

## Lecture 3 BEC

What happens if we subject the condensate to an action of a time dependent potential? In the mean field approximation we may expect that the single particle wave function will satisfy the time dependent version of the GP equation. A more formal justification will come when we turn to a quantum field description of the dynamics. Thus

$$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \varphi + V_r \varphi + Ng |\varphi|^2 \varphi \quad (1)$$

is the time dependent GP equation. We may expect that if the condensate is not shaken too rapidly it still behaves as a condensate, a droplet of quantum matter. Many useful results were obtained based on the time dependent GP equation.

It is instructive to reformulate the condensate dynamics in terms of the hydrodynamic formulation.

We represent the complex wave function in terms of two real functions: the amplitude and the phase:

$$\psi(\vec{r}) = \sqrt{\rho} \exp[i\varphi] \quad (2)$$

As in my QM3 and QM4, it is useful to replace the phase by its gradient:

$$\vec{v} = \frac{\hbar}{m} \nabla \varphi \quad (3)$$

In the present, nonlinear case the hydrodynamic equations take a form:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \text{div}[\rho \vec{v}] &= 0 \\ m \frac{\partial \vec{v}}{\partial t} &= -\nabla[V_r - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} + \frac{1}{2} m v^2 + gN\rho] \end{aligned} \quad (4)$$

Only the last term indicates our nonlinear condensate problem. The TF approximation consists in omitting the “quantum pressure term” from equation (4). In the static case we get back the parabolic TF solution. In the full dynamics static case  $\vec{v} = 0$ . In a small region where we can neglect the change of the trapping potential, the equilibrium requires that the quantum pressure term is canceling the effect of nonlinearity:

$$\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = gN\rho \quad (5)$$

A typical situation is in a box container with walls being an infinite potential. In such a container the condensate density must grow from zero to its bulk value over a distance called the healing length. A very rough estimate gives:

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$$\frac{\hbar^2}{2m} \frac{1}{\xi^2} \approx \frac{4\pi\hbar^2 aN}{mV} \quad (6)$$

Assuming the condensate is spread evenly in the TF sphere we obtain:

$$\xi = \frac{15^{3/10} d^{6/5}}{6^{1/2} a^{1/5} N^{1/5}} \quad (7)$$

So the healing length divided by the TF radius is:

$$\frac{\xi}{R} = \frac{15^{1/10}}{6^{1/2}} \left[ \frac{d}{aN} \right]^{2/5} \quad (8)$$

and for realistic condensate is tiny.

The hydrodynamic equations may be simplified in TF approximation also for not so violent dynamics of the condensate:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \text{div}[\rho \vec{v}] &= 0 \\ m \frac{\partial \vec{v}}{\partial t} &= -\nabla \left[ \frac{1}{2} m \omega^2(t) r^2 + \frac{1}{2} m v^2 + gN\rho \right] \end{aligned} \quad (9)$$

We intend to check the behavior of the condensate under the influence of changing frequency of the trapping potential. Suppose  $\varphi(r)$  is the TF static solution with the initial value of the trapping frequency. Several authors found that in the TF approximation the time dependent solution of equations (9) is of the self-similar form:

$$\varphi_b(r,t) = \frac{1}{b^{3/2}(t)} \varphi\left(\frac{r}{b(t)}\right) \exp[i\chi(r,t)] \quad (10)$$

The phase is necessary since the condensate must move.

$$\vec{v} = \vec{r} \frac{\dot{b}}{b} \quad (11)$$

and the scale factor satisfies:

$$\ddot{b} = -\omega^2(t)b + \frac{\omega_0^2}{b^4} \quad (12)$$

which has a form of Newton equation for the scale factor. Let us consider two cases:

1. Free expansion of the condensate. At  $t=0$  we turn-off the trapping potential and let the gas expand. The scale factor satisfies:

$$\ddot{b} = \frac{\omega_0^2}{b^4} \text{ and the motion is like sliding from a potential barrier } V = \frac{\omega_0^2}{3b^3}. \text{ The initial condition}$$

is  $b(0) = 1$ . Thus the ‘‘conservation of energy’’ gives for the asymptotic speed:

$$\dot{b}(\infty) = \omega_0 \sqrt{\frac{2}{3}} \quad (13)$$

So speed is higher for stronger confinement. Thus expanding asymmetric condensate changes its aspect ratio (wait for my afternoon presentation).

2. Now, suppose we shake the condensate and then return the frequency to its initial value. The equation takes a form:

$$\ddot{b} = -\omega^2 b + \frac{\omega^2}{b^4} \quad (14)$$

So it is a “potential” motion in  $V(b) = \omega^2 \left[ \frac{b^2}{2} + \frac{1}{3b^3} \right]$ .

This potential has a minimum at  $b=1$  and expansion around this minimum gives a frequency of small vibrations:  $\omega = \omega_0 \sqrt{5}$ . Very large vibrations have a frequency  $\omega = 2\omega_0$ .

Castin and Dum has extended the self-similar expansion to the asymmetric trap introducing separate equation for scale factors in all three dimensions. As a result they obtained a set of 3 “Newton like” coupled equations.

### Small vibrations

Equations (9) may be linearized around the static TF parabolic distribution:

$$\rho = \rho_0 + \delta\rho = \frac{1}{gN}(\mu - V_r) + \delta\rho \quad (15)$$

$$\vec{v} = \delta\vec{v}$$

We get:

$$\begin{aligned} \frac{\partial \delta\rho}{\partial t} + \text{div}[\rho_0 \delta\vec{v}] &= 0 \\ m \frac{\partial \delta\vec{v}}{\partial t} &= -\nabla[gN\delta\rho] \end{aligned} \quad (16)$$

Eliminating the velocity:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \text{div}\left[\frac{\rho_0(r)gN}{m} \text{grad}(\delta\rho)\right] \quad (17)$$

we get a compact equation (17) for the small deformations. For a homogeneous system

$\rho_0 = \frac{1}{V}$  and the equation (17) becomes a wave equation:

$$\frac{1}{c^2} \frac{\partial^2 \delta\rho}{\partial t^2} - \Delta \delta\rho = 0 \quad (18)$$

with the sound velocity:  $c = \sqrt{\frac{gn}{m}}$  where  $n = \frac{N}{V}$  is the number density.

Back to the harmonic trap:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \nabla \left[ \left( \mu - \frac{1}{2} m \omega^2 r^2 \right) \nabla (\delta\rho) \right] \quad (19)$$

We may look for harmonic time dependence of the radial function. This equation has spherical symmetry so we can look for the solutions in the spherical harmonics expansion:

$$\delta\rho = P_l(r/R) r^l Y_{lm}(\vartheta, \varphi) \quad (20)$$

This is a second order equation for the radial function subject to boundary conditions: nonsingular at the origin and zero at the TF radius. This leads to a quantization condition for the eigenfrequencies. They DO NOT DEPEND on the interaction:

$$\omega(n_r, l) = \omega(2n_r^2 + 2n_r l + 3n_r + l)^{1/2} \quad (21)$$

s-wave, lowest solution is  $\omega(1,0) = \sqrt{5}\omega$  again.

It is worth noting the free oscillator frequencies expressed in terms of spherical coordinates:

$$\omega(n_r, l) = \omega(2n_r + l) \quad (22)$$

The lowest order frequencies (21) were tested in experiments. More precisely they tested generalization to axial and not spherical symmetric traps. Separate problem is a deviation due to temperature. More about it in my afternoon lecture on Wednesday.

All results based on the time dependent GP equations are referred to as the **mean field results**.

### Splitting of the condensate

Let us look at a highly simplified model of rapid splitting of the condensate (by a knife of the laser beam). The orbital of the condensed atoms is split into 2 parts, the left and the right:

$$\varphi(\vec{r}) = \varphi_L(\vec{r}) + \varphi_R(\vec{r}) \quad (23)$$

Thus the multiatom wave function takes a form:

$$\Psi(r_1, r_2, \dots, r_N) = \prod_{i=1}^N (\varphi_L(r_i) + \varphi_R(r_i)) \quad (24)$$

This product is a sum with varying number of atoms on the left and remaining, also varying

on the right. There are  $\binom{N}{k} = \frac{N!}{k!(N-k)!}$  terms with  $k$  atoms on the left and  $(N-k)$  on the

right. So the state (24) may be written in the form:

$$|\Psi\rangle = \sum_{k=0}^N c(N,k) |k, N-k\rangle \quad (25)$$

$$p = \langle \varphi_L | \varphi_L \rangle$$

with obvious meaning of the  $|k, N-k\rangle$  state. If  $q = \langle \varphi_R | \varphi_R \rangle$  the coefficients

$$p + q = 1$$

$$c(N,k) = \sqrt{\binom{N}{k}} p^{k/2} q^{(N-k)/2} \quad (26)$$

Which, as it is easy to check, is properly normalized. If we let the two separate condensates evolve in time, each term acquires its own phase factor:

$$|\Psi(t)\rangle = \sum_{k=0}^N \sqrt{\binom{N}{k}} p^{k/2} q^{(N-k)/2} \exp\left[-\frac{it}{\hbar}(E(k) + E(N-k))\right] |k, N-k\rangle \quad (27)$$

For the ideal Bose gas, the energy is a linear function of the number of atoms and the evolution would be just a common phase factor. For the interacting gas there is a nonlinearity and, as a result, the terms in (27) undergo a dephasing. We have a decoherence. For a fifty-fifty splitting, the distribution (26) is sharply peaked at  $N/2$ . So the decoherence is well approximated by the small parameter expansion in the exponent.

$$E(k) + E(N-k) = 2E(N/2) + E''(N/2)(N/2 - k)^2 \quad (28)$$

So

$$|\Psi(t)\rangle = \frac{1}{2^{N/2}} \exp\left[-\frac{2itE(N/2)}{\hbar}\right] \sum_{k=0}^N \sqrt{\binom{N}{k}} \exp\left[-\frac{itE''(N/2)(N/2 - k)^2}{\hbar}\right] |k, N-k\rangle$$

Hence, upon the removal of the splitting laser beam, the interference fringes are gradually having less contrast. If, however, you wait long enough, the coherence will come back. It has been observed in a famous experiment of Immanuel Bloch. Some more info about it on Wednesday.

### One more example of single particle density matrix

On Friday we have seen how the Onsager-Penrose definition of the condensate wave function works for an ideal Bose gas at zero temperature. Now, just to get better intuitions here is another example. Imagine the ideal gas has  $N$  atoms, of which  $k$  are in a normalized state  $\varphi_1$  and  $N-k$  are in an orthogonal state  $\varphi_2$ . The normalized multiparticle wave function has a form:

$$\Psi(1, \dots, N) = \binom{N}{k}^{-1/2} \sum_{\{i\}} \varphi_1(i_1) \dots \varphi_1(i_k) \varphi_2(i_{k+1}) \dots \varphi_2(i_N) \quad (29)$$

where the sum is extended over all splittings of  $N$  particle indices into two groups. Of course different terms are mutually orthogonal. Now we first construct the  $N$  body density matrix as a projector on (29). To compute a one body density matrix we need to integrate over  $N-1$  variables. Nonzero terms are only those that are proportional to projectors on the state 1 and on the state 2. Moreover, the integrals are nonzero only if all the remaining variables will be in the same function. This way we get

$$\rho(1,1') = \frac{k!(N-k)!}{N!} \left[ \frac{(N-1)!}{(k-1)!(N-k)!} \varphi_1(1)\varphi_1^*(1') + \frac{(N-1)!}{k!(N-k-1)!} \varphi_2(1)\varphi_2^*(1') \right]$$

or, as to be expected:

$$\rho(1,1') = \left[ \frac{k}{N} \varphi_1(1)\varphi_1^*(1') + \frac{N-k}{N} \varphi_2(1)\varphi_2^*(1') \right]$$