that including the width of the η particle in its tree level propagator by the substitution

$$\frac{i}{q^2 - M_\eta^2} \qquad \rightarrow \qquad \frac{i}{q^2 - M_\eta^2 + iM_\eta\Gamma_\eta}\,,$$

(with Γ_η computed in the lowest order) restores unitarity of the scattering amplitude saturating (up to nonresonant terms) the basic unitarity relation (for $l = 0$).

Problem VI.9
Consider the interaction of the charged particles π of mass M_π with a massive spin one particle of mass M_V

$$\mathcal{L}_{\rm int} = -igV^\mu\left(\pi^*\partial_\mu\pi - \partial_\mu\pi^*\pi\right) - \lambda(\pi^*\pi)^2\,.$$

Find the partial wave amplitudes $\mathcal{T}^{(l)}(s)$ of the elastic $\pi^+\pi^-$ scattering. As in the preceding problem investigate the constraints imposed by unitarity on the partial amplitudes $\mathcal{T}^{(l)}(s)$.

Problem VI.10
Imposing the unitarity bounds on the pion scattering amplitudes determine the range of energies for which the interaction (see Problem VI.7)

$$\mathcal{L} = \frac{f_\pi^2}{4}{\rm tr}\left(\partial_\mu U\partial_\mu U^{-1}\right) + \dots\,,$$

can be used in the tree level approximation.

Problem VI.11
Find the amplitudes of binary pion scatterings as in Problem VI.7 but taking into account finite pion masses by using the Lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{4}\,{\rm tr}\left(\partial_\mu U\partial_\mu U^{-1}\right) + \frac{f_\pi^2 M_\pi^2}{4}\,{\rm tr}\left(U + U^{-1}\right) + \dots\,,$$

In the lowest order find the pion scattering phase shifts $\delta_I^{(l)}$ and the pion scattering lengths.

Problem VI.12
Using the interaction $\mathcal{H}_{\rm int}(x)$ found in Problem III.11 write down the lowest order (in the coupling constant) amplitudes of the pion-nucleon scattering. Check by direct calculation that $S_{I',I_3';I,I_3} = S^I\delta_{I',I}\delta_{I_3',I_3}$.

Problem VI.13
Do the same as in Problem VI.12 for the nucleon-antinucleon annihilation into two pions.

Problem VI.14
Consider the Yukawa interaction $\mathcal{H}_{\text{int}} = h\,\bar{\psi}\psi\varphi$, where h is the coupling constant, of a spinless neutral particle η with fermions f (and their antifermions $\bar{f}$). Calculate in the lowest order in h the differential cross section for elastic scattering $\eta\bar{f} \to \eta\bar{f}$. Assume that the initial antifermions are unpolarized and the final antifermion spin is not measured.

Problem VI.15
Let ψ_a and ψ_b be the field operators of fermions (antifermions) f_a ($\bar{f}_a$) and f_b ($\bar{f}_b$) with masses m_a and m_b, respectively. Using the interaction $\mathcal{L}_{\text{int}} = -ig_a\bar{\psi}_a\gamma^5\psi_a\varphi - ig_b\bar{\psi}_b\gamma^5\psi_b\varphi$ compute in the CMS the differential and total cross sections for processes: $f_a\bar{f}_a \to f_b\bar{f}_b$, $f_a\bar{f}_b \to f_a\bar{f}_b$ and $f_a\bar{f}_a \to f_a\bar{f}_a$. Assuming that m_a, $m_b \ll M$, where M is the mass of the neutral spinless particle described by φ write down in each case the effective Lagrangian with contact interaction of fermions which reproduces the scattering amplitude for energies of the colliding fermions much smaller than M.

Problem VI.16
In quantum electrodynamics compute (in the lowest order in e) the CMS differential and total cross sections for production of a $\mu^-\mu^+$ pair in the e^-e^+ collision. Compare the angular distribution of the produced μ^- with the distribution of (hypotetical) spinless $\tilde{\mu}$ particles produced in the e^-e^+ collision.

Problem VI.17
Explain the dependence on the scattering angle of the cross sections computed in Problem 16 by studing in the high energy limit (negligible particle masses) annihilation and production of particles of definite helicities.

Problem VI.18
In quantum electrodynamics of electrons and photons write down the lowest order amplitude for elastic γe^- scattering (the Compton process) and check that it is gauge invariant, that is, it vanishes when any of the two photon polarization vectors $\epsilon_\mu(\mathbf{k}_i, \lambda_i)$ is replaced by the four-momentum k_i^μ of the corresponding photon. Compute the differential (in the laboratory frame) and total cross sections.

Problem VI.19
Supersymmetric theories predict the existence of a spin 0 partner for each fermion (e.g. the supersymmetric partners of $e^\pm$ are the *selectrons* $\tilde{e}^\pm$) and of neutral fermions N^0 called neutralinos (which are supersymmetric partners of the Higgs boson and gauge bosons). Calculate the differential cross section for the process $\gamma N^0 \to e^-\tilde{e}^+$. Assume the most general (not necessarily parity conserving) form of the neutralino-electron-selectron vertex (photon interaction vertices are standard). Fix the relative sign between the two amplitudes contributing in the lowest order by appealing to the gauge invariance.

Problem VI.20
Consider scattering of photons on charged (charge Q in units $e > 0$) spinless particles of

mass M in the Laboratory frame. The interaction is

$$\mathcal{L} = ieQA^\mu(\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi) + e^2Q^2A^\mu A_\mu\phi^\dagger\phi\,.$$

Compute the differential cross section for finding the scattered photon at an angle θ (with respect to the direction of the initial photon) with polarization λ_2, if the initial photon has momentum $\mathbf{k}_1$ and polarization λ_1. Find also the differential cross section averaged over polarizations of the initial photon in the case the polarization of the final photon is not measured. To compute the latter cross section, construct explicitly the polarization vectors of the photons choosing them to be purely spatial (this eliminates two of the three terms in the amplitude) and perform the necessary sumation over polarizations using these vectors. To appreciate, how more efficient this approach is, recover the same result using the Feynman trick $\sum_\lambda \epsilon_\mu(\mathbf{k},\lambda)\epsilon^*_\nu(\mathbf{k},\lambda) \to -g^{\mu\nu}$.

Problem VI.21

Consider the production process $S_1(\mathbf{k}_1) + S_2(\mathbf{k}_2) \to \tilde{S}_1(\mathbf{p}_1) + \tilde{S}_2(\mathbf{p}_2) + \tilde{S}_3(\mathbf{p}_3)$ where all S_i and $\tilde{S}_i$ are spinless particles. The proces occurs (in the tree Feynman diagram) through the s-channel annihilation of S_1S_2 into a (virtual) spinless particle of mass m which goes into $\tilde{S}_3$ and another spinless particle S^* of mass M and width $\Gamma_{\rm tot}$ which decays producing $\tilde{S}_2$ and $\tilde{S}_1$ (there may also be other Feynman diagrams contributing to the total amplitude $\mathcal{A}[S_1(\mathbf{k}_1) + S_2(\mathbf{k}_2) \to \tilde{S}_1(\mathbf{p}_1) + \tilde{S}_2(\mathbf{p}_2) + \tilde{S}_3(\mathbf{p}_3)]$). Show that if $\Gamma_{\rm tot} \ll M$ (S^* is a narrow width resonance) then for $\sqrt{s} > M$ the cross section $\sigma(S_1S_2 \to \tilde{S}_1\tilde{S}_2\tilde{S}_3)$ can be approximated by

$$\sigma(S_1S_2 \to \tilde{S}_1\tilde{S}_2\tilde{S}_3) \approx \sigma(S_1S_2 \to \tilde{S}_3S^*) \times {\rm Br}(S^* \to \tilde{S}_1\tilde{S}_2)\,.$$

Compare this approximation with the full $\sigma(S_1S_2 \to \tilde{S}_1\tilde{S}_2\tilde{S}_3)$ cross section numerically by taking the initial and final particles to be massless (so that the results of the Problem V.2 for the final phase space can be used). Take e.g. $m = 100$ GeV, $M = 10$ GeV (so that the peak associated with the s-channel resonanse of mass m does not distort the cross section appreciably) and plot both cross sections as a function of $\sqrt{s}$ for 1 GeV $< \sqrt{s} <$ 30 GeV and several values of $\Gamma_{\rm tot}$.

Problem VI.22

Let the interaction of a massive gauge boson Z^0 with electrons be $\mathcal{L}_{\rm int} = -gZ^0_\mu\bar{\psi}_e\gamma^\mu\psi_e$. Show that at the tree level the following relation holds

$$\sigma(e^-e^+ \to Z) = \frac{12\pi^2}{M_Z}\Gamma(Z^0 \to e^-e^+)\;\delta(s - M_Z^2)\,,$$

where $\sqrt{s}$ is the energy in the e^-e^+ center of mass system and M_Z is the Z mass. Show also that the sum over the three polarizations of the Z boson can be done using $-g_{\mu\nu}$ instead of $-g_{\mu\nu} + q_\mu q_\nu/M_V^2$.

Problem VI.23

Assume the coupling of the massive spin 1 boson Z^0 to leptons ℓ of the following general form

$$\mathcal{L}_{\text{int}} = -\bar{\psi}_\ell \gamma^\kappa (c_L \mathbf{P}_L + c_R \mathbf{P}_R) \psi_\ell Z^0_\kappa \, .$$

Compute in the lowest order the forward-backward asymmetry of the e^+e^- scattering into $\mu^+\mu^-$ defined in the center of mass system:

$$A_{\text{FB}} = \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} \, ,$$

where

$$\sigma_+ = \int_0^{+1} d(\cos\theta) \, \frac{d\sigma}{d(\cos\theta)} \, , \qquad \sigma_- = \int_{-1}^{0} d(\cos\theta) \, \frac{d\sigma}{d(\cos\theta)} \, ,$$

Express A_{FB} through $c^\ell_{L,R}$.