Bose statistics and classical fields

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the most relevant question: where to cut?

- testing classical fields on a soluble model desirable
- ideal gas .... **ideal**
- ideal gas does not thermalize
- equilibrium thermodynamics *via* statistical ensemble is a good testing ground

1D ideal Bose gas

calculating the canonical partition function:

$$Z(N,\beta) = \sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\dots\sum_{n_k=0}^{\infty}\exp\left[-\beta\hbar\omega\sum_{k=0}^{\infty}kn_k\right]\delta_{\sum_{k=0}^{\infty}n_k,N}$$

$$\delta_{\sum n_k,N} = \frac{1}{2\pi}\int_0^{2\pi}\exp[i\eta\sum n_k]\exp[-i\eta N]d\eta$$

$$\xi = \exp(-\beta\hbar\omega) \qquad z = \exp[-i\eta]$$

$$Z(N,\beta) = \frac{1}{2\pi i}\oint dz z^{N-1}\prod_{k=0}^{\infty}\frac{z}{z-\xi^k}$$

$$Z(N,\beta) = \sum_{j=0}^{\infty}\xi^{jN}\prod_{k\neq j}^{\infty}\frac{1}{1-\xi^{k-j}}$$

... the same with classical fields:

$$\sum_{n=0}^{\infty} \rightarrow \frac{1}{\pi}\int d^2\alpha$$

cut-off

$$Z(N,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \exp(-iN\eta)\prod_{j=0}^{K}\int \frac{d^2\alpha}{\pi}\exp[-(\beta\hbar\omega j - i\eta)|\alpha|^2]$$

$$Z(N,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \exp(-iN\eta)\prod_{j=0}^{K}\frac{1}{\beta\hbar\omega j - i\eta}$$

$$Z(N,\beta) = \sum_{j=0}^{K}\xi^{jN}\prod_{k\neq j}^{K}\frac{1}{(\ln\xi)(j-k)}$$

canonical partition function of the excited atoms

no contribution of the ground state

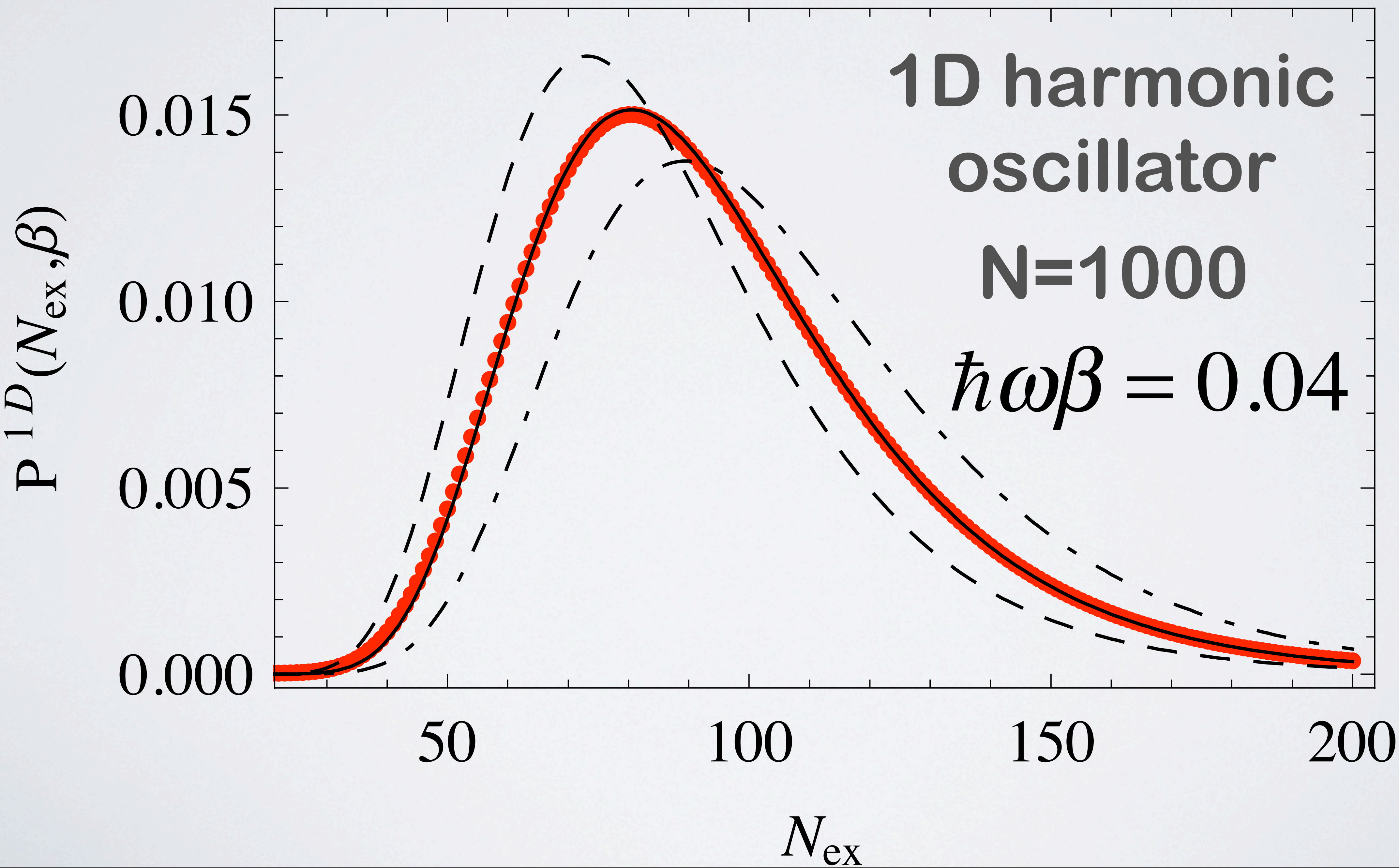
$$Z_{ex}(N_{ex},\beta)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\ldots\sum_{n_k=0}^{\infty}\ldots\exp\left[-\beta\hbar\omega\sum_{k=0}^{\infty}kn_k\right]\delta_{\sum_{k=0}^{\infty}n_k,N_{ex}}$$

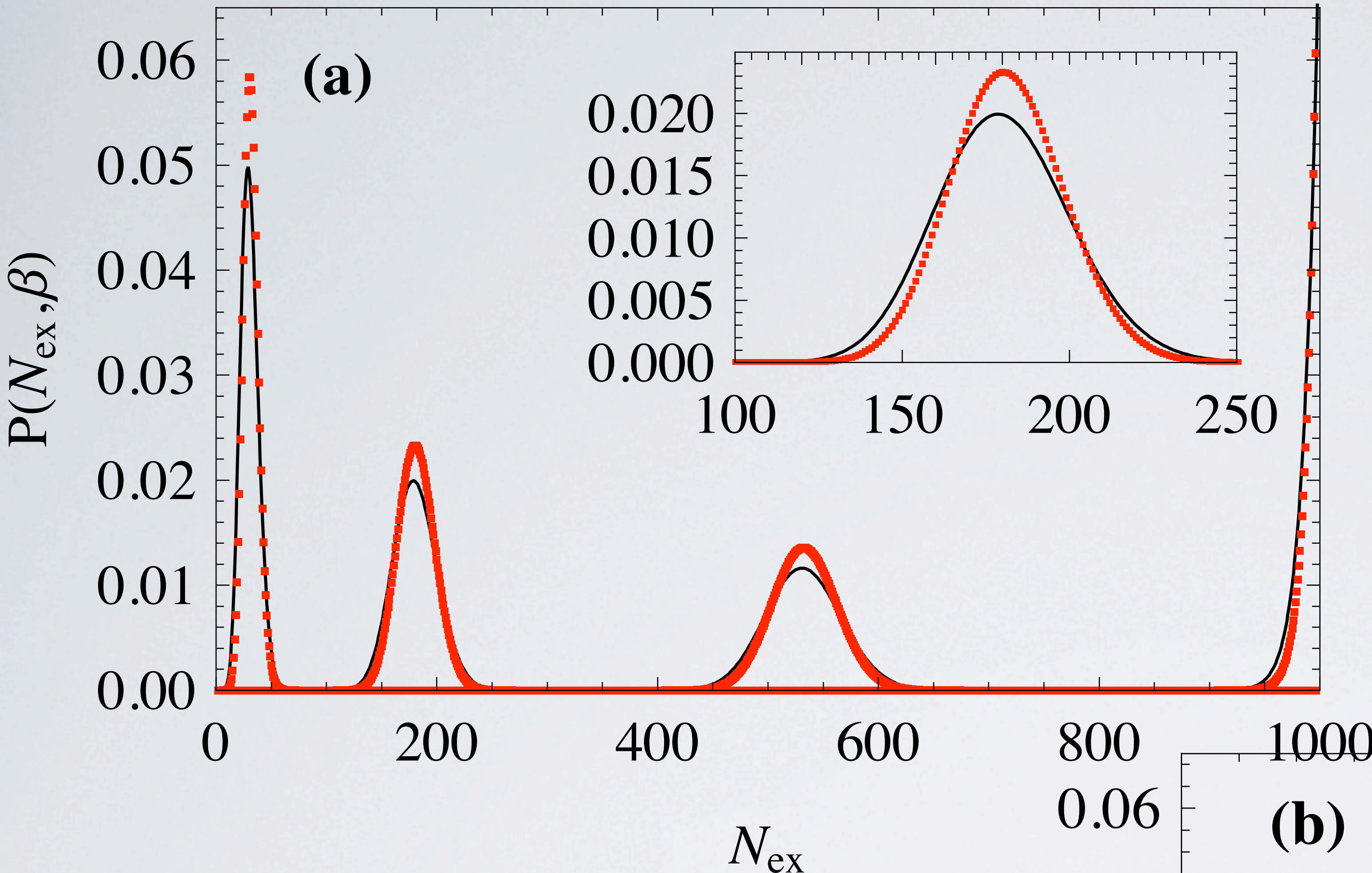
$$Z_{ex}(N_{ex},\beta)=\sum_{j=1}^{\infty}\xi^{jN_{ex}}\prod_{k\neq j}^{\infty}\frac{1}{1-\xi^{k-j}}$$

no contribution of the ground state

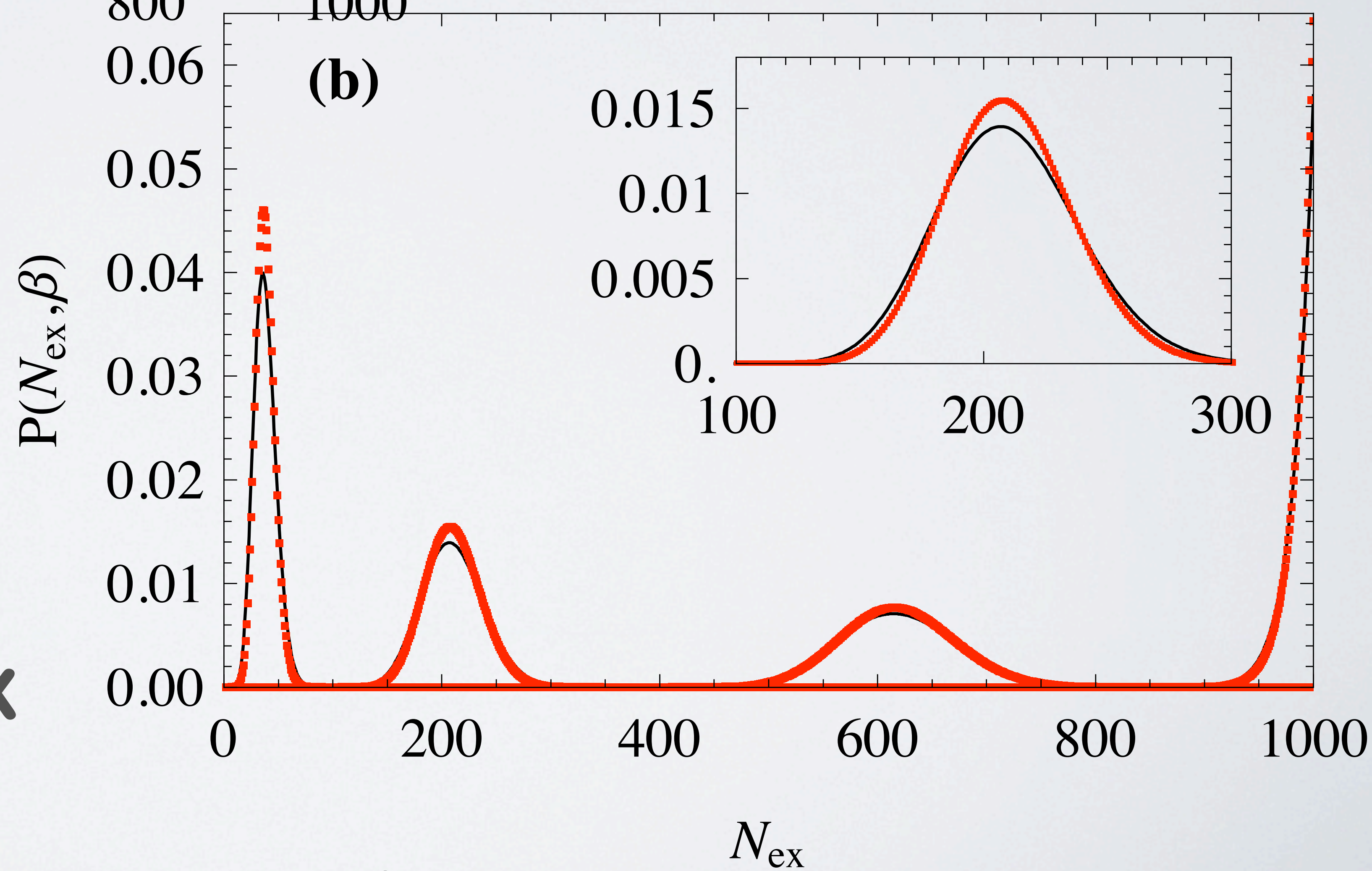
Probability distribution of the number of excited atoms

$$P(N_{ex},\beta) = \frac{Z_{ex}(N_{ex},\beta)}{Z(N,\beta)}$$





3D harmonic



3D box

E. Witkowska, M. Gajda, and K. Rzążewski, *Phys Rev* A **79**, 033631 (2009)

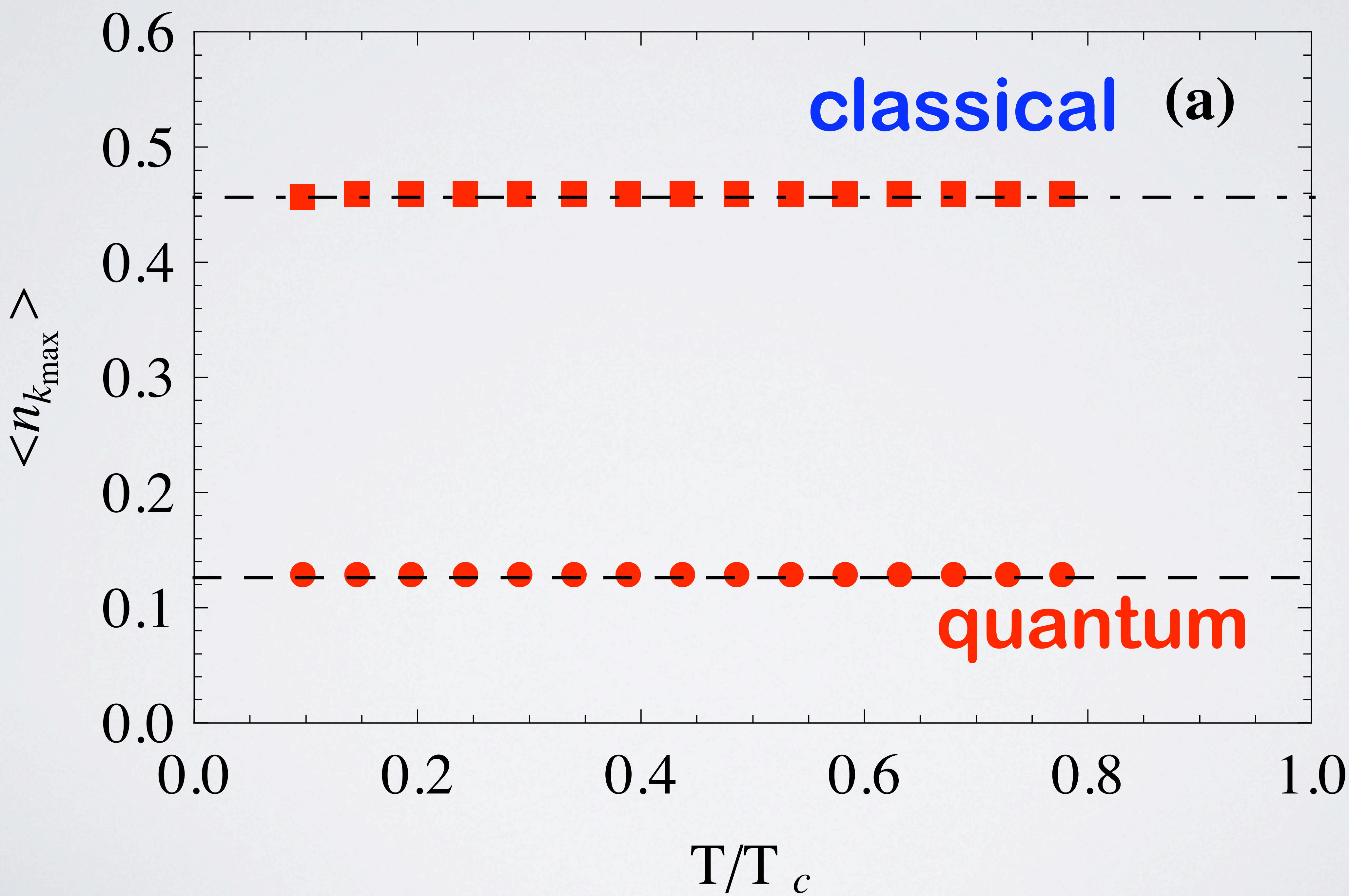
analytic 3D results following

V. V. Kocharovsky, Vl. V. Kocharovsky, and Marlan O. Scully, *Phys. Rev. Lett.* **84**, 2306, (2000)

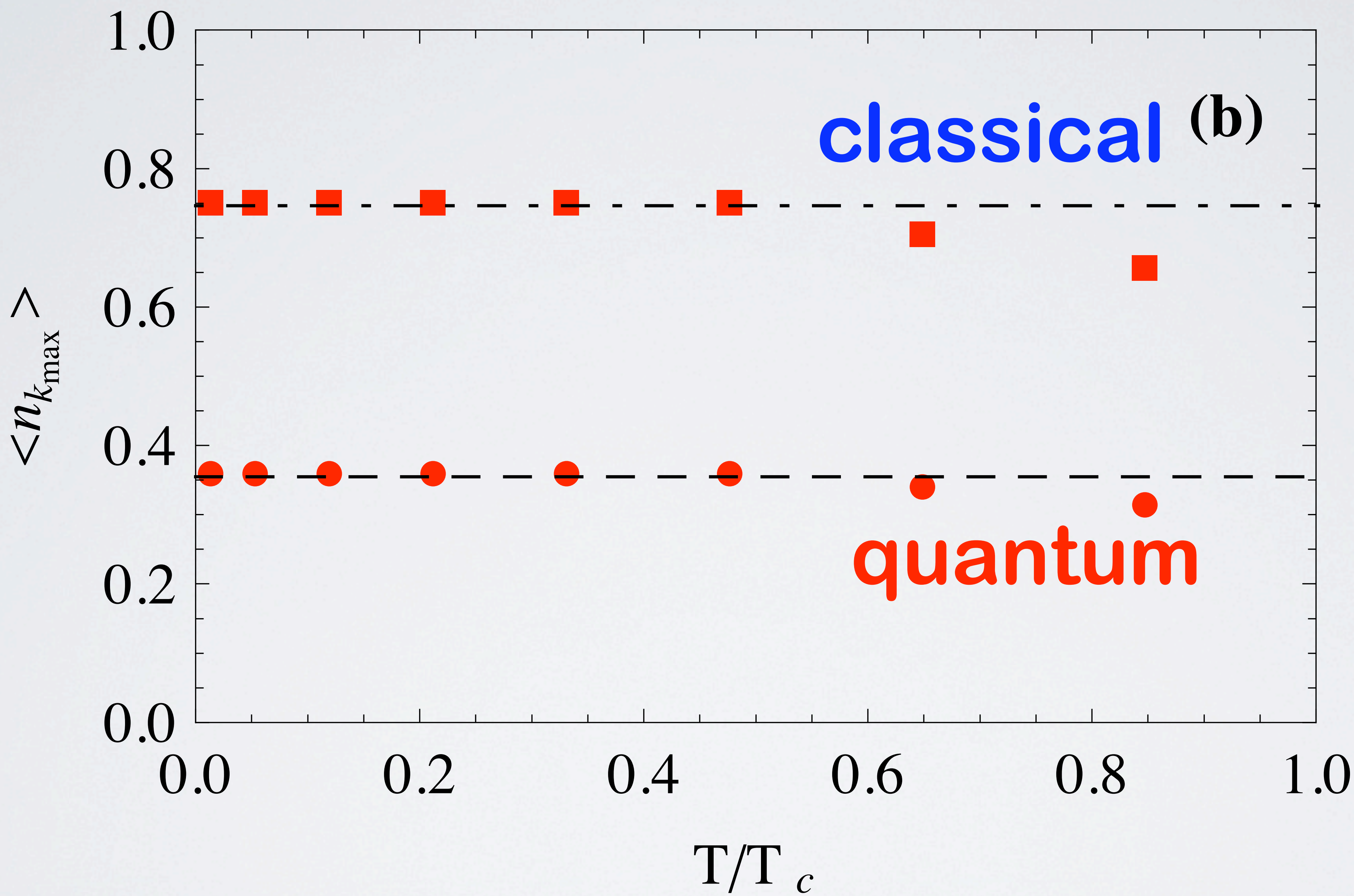
optimal cut-off for D-dimensional harmonic oscillator:

$$\hbar\omega K_{\max}\beta = \begin{cases} 1 & D = 1 \\ \left[\zeta(D)(D-1)(D-1)!\right]^{1/(D-1)} & D \geq 2 \end{cases}$$

mean occupation of the highest retained mode for 3D harmonic oscillator



mean occupation of the highest retained mode for 3D box



$$\frac{\hbar^2 k_{max}^2}{2m}\beta = \pi\left[\frac{\zeta(3/2)}{4}\right]^4$$

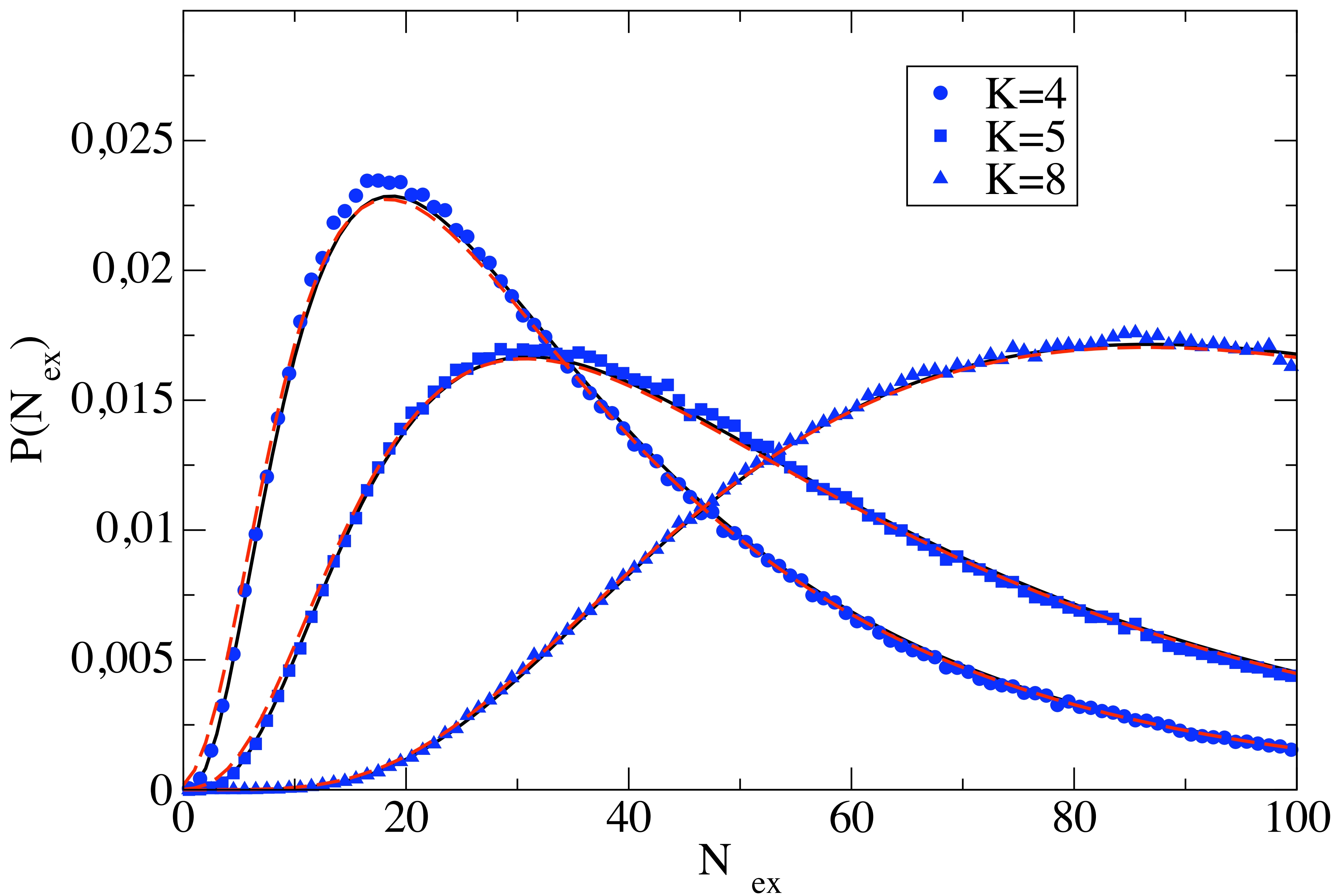
For weakly interacting Bose gas we have a finite dimensional classical system!

$$P(\{\alpha_j\}) = \frac{1}{Z}\exp\left[-\beta\left(\sum_k \varepsilon(k)\,|\alpha_k|^2 + E_{\mathrm{int}}(\{\alpha_j\})\right)\right]$$

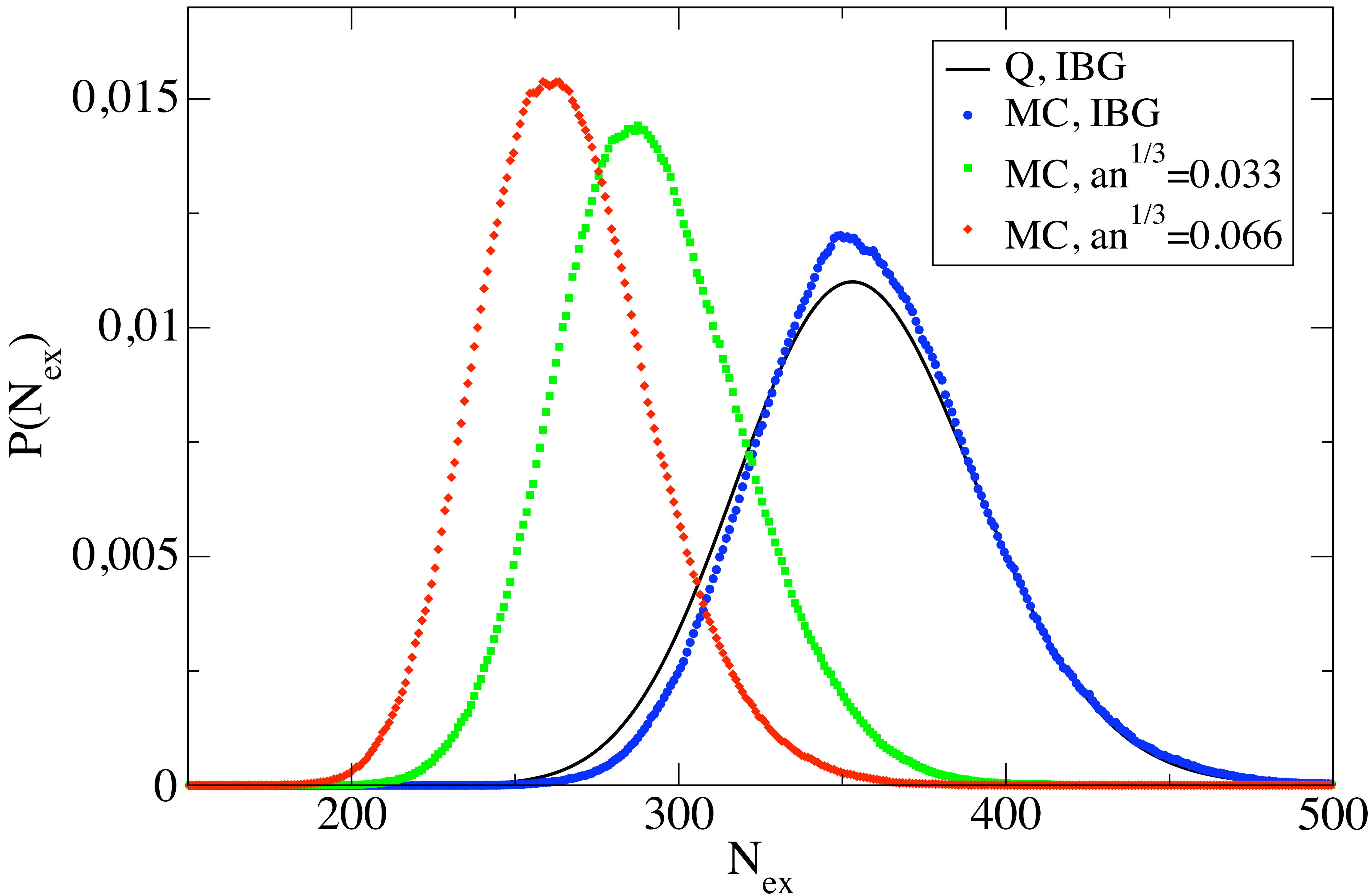
$$\sum_{k=0}^{k_{\max}} |\alpha_k|^2 = N$$

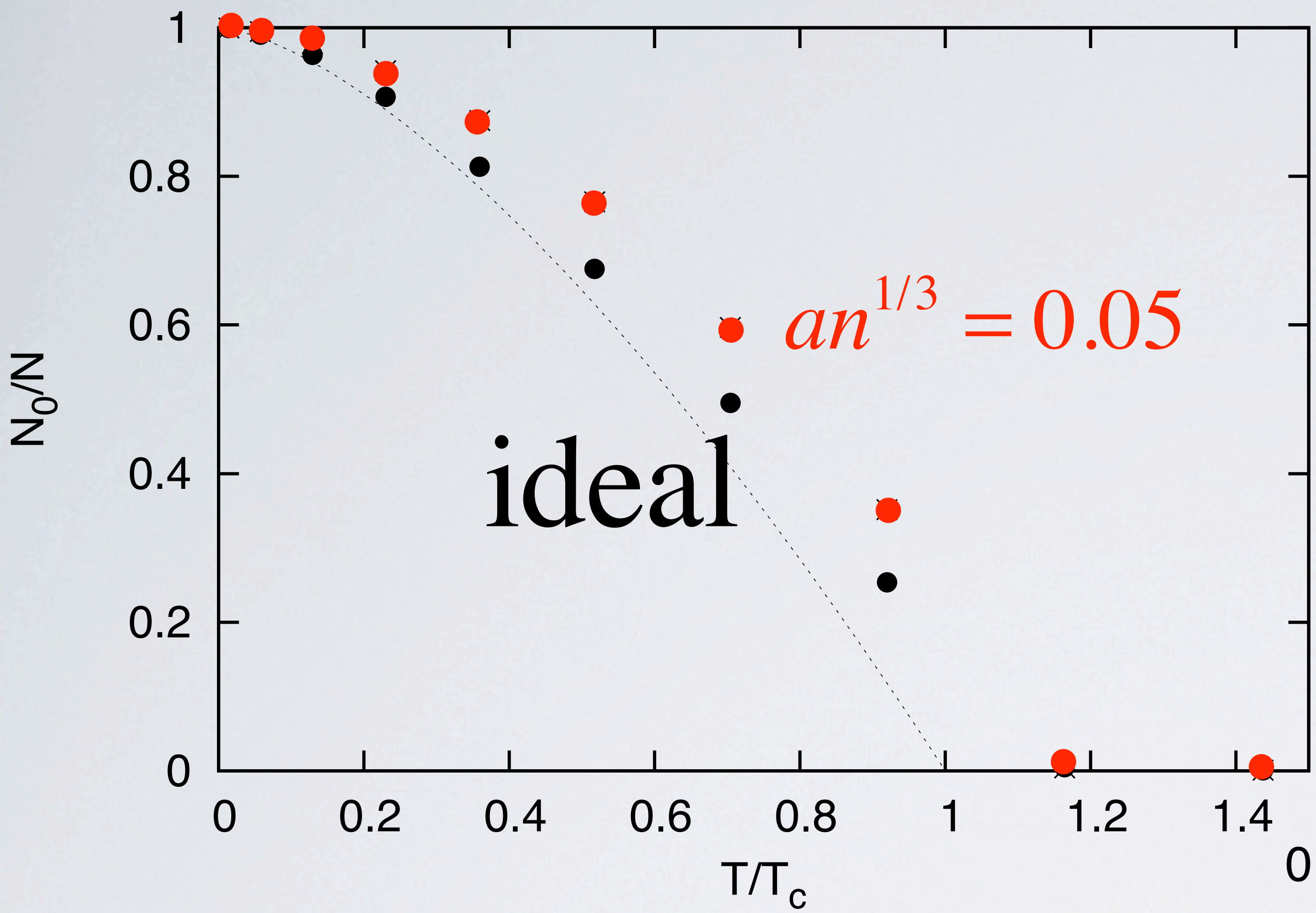
Metropolis algorithm may be used to generate this classical probability distribution

testing the method on 1D box with N=1000 noninteracting atoms



3D box with N=1000 atoms

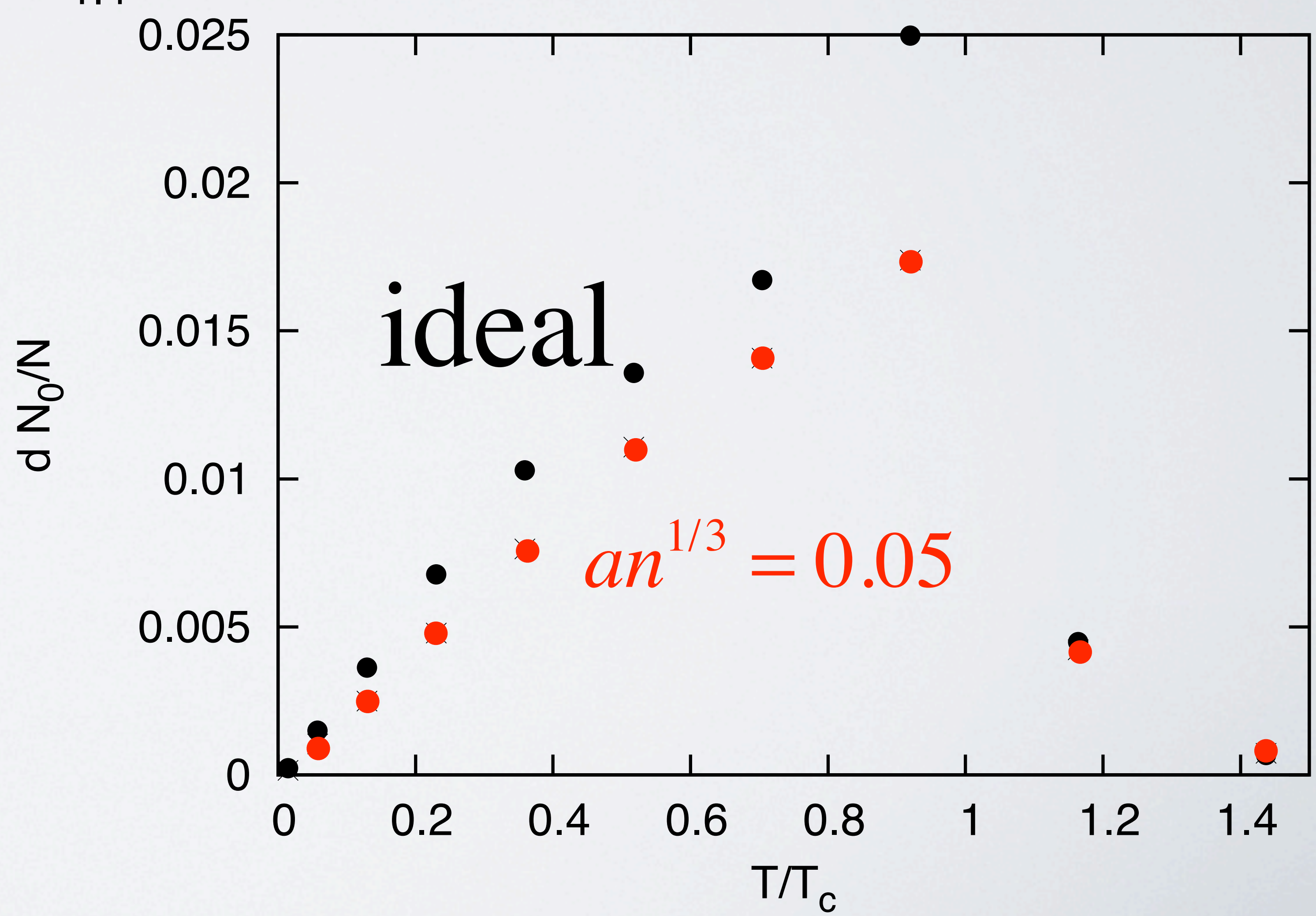




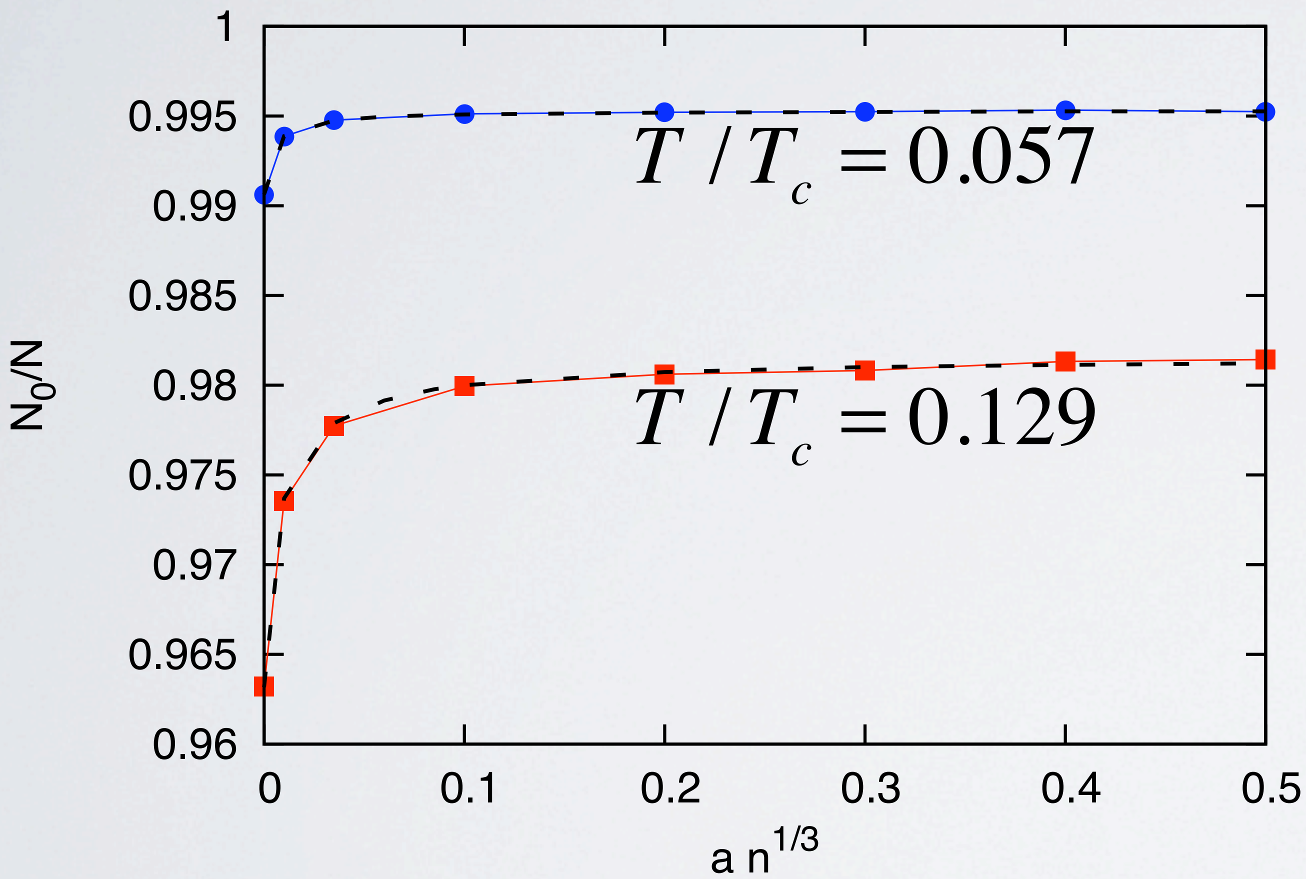
N=10 000

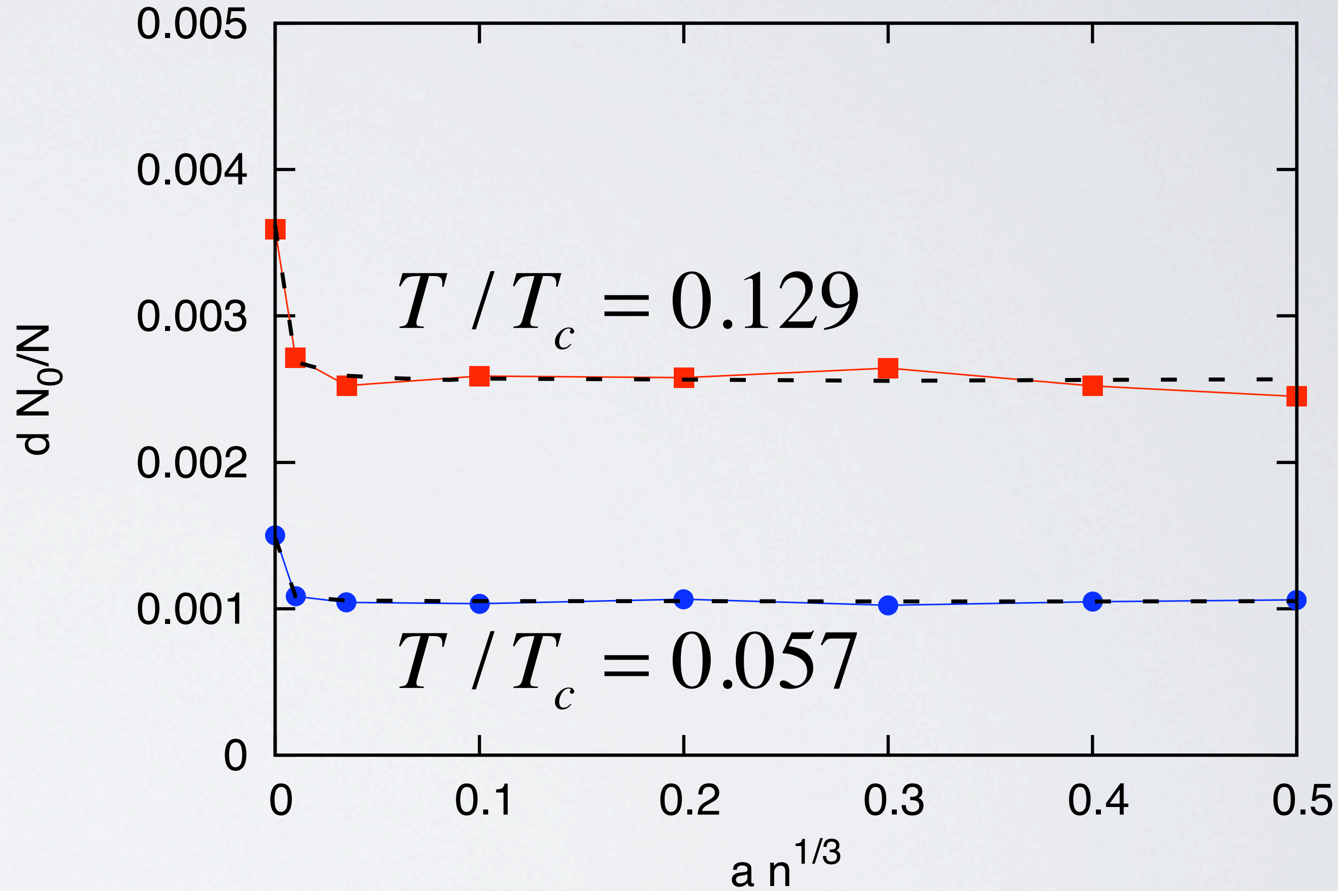
3D box

8 to 8000
modes



saturation with interaction:





summary

with optimal choice of the cut-off, classical fields are able to match full statistical properties of the ideal Bose gas

the finite weakly interacting classical system may be studied with the help of Monte Carlo methods