Narodowe
Laboratorium
Technologii
Kwantowych

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„Informatyka kwantowa: podstawy implementacji optycznych” w ramach kursu zaawansowanego organizowanego dniach 29.06 – 24.07.09 będącego kontynuacją szkoleń z zakresu zarządzania dużą infrastrukturą badawczą organizowanego przez Narodowe Laboratorium Technologii Kwantowych

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I POLARIZATION

A1.

Plane electromagnetic wave propagating along z axis. Electric field:

$$\vec{E}(z,t) = \vec{e}_x E_x(z,t) + \vec{e}_y E_y(z,t)$$

where $\vec{e}_x, \vec{e}_y$ are unit vectors. Let ω_0 be frequency, k_0 - the wave vector.

$$E_x(z,t) = E_{0x} \cos(k_0 z - \omega_0 t + \varphi_x)$$

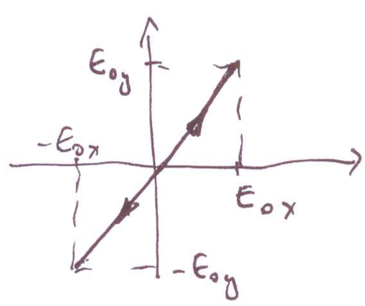
$$E_y(z,t) = E_{0y} \cos(k_0 z - \omega_0 t + \varphi_y)$$

We can always redefine time such that $\varphi_x = 0$. The electric field vector at $z=0$:

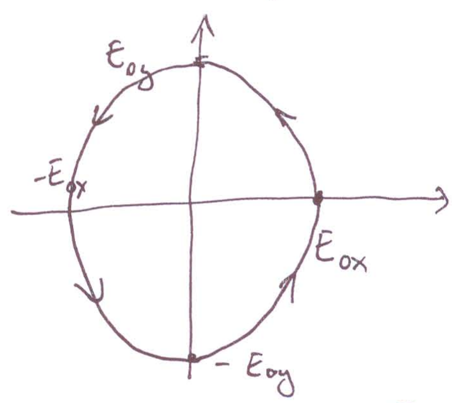
$$E_x(z=0,t) = E_{0x} \cos \omega_0 t$$

$$E_y(z=0,t) = E_{0y} \cos(\omega_0 t - \varphi_y)$$

$\varphi_y = 0$: linear polarisation



$\varphi_y = \frac{\pi}{2}$ elliptical polarisation:



In particular, when $E_{0x} = E_{0y}$ we get circular polarization.

For arbitrary φ_y polarization also ~~circular~~, elliptical, only principal axes aligned differently.

II JONES VECTORS AND MATRICES

Let's write:

$$\vec{E}(z,t) = \mathrm{Re}\left(\vec{\mathcal{E}}\, e^{ik_0 z - i\omega_0 t} \right)$$

where

$$\vec{\mathcal{E}} = \begin{pmatrix} E_{0x} \\ E_{0y}\, e^{-i\varphi_y} \end{pmatrix}$$

(3rd coordinate always $= 0$).

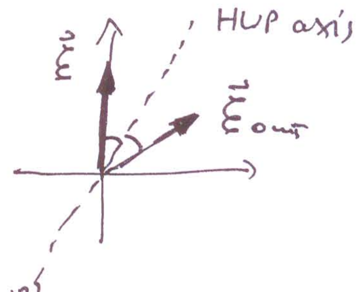
This is <u>Jones vector</u>

A birefringent element introduces a phase shift between two distinguished orthogonal axes. If aligned along xy coordinates,

$$\vec{\mathcal{E}} = \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} \longmapsto \begin{pmatrix} \mathcal{E}_x \\ e^{-i\xi}\mathcal{E}_y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\xi} \end{pmatrix}}_{\text{JONES MATRIX}} \vec{\mathcal{E}}$$

Examples:

$\xi = \pi$: half-wave plate (HWP)
("rotates" polarization)

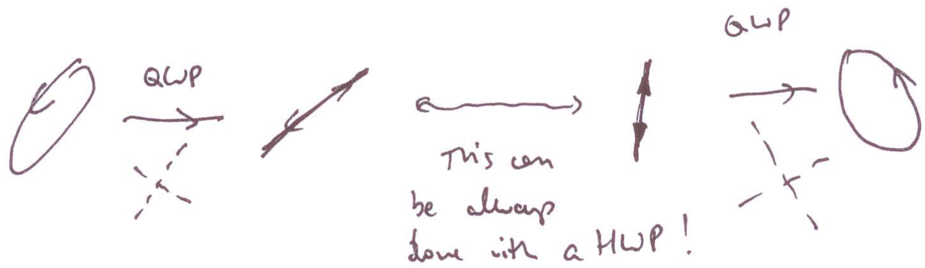


$\xi = \frac{\pi}{2}$ quarter-wave plate: (QWP)

III POLARIZATION TRANSFORMATIONS

A 3

Task: change one elliptic polarization into another one (without losses!).



QWP + HWP + QWP is universal!

What a lossless transformation means?

(Energy) Intensity of a beam: $|\mathcal{E}_x|^2 + |\mathcal{E}_y|^2 = \vec{\mathcal{E}}^\dagger \vec{\mathcal{E}}$

Transformation: $\vec{\mathcal{E}} \longmapsto \hat{U}\vec{\mathcal{E}}$.

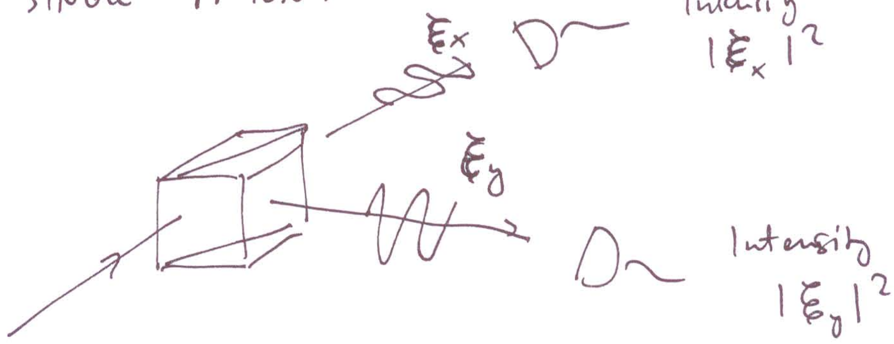
Preserved intensity:

$$\vec{\mathcal{E}}^\dagger \vec{\mathcal{E}} = \vec{\mathcal{E}}^\dagger \hat{U}^\dagger \hat{U} \vec{\mathcal{E}} \quad \text{for any } \vec{\mathcal{E}}$$

$\Rightarrow$ $\hat{U}$ is unitary

We will never worry about the overall phase, as all we measure is intensity.

IV SINGLE PHOTON.



How does a detector operate? Generates photoelectrons. To make a long story short, photoelectrons are generated by individual portions - quanta - of light called photons. After a polarizer, we measure numbers of photons that went one or another way.

The property of polarization can be attributed to individual photons. Suppose that we isolated only a single photon from a beam with a definite polarization. What will happen to it on a polarizer? We do not know for sure. All we can predict is the probability that the photon will go one or another way. The probabilities are such that they reproduce intensities for large number of incident photons.

V STATE VECTOR.

Let's reinterpret Jones vector

CLASSICALLY:

$$\vec{\mathcal{E}} = \begin{pmatrix}\mathcal{E}_x \\ \mathcal{E}_y\end{pmatrix}$$

Intensities: $|\mathcal{E}_x|^2, |\mathcal{E}_y|^2$ for transmitted and reflected

QUANTUMLY

$$|\psi\rangle \equiv \begin{pmatrix}\psi_x \\ \psi_y\end{pmatrix}$$

Probabilities detecting: $|\psi_x|^2, |\psi_y|^2$ for transmitted and reflected.

Total probability:

$$|\psi_x|^2 + |\psi_y|^2 = 1.$$

Wave plates transform state vectors exactly as wave vectors.

Some popular states:

$$|\leftrightarrow\rangle \equiv \begin{pmatrix}1 \\ 0\end{pmatrix} \qquad |\updownarrow\rangle \equiv \begin{pmatrix}0 \\ 1\end{pmatrix}$$

$$|\nearrow\rangle \equiv \begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix} \qquad |\searrow\rangle \equiv \begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$|\circlearrowleft\rangle \equiv \begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}}\end{pmatrix} \qquad |\circlearrowright\rangle \equiv \begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}}\end{pmatrix}.$$

Note that we can discriminate between these pairs using a polarizer, possibly preceded by some wave-plates.

Scalar product: $\langle\psi| = (|\psi\rangle)^\dagger = (\psi_x^*, \psi_y^*)$

Two distinguishable states:

$$\langle\leftrightarrow|\updownarrow\rangle = \langle\nearrow|\searrow\rangle = \langle\circlearrowleft|\circlearrowright\rangle = 0.$$

Can we distinguish $|\leftrightarrow\rangle$ from $|\nearrow\rangle$?

This turns out to be a <u>big question</u>

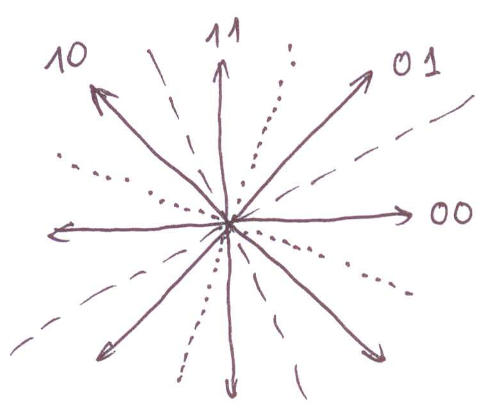
VI QUBIT.

In the polarization state of one photon we can send one bit of information, for example "0" $\equiv |\leftrightarrow\rangle$, "1" $\equiv |\updownarrow\rangle$. But we can also prepare superposition: $|\psi\rangle = \psi_x |\leftrightarrow\rangle + \psi_y |\updownarrow\rangle$ - QUBIT

This is something unimaginable classically.

Is it useful? It turns out that not in sending information in the simplest scenario (proof complicated). But... suppose Alice needs to send 2 bits, Bob will need only one of them. (Although he does not know which one in advance).

If one bit, correct value in: $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$ cases. If one qubit:



--- good measurement to read out the value 1st bit

.... good measurement to read out the value of 2nd bit

QUANTUM RANDOM ACCESS CODE.

VII POL. MEASUREMENT

Probability of detecting a photon transmitted or detected, written in a VERY COMPLICATED WAY:

$$P_{\leftrightarrow} = |\psi_x|^2 = |\langle\leftrightarrow|\psi\rangle|^2 = \langle\psi|\leftrightarrow\rangle\langle\leftrightarrow|\psi\rangle = \langle\psi|\hat{P}_{\leftrightarrow}|\psi\rangle$$

$$P_{\updownarrow} = |\psi_y|^2 = |\langle\updownarrow|\psi\rangle|^2 = \langle\psi|\underbrace{\updownarrow\rangle\langle\updownarrow}|\psi\rangle = \langle\psi|\hat{P}_{\updownarrow}|\psi\rangle$$

ψ_x, ψ_y can be written as scalar products.

Property of scalar product

can be viewed as 2×2 matrices.

$$\hat{P}_{\leftrightarrow} = |\leftrightarrow\rangle\langle\leftrightarrow| \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_{\updownarrow} = |\updownarrow\rangle\langle\updownarrow| \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

~~Two properties~~:

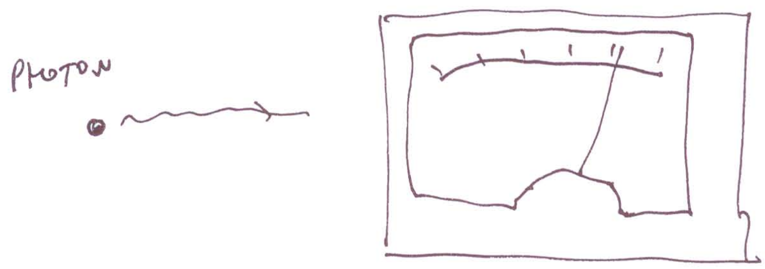
Expression of the form $\langle\psi|\hat{A}|\psi\rangle$ is called an expectation value

Two properties of our operators:

$\langle\psi|\hat{P}_i|\psi\rangle \geq 0$ — probability needs to be positive definite

$\hat{P}_{\leftrightarrow} + \hat{P}_{\updownarrow} = \hat{\mathbb{1}}$ — this guarantees that the probabilities sum up to 1.

VIII GENERALIZED MEASUREMENT



A set of outcomes labelled with r

Postulates:

- probability of obtaining an outcome r is given by an expression of the form:

$$p_r = \langle\psi|\hat{M}_r|\psi\rangle$$

- to make sense, $\hat{M}_r$ needs to be hermitian and nonnegative.

$$\sum \hat{M}_r = \hat{\mathbb{1}}$$ – for the probabilities to sum up to one.

That's it, let's not worry for now how to realize it (cf. tomorrow).

Useful identity: $\hat{\mathbb{1}} = \sum_{j=0,1} |j\rangle\langle j|$.

Then:

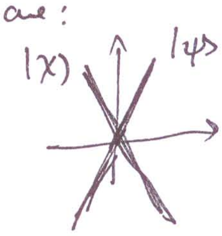
$$\langle\psi|\hat{M}_r|\psi\rangle = \langle\psi|\hat{M}_r\,\hat{\mathbb{1}}|\psi\rangle = \sum_j \langle\psi|\hat{M}_r|j\rangle\langle j|\psi\rangle$$

$$= \sum_j \underbrace{\langle j|\psi\rangle\langle\psi|\hat{M}_r|j\rangle}_{\text{PRODUCT OF TWO OPERATORS}} = \mathrm{Tr}\left(|\psi\rangle\langle\psi|\hat{M}_r\right)$$

IX MINIMUM ERROR

Suppose for concreteness that states are:

$$|\psi\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \qquad |\chi\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$



Scalar product: $\langle\psi|\chi\rangle = \cos\theta$

Game: ± 1 € if guessed (in)correctly.

Both states given with the same probability.

Measurement with two outcomes:

" THAT WAS ψ " $\equiv \hat{M}_\psi$

" THAT WAS χ " $\equiv \hat{M}_\chi$

AVG PAYOFF:

$$P = \frac{1}{2}\langle\psi|\hat{M}_\psi|\psi\rangle + \frac{1}{2}\langle\chi|\hat{M}_\chi|\chi\rangle - \frac{1}{2}\langle\chi|\hat{M}_\psi|\chi\rangle - \frac{1}{2}\langle\psi|\hat{M}_\chi|\psi\rangle$$

$$= \frac{1}{2}\mathrm{Tr}\left[(|\psi\rangle\langle\psi| - |\chi\rangle\langle\chi|)(\hat{M}_\psi - \hat{M}_\chi)\right]$$

Now use: $\hat{M}_\chi = \hat{\mathbb{1}} - \hat{M}_\psi$

$$|\psi\rangle\langle\psi| - |\chi\rangle\langle\chi| = \hat{\sigma}_1 \sin\theta, \quad \text{where } \hat{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |\nearrow\rangle\langle\nearrow| - |\searrow\rangle\langle\searrow|$$

All this gives:

$$P = \sin\theta \Big(\underbrace{\langle\nearrow|\hat{M}_\psi|\nearrow\rangle}_{\substack{\text{Maximize} \\ =1}} - \underbrace{\langle\searrow|\hat{M}_\psi|\searrow\rangle}_{\substack{\text{minimize} \\ =0.}} \Big)$$

$$P = \sin\theta = \sqrt{1 - |\langle\psi|\chi\rangle|^2}$$

$$\hat{M}_\psi = |\searrow\rangle\langle\searrow|, \quad \hat{M}_\chi = |\nearrow\rangle\langle\nearrow|$$

X UNAMBIGUOUS MEASUREMENT

Three outcomes.

THAT WAS $\|\psi\rangle$ FOR SURE	THAT WAS $\|\chi\rangle$ FOR SURE	DIDN'T MANAGE
$\hat{M}_\psi$	$\hat{M}_\chi$	$\hat{M}_?$

All nonnegative, $\hat{M}_\psi + \hat{M}_\chi + \hat{M}_? = \hat{1}$.

Now we require that

$$\langle\chi|\hat{M}_\psi|\chi\rangle = \langle\psi|\hat{M}_\chi|\psi\rangle = 0.$$

Reminder: eigenvalue and eigenvectors.

$$\hat{M}_\psi|u_i\rangle = \lambda_i|u_i\rangle$$

$i = 1, 2$, eigenvectors mutually orthogonal.

$$\hat{M}_\psi = \hat{M}_\psi\hat{1} = \sum_i \lambda_i |u_i\rangle\langle u_i| \qquad |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| = \hat{1}.$$

This means that

$$(*) \qquad \langle\chi|\hat{M}_\psi|\chi\rangle = \lambda_1|\langle u_1|\chi\rangle|^2 + \lambda_2|\langle u_2|\chi\rangle|^2.$$

$\lambda_i \geq 0$ to ensure that $\hat{M}_\psi$ nonnegative.

Suppose that $\lambda_1 > 0$. For $(*)$ to be zero this means that $\langle u_1|\chi\rangle = 0$, i.e.

$$|u_1\rangle = |\chi^\perp\rangle, \qquad |\chi^\perp\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}.$$

But then $|u_2\rangle = |\chi\rangle \Rightarrow \lambda_2 = 0$.

Thus:

$$\hat{M}_\psi = \lambda|\chi^\perp\rangle\langle\chi^\perp|$$

Analogous argument — In principle could be different. Handwaving symmetry argument.

$$\hat{M}_\chi = \lambda|\psi^\perp\rangle\langle\psi^\perp|, \qquad |\psi^\perp\rangle = \begin{pmatrix}-\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

So far so good - but ~~how~~ what about λ? A(11).

$$\hat{M}_? = \hat{\mathbb{1}} - \hat{M}_\psi - \hat{M}_\chi \equiv \begin{pmatrix} 1-\lambda(1+\cos\theta) & 0 \\ 0 & 1-\lambda(1-\cos\theta) \end{pmatrix}$$

Eigenvalues: $1-\lambda(1\pm\cos\theta)$. Nonnegativity means that:

$$\lambda \leq \frac{1}{1\pm\cos\theta} \quad \text{or equivalently} \quad \lambda \leq \frac{1}{1+|\cos\theta|}$$

We should of course take the maximum value. Probability of correct identification:

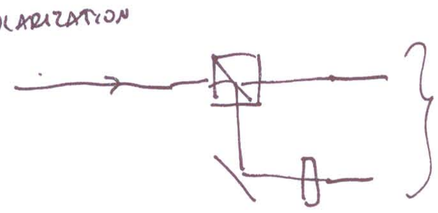
$$\langle\psi|M_\psi|\psi\rangle = \lambda\,|\langle\chi^\perp|\psi\rangle|^2 = \frac{\sin^2\theta}{1+|\cos\theta|} =$$

$$= 1 - |\cos\theta|$$

The probability of an inconclusive result:

$$\langle\psi|\hat{M}_?|\psi\rangle = \langle\chi|\hat{M}_?|\chi\rangle = |\cos\theta|.$$

I DUAL-RAIL QUBIT



$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

Identical polarization, but different paths. Photon can be detected in the upper or in the lower path, probabilities $|\psi_1|^2$ and $|\psi_2|^2$ respectively.

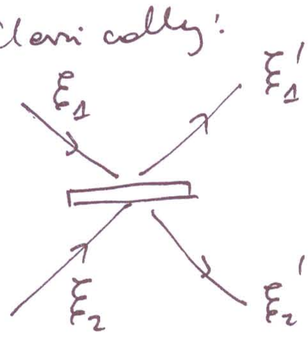
What can we do? Insert for example a phase shifting element - act the same way as for classical fields.

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longmapsto \begin{pmatrix} e^{i\phi}\psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

How to "mix" the two amplitudes?

BEAM SPLITTER

Classically:



$$\mathcal{E}_1' = r_1 \mathcal{E}_1 + t_2 \mathcal{E}_2$$

$$\mathcal{E}_2' = t_1 \mathcal{E}_1 + r_2 \mathcal{E}_2$$

Matrix form:

$$\begin{pmatrix} \mathcal{E}_1' \\ \mathcal{E}_2' \end{pmatrix} = \begin{pmatrix} r_1 & t_2 \\ t_1 & r_2 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix}$$

B2.

Any beam splitter, provided that lossless:

$\equiv$

$$\begin{pmatrix} \cos\frac{\vartheta}{2} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix}$$

We will usually write phase shifts separately.

Quantumly:

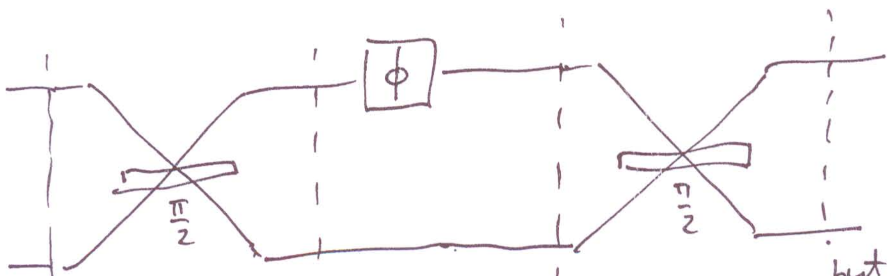
$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\frac{\vartheta}{2} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

Another realization of a qubit:

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

and we have a full set of transformations!

Let's put together an interferometer.



$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \begin{pmatrix} \frac{e^{i\phi}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

but difference is here

$$\begin{pmatrix} \cos\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} \end{pmatrix}.$$

at this stage, detection in upper and lower paths would be insensitive to phase....

PROBABILITY AMPLITUDES ARE <u>MORE</u> THAN JUST PROBABILITIES.

B3

II NETWORKS.

Let's go back to states distinguishability. Minimum-error was easy. What about unambiguous?

A plate oriented at the Brewster angle: horizontal polarization transmitted fully, t of vertical transmitted, r reflected.

Transmitted: $$|\psi\rangle \longrightarrow \begin{pmatrix} \sin\frac{\Theta}{2} \\ t\cos\frac{\Theta}{2} \end{pmatrix}$$

$$|\chi\rangle \longrightarrow \begin{pmatrix} -\sin\frac{\Theta}{2} \\ t\cos\frac{\Theta}{2} \end{pmatrix}$$

Reflected: in both cases $\begin{pmatrix} 0 \\ r\cos\frac{\Theta}{2} \end{pmatrix}$.

Let us take $t = \tan\frac{\Theta}{2}$. Then transmitted states:

$$|\psi\rangle \to \sin\frac{\Theta}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |\chi\rangle \to \sin\frac{\Theta}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

These states are ORTHOGONAL! can be distinguished in X basis.

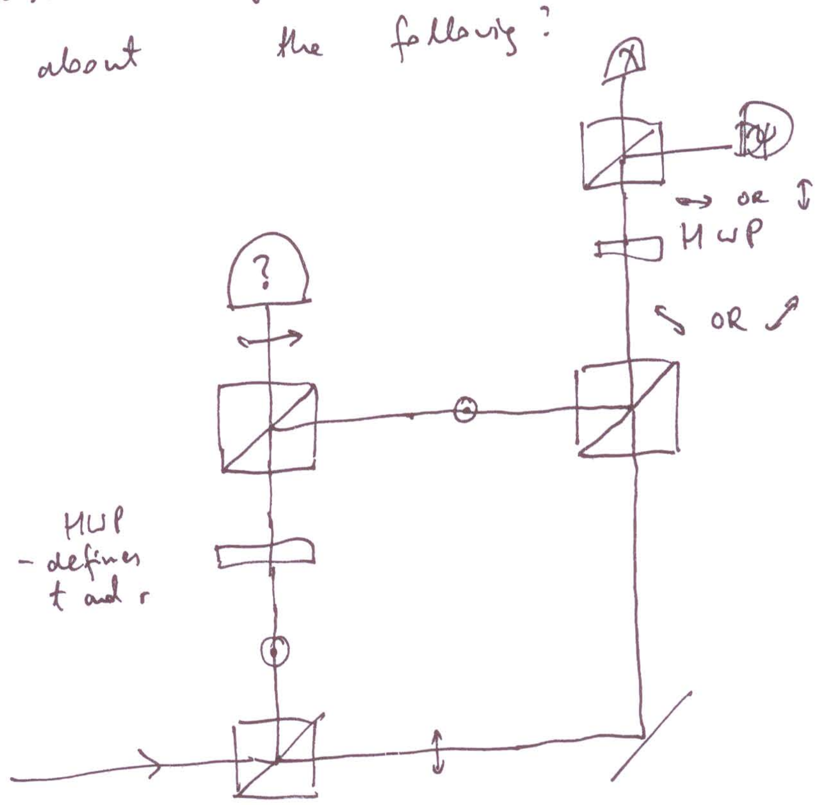
Note that they are not normalized because our photon could get reflected.

Probability of detecting transmitted photon is $2\sin^2\frac{\Theta}{2} = 1 - \cos\Theta$.

Optimal measurement.

B4

Finding a Brewster plate with right transmission may be tricky.
What about the following?



III BLOCH REPRESENTATION

Let's go back to polarization qubit, but will be applicable to any other qubit as well.

$$|\psi\rangle = \psi_x |\leftrightarrow\rangle + \psi_y |\updownarrow\rangle$$

Overall phase does not matter, normalization, $\Rightarrow$ we can parametrize $\psi_x = \cos\frac{\theta}{2}$ and $\psi_y = e^{i\varphi}\sin\frac{\theta}{2}$.

Expectation values could be written using an operator:

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(\hat{\mathbb{1}} + s_1\hat{\sigma}_1 + s_2\hat{\sigma}_2 + s_3\hat{\sigma}_3\right)$$

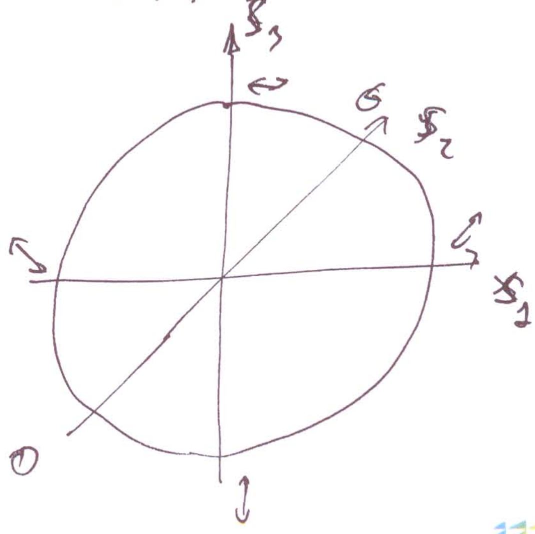
where: <u>PAULI MATRICES</u> ARE GIVEN BY:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\vec{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Unit length



THESE ARE NOT PHYSICAL COORDINATES

B6

Properties of Pauli matrices:

$$\mathrm{Tr}\,\hat{\sigma}_k = 0, \qquad \mathrm{Tr}(\hat{\sigma}_k \hat{\sigma}_l) = 2\delta_{kl}$$

This means that

$$\langle\psi|\hat{\sigma}_k|\psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|\hat{\sigma}_k) = S_k.$$

Scalar product: let $\vec{s}$ and $\vec{s}\,'$ be associated with $|\psi\rangle$ and $|\chi\rangle$.

$$|\langle\psi|\chi\rangle|^2 = \mathrm{Tr}(|\psi\rangle\langle\psi|\chi\rangle\langle\chi|) = \frac{1}{2}(1+\vec{s}\cdot\vec{s}\,').$$

An orthogonal state is located on an antipodal point.

Transformations: a wave plate $\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\xi} \end{pmatrix}$ correspond to $\varphi \to \varphi - \xi$.

Bloch representation: rotation about s_3 by ξ.

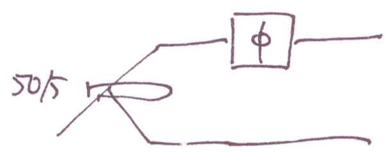
Rotated wave plate: certain states of linear polarization will be invariant - those aligned along principal axes of the plate. Rotation will be about the axis defined by these two states in the Bloch sphere.

B7.

Task: send states from a great circle of the Bloch sphere as easily as possible.

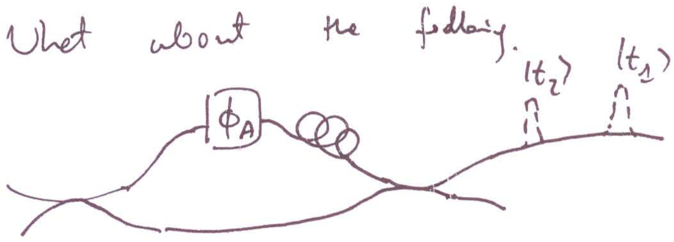
POLARIZATION linear polarizations, put a HWP.

DUAL-RAIL:



Good, except sensitive to phase drift.

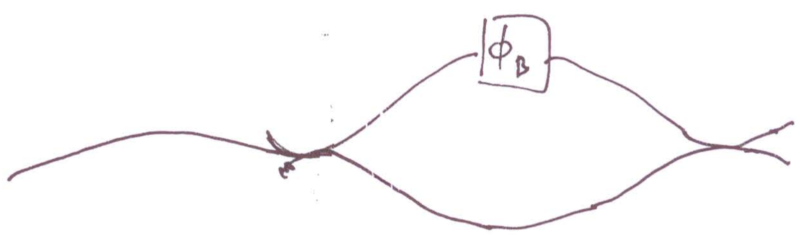
What about the following.



$|t_1\rangle + e^{i\phi}|t_2\rangle$

PROBLEM: losses.

Readout:



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C1

I PROPAGATION IN A MATERIAL MEDIUM.

Plane waves, one spatial dimension; wave equation:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 P}{\partial t^2}$$

Assume that the wave oscillates with frequency ω.

First, linear polarization:

$$P^L(z,t) = \epsilon_0 \chi(\omega) E(z,t)$$

Then:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1+\chi(\omega)}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Solution:

$$E_\omega(z,t) = A e^{ik(\omega)z - i\omega t} + c.c.$$

where the wave vector $k(\omega) = \frac{\sqrt{1+\chi(\omega)}}{c}\omega$ Refractive index $n(\omega) = \sqrt{1+\chi(\omega)}$

Let us assume that $P = P^L + P^{NL}$,

where $P^{NL} = \epsilon_0 \chi^{(2)} E^2$

Usually, this can be treated as a perturbation.

Let's send into the medium:

$$A e^{ik(\omega_1)z - i\omega_1 t} + B e^{ik(\omega_2)z - i\omega_2 t} + c.c.$$

Nonlinear polarization will have a component oscillating at frequency $\omega_3 = \omega_1 + \omega_2$ Exact form:

$$P^{NL}_{\omega_3} = 2\epsilon_0 \chi^{(2)} A B e^{i[k(\omega_1)+k(\omega_2)]z - i\omega_3 t}$$

This can emit light at frequency ω_3.

Let's assume that in addition to the two fields A i B we also have:

$$E_{\omega_3}(z,t) = \mathcal{E}(z) e^{ik(\omega_3)z - i\omega_3 t} + c.c.$$

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C2

It will be convenient to rewrite the wave equation at frequency ω_3 as:

$$\frac{\partial^2 E_{\omega_3}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E_{\omega_3} + \frac{1}{\epsilon_0} P^{L}_{\omega_3}\right) = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 P^{NL}_{\omega_3}}{\partial t^2}$$

First, let us take care of the left-hand side

$$\frac{\partial^2 E_{\omega_3}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E_{\omega_3} + \frac{1}{\epsilon_0} P^{L}_{\omega_3}\right) =$$

$$= \left(\frac{\partial^2 C}{\partial z^2} + 2ik(\omega_3)\frac{\partial C}{\partial z} \underbrace{- k^2(\omega_3)\, C + \frac{1+\chi(\omega)}{c^2}\omega^2 C}_{\text{These two terms cancel each other}}\right) e^{ik(\omega_3)z - i\omega_3 t}$$

If variation of C is small on the scale of a wavelength, then we can neglect 2^{nd} derivative.

Finally:

$$2ik(\omega_3)\frac{dC}{dz} = -2\frac{\omega_3^2}{c^2}\chi^{(2)} AB\, e^{i[k(\omega_1)+k(\omega_2)-k(\omega_3)]z}$$

This gives:

$$\frac{dC}{dz} = \frac{i\omega_3 \chi^{(2)}}{c\, n(\omega_3)} AB\, e^{i\Delta k z}$$

Let's assume that initially $C(z=0) = 0$.

Integration:

$$C(z) = \frac{i\omega_3 \chi^{(2)}}{c\, n(\omega_3)} AB\, \frac{e^{i\Delta k z} - 1}{i\Delta k}$$

Intensity:

$$|C(z)|^2 \propto |A|^2 |B|^2 \frac{\sin^2 \frac{\Delta k \cdot z}{2}}{(\Delta k)^2}$$

Physics: contributions from polarization need to add coherently. Best signal when $\Delta k = 0$.

C3

$\Delta k = 0$ is called <u>phase matching condition</u>. It's not that easy to achieve, but doable. One way: use a birefringent material and choose different polarization for the three frequencies. One needs to ensure that the modes couple there through non-linear $\chi^{(2)}$ interaction. <u>Doable</u>

So far, we considered the case when A and B is strong, and C could be viewed as perturbative.

More generally:

$$E(z,t) = A(z)\, e^{ik(\omega_1)z - i\omega_1 t} + B(z)\, e^{ik(\omega_2)z - i\omega_2 t} + C(z)\, e^{ik(\omega_3)z - i\omega_3 t}$$

Non-linear polarization at frequencies ω_1 and ω_2:

$$P^{NL}_{\omega_1} = 2\epsilon_0 \chi^{(2)} B^*(z) C(z)\, e^{i[k(\omega_3) - k(\omega_2)]z - i\omega_1 t}$$

$$P^{NL}_{\omega_2} = 2\epsilon_0 \chi^{(2)} A^*(z) C(z)\, e^{i[k(\omega_3) - k(\omega_1)]z - i\omega_2 t}$$

Equation for A, B, and C:

$$\frac{dA}{dz} = \frac{i\omega_1 \chi^{(2)}}{cn(\omega_1)} B^*(z) C(z)$$

$$\frac{dB}{dz} = \frac{i\omega_2 \chi^{(2)}}{cn(\omega_2)} A^*(z) C(z)$$

$$\frac{dC}{dz} = \frac{i\omega_3 \chi^{(2)}}{cn(\omega_3)} A(z) B(z)$$

Three coupled nonlinear equations - sounds like a lot of fun!

C4

Rescaling:

$$A(z) = \sqrt{\frac{\hbar\omega_1}{2\epsilon_0 c n(\omega_1)}}\,\alpha(z)$$

$$B(z) = \sqrt{\frac{\hbar\omega_2}{2\epsilon_0 c n(\omega_2)}}\,\beta(z)$$

$$C(z) = -i\sqrt{\frac{\hbar\omega_3}{2\epsilon_0 c n(\omega_3)}}\,\gamma(z)$$

Why this way?

Then the energy flow is proportional to $\hbar\omega_1 |\alpha(z)|^2$ etc.

$|\alpha(z)|^2$ can be viewed as "the number of photons".

Let's rescale the coupling constant as well:

$$\kappa = \chi^{(2)}\sqrt{\frac{\hbar\omega_1\omega_2\omega_3}{2\epsilon_0 c^3 n(\omega_1)\, n(\omega_2)\, n(\omega_3)}}$$

Finally:

$$\frac{d\alpha}{dz} = \kappa\, \beta^*(z)\,\gamma(z)$$

$$\frac{d\beta}{dz} = \kappa\, \alpha^*(z)\,\gamma(z)$$

$$\frac{d\gamma}{dz} = -\kappa\, \alpha(z)\,\beta(z)$$

Quantum mechanically, amplitudes are replaced by annihilation operators: $\alpha \longrightarrow \hat{a}$, $\alpha^* \longrightarrow \hat{a}^+$

$$\frac{d\hat{a}}{dz} = \kappa\, \hat{b}^+(z)\,\hat{c}(z)$$

$$\frac{d\hat{b}}{dz} = \kappa\, \hat{a}^+(z)\,\hat{c}(z)$$

$$\frac{d\hat{c}}{dz} = -\kappa\, \hat{a}(z)\,\hat{b}(z)$$

C8

What spatial dependence of operators means?

Let us recall Heisenberg picture.

Time evolution ($\hbar = 1$)

$$i \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

Formally:

$$|\psi(t)\rangle = \underbrace{e^{-i\hat{H}t}}_{\text{Evolution operator } \mathcal{U}(t)} |\psi(0)\rangle$$

Schrödinger picture

Exp. value at a certain time t:

$$\langle A(t) \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle = \langle \psi(0) | \hat{\mathcal{U}}^+(t) \hat{A} \hat{\mathcal{U}}(t) | \psi(0) \rangle$$

State vector can be kept fixed,
we can evolve operators instead.

$$\hat{A}(t) = \hat{\mathcal{U}}^+(t) \hat{A} \, \mathcal{U}(t)$$

Equation of motion for an operator:

$$\frac{dA}{dt} = i\hat{H}\hat{A}(t) - i\hat{A}(t)\hat{H} = -i[\hat{A}(t), \hat{H}].$$

Here we have evolution over z rather than time
– this way we relate what happens at the
input and at the output of the medium.

Our equations can
be written as:

$$\frac{d\hat{a}}{dz} = -i[\hat{a}, \hat{H}]$$

$$\frac{d\hat{b}}{dz} = -i[\hat{b}, \hat{H}]$$

$$\frac{d\hat{c}}{dz} = -i[\hat{c}, \hat{H}]$$

where:

$$\hat{H} = i\kappa(\hat{a}^+\hat{b}^+\hat{c} - \hat{a}\hat{b}\hat{c}^+)$$

Interpretation:

$$c \longleftrightarrow \underbrace{a + b}_{\text{pair of photons}}.$$

C6

Let's assume that $\hat{c}$ is strong, we can replace it then by a c-number γ (REAL). Back to Schrödinger picture

$$\hat{U}(z) = e^{-i\hat{H}z} = e^{r(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})} \qquad r = k\gamma z$$

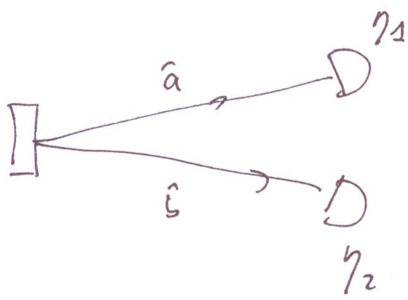
Initially nothing in modes $\hat{a}$ and $\hat{b}$.

$$\hat{U}(z)|00\rangle \approx (1 + r\hat{a}^\dagger\hat{b}^\dagger - r\hat{a}\hat{b})|00\rangle$$

$$= |00\rangle + r|1_a 1_b\rangle$$

With a small probability we generate a pair of photons.

Basis for expts on testing Bell's inequalities, teleportation, dense coding — you name it...
(typically non-degenerate spatially).
Application: absolute calibration of detector efficiency.



$$R_1 = \eta_1 R_{source}$$

$$R_2 = \eta_2 R_{source}$$

$$R_{coinc} = \eta_1 \eta_2 R_{source}$$

$$\frac{R_{coinc}}{R_2} = \eta_1$$

C7

Frequency conversion

Suppose that $\hat{b}$ is classical. Then:

$\hat{b} \longrightarrow \beta$ (real)

$$\frac{d\hat{a}}{dz} = \kappa\beta\, \hat{c}(z)$$

$$\frac{d\hat{c}}{dz} = -\kappa\beta\, \hat{a}(z)$$

Solution:

$$\hat{a}(z) = \hat{a}(0)\cos\theta + \hat{c}(0)\sin\theta$$

$$\hat{c}(z) = -\hat{a}(0)\sin\theta + \hat{c}(0)\cos\theta$$

$$\boxed{\theta = \kappa\beta z}$$

If we take $z = \frac{\pi}{2\kappa\beta}$, then:

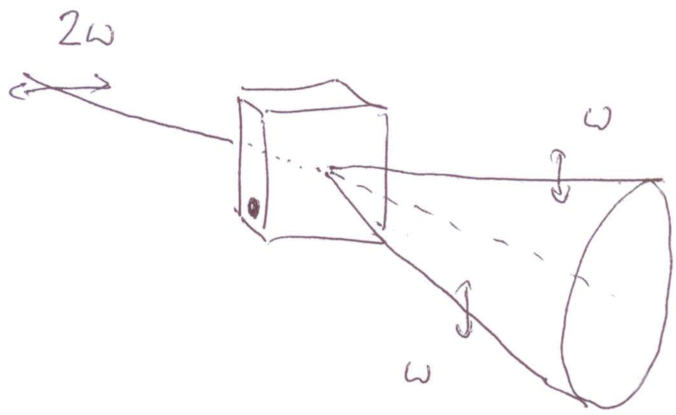
$$\hat{a}\left(\frac{\pi}{2\kappa\beta}\right) = \hat{c}(0)$$

$$\hat{c}\left(\frac{\pi}{2\kappa\beta}\right) = -\hat{a}(0)$$

The beam at frequency ω_3 "takes over" the quantum state initially at frequency ω_1.

D1

Type-I process:

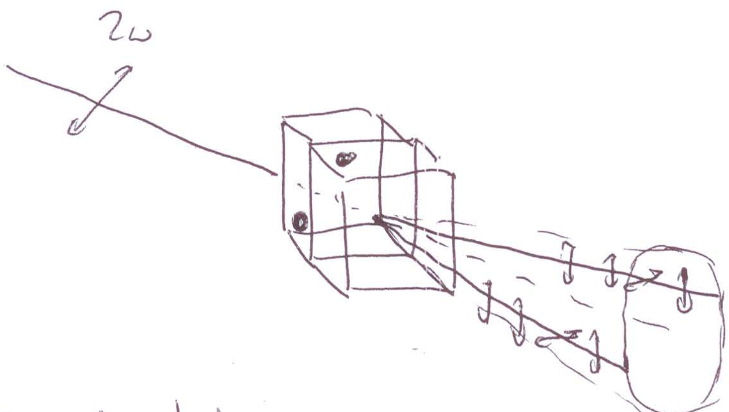


Photons emitten over a cone, both with identical polarization.

Photons appear in pairs on the opposite ends of the cone, polarization fixed to vertical.

$|vac\rangle + r|\updownarrow\updownarrow\rangle$

Imagine that we stack two crystals rotated by 90°. Pump should have both horizontal and vertical components:



Output state:

$$|vac\rangle + \underbrace{r|\updownarrow\updownarrow\rangle + r|\leftrightarrow\leftrightarrow\rangle}_{\sqrt{2}r\cdot\frac{1}{\sqrt{2}}\left(|\updownarrow\updownarrow\rangle + |\leftrightarrow\leftrightarrow\rangle\right)}$$

What can we do? Let's measure polarization..

$$|\theta_A\rangle = \cos\theta_A |\leftrightarrow\rangle + \sin\theta_A |\updownarrow\rangle$$
$$|\theta_B\rangle = \cos\theta_B |\leftrightarrow\rangle + \sin\theta_B |\updownarrow\rangle$$

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(|\leftrightarrow\leftrightarrow\rangle + |\updownarrow\updownarrow\rangle\right)$$

Now:
$$\left(\langle\theta_A|\langle\theta_B|\right)|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B\right)$$
$$= \frac{1}{\sqrt{2}}\cos(\theta_A - \theta_B)$$

Probability of detecting photon at angles θ_A and θ_B:
$$P(\theta_A;\theta_B) = \frac{1}{2}\cos^2(\theta_A - \theta_B)$$

Probability of detecting a single photon at a given polarization: $P(\theta_A) = P(\theta_B) = \frac{1}{2}$.

Conditional probability:
$$P(\theta_B|\theta_A) = \frac{P(\theta_A;\theta_B)}{P(\theta_A)} = \cos^2(\theta_B - \theta_A)$$

— analog of Malus' law.

D3

CONDITIONAL STATE VECTOR

$$|\Phi_+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\leftrightarrow\rangle_A|\leftrightarrow\rangle_B + |\updownarrow\rangle_A|\updownarrow\rangle_B\right)$$

$$|\tilde{\phi}_+\rangle_B = {}_A\langle\theta_A|\Phi_+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\cos\theta_A|\leftrightarrow\rangle_B + \sin\theta_A|\updownarrow\rangle_B\right)$$

- $|\tilde{\phi}_+\rangle_B$ can be used to calculate results of measurements on photon B **provided** that the photon A was found in state $|\theta_A\rangle$
- ${}_B\langle\tilde{\phi}_+|\tilde{\phi}_+\rangle_B$ gives the probability of measuring photon A in the state $|\theta_A\rangle$:

$${}_B\langle\tilde{\phi}_+|\tilde{\phi}_+\rangle_B = \langle\Phi_+|\left(|\theta_A\rangle_A\langle\theta_A|\otimes\mathbb{1}_B\right)|\Phi_+\rangle$$

"Collapse of the wave function" does a measurement on the subsystem A change the state of the subsystem B???? Overinterpretation of the quantum-mechanical formalism: a wave function is not a quantity that can be measured directly. What matters are **correlations** between measurement outcomes.

D4.

QUANTUM DENSE CODING.

Take state $|\Phi_+\rangle$ and insert a HWP at ~~45~~ 0°

We obtain:

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}\left(|\leftrightarrow\leftrightarrow\rangle - |\updownarrow\updownarrow\rangle\right)$$

$$\boxed{\text{BELL STATES}}$$

Follow this with a HWP @ 45°:

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left(|\leftrightarrow\updownarrow\rangle + |\updownarrow\leftrightarrow\rangle\right)$$

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|\leftrightarrow\updownarrow\rangle - |\updownarrow\leftrightarrow\rangle\right)$$

These four states have some interesting properties:

- individual photons are in a maximally mixed state.
- they are mutually orthogonal
- they can be transformed one into another by manipulating <u>one photon</u> only.

Protocol:

ALICE $|\Phi_+\rangle$ BOB

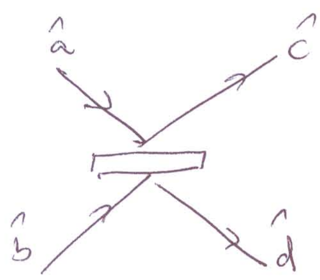
Alice transmits <u>two bits</u> of information by preparing one of four Bell states, and sending her <u>one</u> qubit to Bob. The resource required another transmission, but <u>prior</u> to information received by Alice.

D5

PROOF-OF-PRINCIPLE EXPERIMENT.

How to discriminate between Bell's states?

First, Hong-Ou-Mandel interferometer



$$\begin{pmatrix}\hat c\\ \hat d\end{pmatrix} = \frac{1}{\sqrt2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\hat a\\ \hat b\end{pmatrix}$$

$$|1_a 1_b\rangle = \hat a^\dagger \hat b^\dagger |vac\rangle =$$

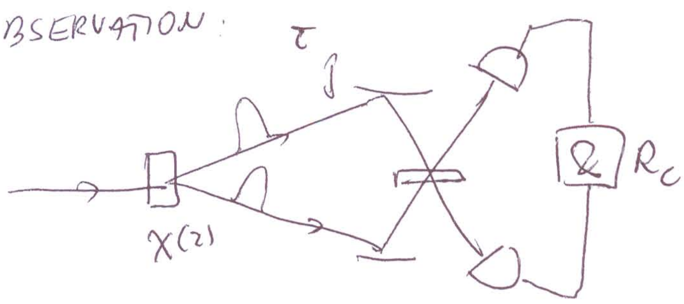
$$\frac12(\hat a^\dagger - \hat b^\dagger)(\hat a^\dagger + \hat b^\dagger)|vac\rangle = \frac12((\hat a^\dagger)^2 - (\hat b^\dagger)^2)|vac\rangle$$

$$= \frac{1}{\sqrt2}(|2_a 0_b\rangle - |0_a 2_b\rangle).$$

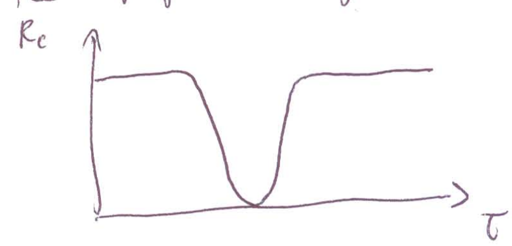
"DIAGRAMMATIC" REPRESENTATION:

THESE TWO PROBABILITY AMPLITUDES CANCEL EACH OTHER.

OBSERVATION:



Single photons are wave packets (duration defined by properties of the pump & crystal)



D6

What about a singlet state at a beam splitter?

$$\begin{pmatrix} \hat{c}_{\leftrightarrow} \\ \hat{d}_{\leftrightarrow} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{\leftrightarrow} \\ \hat{b}_{\leftrightarrow} \end{pmatrix} \qquad \begin{pmatrix} \hat{c}_{\updownarrow} \\ \hat{d}_{\updownarrow} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{\updownarrow} \\ \hat{b}_{\updownarrow} \end{pmatrix}$$

Projection on:

$$\langle vac| \hat{c}_{\leftrightarrow} \hat{d}_{\updownarrow} = \langle vac| \frac{1}{\sqrt{2}} (\hat{a}_{\leftrightarrow} + \hat{b}_{\leftrightarrow}) \frac{1}{\sqrt{2}} (-\hat{a}_{\updownarrow} + \hat{b}_{\updownarrow})$$

$$= \frac{1}{2} \langle vac| (-\hat{a}_{\leftrightarrow} \hat{a}_{\updownarrow} + \hat{a}_{\leftrightarrow} \hat{b}_{\updownarrow} - \hat{a}_{\updownarrow} \hat{b}_{\leftrightarrow} + \hat{b}_{\leftrightarrow} \hat{b}_{\updownarrow})$$

we never send 2 photon into the same input port

$$= \frac{1}{\sqrt{2}} \langle \Psi_- | .$$

Not full projection, but we also have $\langle vac| \hat{c}_{\updownarrow} \hat{d}_{\leftrightarrow}$.

Now:

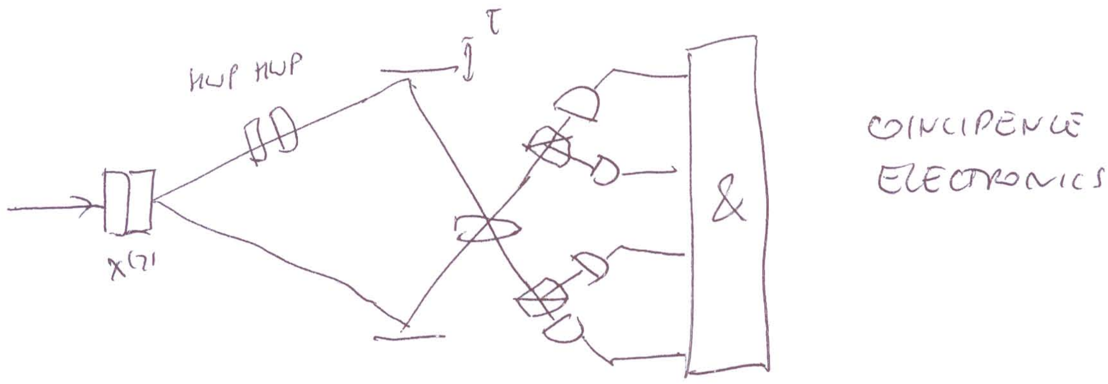
$|\Psi_-\rangle$ = $c_{\leftrightarrow} d_{\updownarrow}$ OR $c_{\updownarrow} d_{\leftrightarrow}$

$|\Psi_+\rangle$ = $c_{\leftrightarrow} c_{\updownarrow}$ OR $d_{\leftrightarrow} d_{\updownarrow}$

$$\left.\begin{matrix} |\Phi_+\rangle \\ |\Phi_-\rangle \end{matrix}\right\}$$ $2\times c_{\leftrightarrow}$ OR $2\times c_{\updownarrow}$ OR $2\times d_{\leftrightarrow}$ OR $2\times d_{\updownarrow}$.

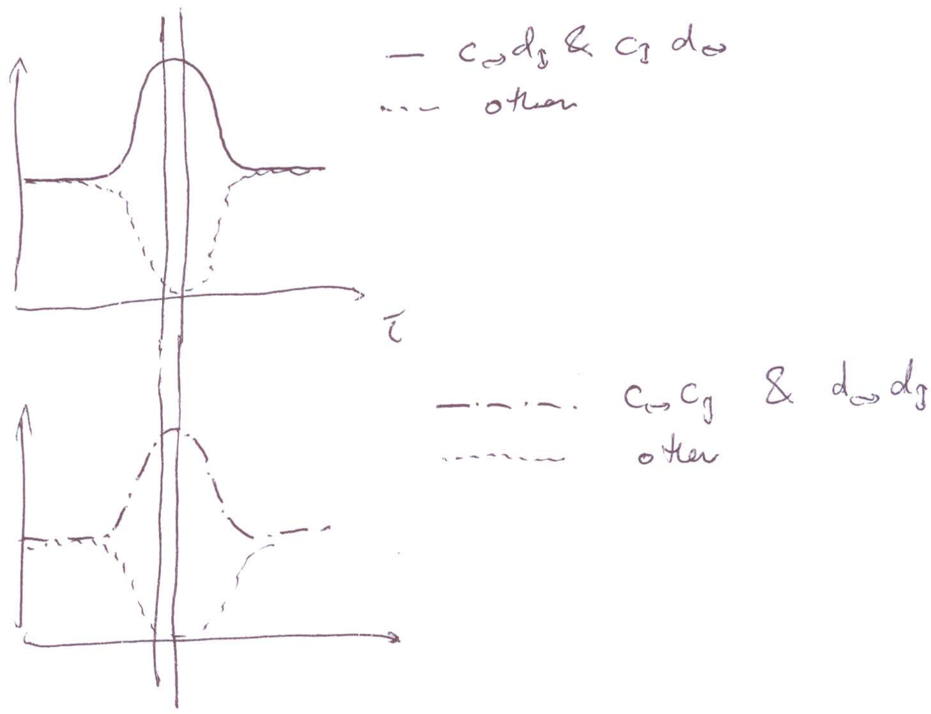
WE CAN DETECT A TRIT ($\log_2 3$ INFORMATION PER PAIR).

EXPERIMENTAL REALIZATION:



SINGLET: $|\Psi_-\rangle$

$|\Phi_+\rangle$



PROBLEMS:
- SOURCE NOT DETERMINISTIC
- LOSSES
- LOOKING ONLY AT COINCIDENCE EVENTS AT THE OUTPUT "COINCIDENCE BASIS MEASUREMENTS"

D8

TELEPORTATION

Change in notation: $|\leftrightarrow\rangle \equiv |0\rangle$ ← THESE ARE NOT
$|\updownarrow\rangle \equiv |1\rangle$ ← PHOTON NUMBERS

$$|\Phi_+\rangle_{A'B} = \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$

Suppose that Alice receives a qubit $|\psi_A\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$
And receives qubits AA' in the Bell basis:

$$|\tilde{\Phi}_+\rangle_B = {}_{AA'}\langle\Phi_+|\left(|\psi\rangle_A|\Phi_+\rangle_{A'B}\right) = \frac{1}{2}\left(\alpha|0\rangle_B + \beta|1\rangle_B\right)$$

$$|\tilde{\Phi}_-\rangle_B = \frac{1}{2}\left(\alpha|0\rangle_B - \beta|1\rangle_B\right)$$

$$|\tilde{\Psi}_+\rangle_B = \frac{1}{2}\left(\alpha|1\rangle_B + \beta|0\rangle_B\right)$$

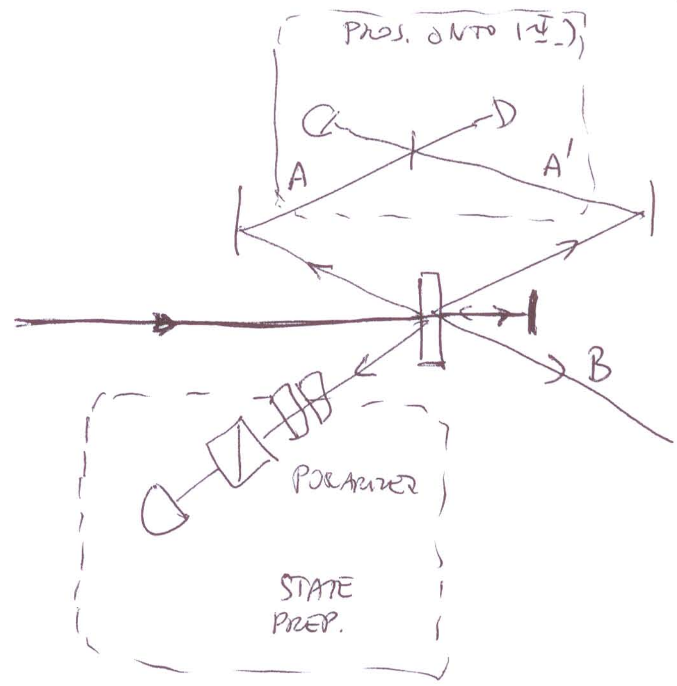
$$|\tilde{\Psi}_-\rangle_B = \frac{1}{2}\left(\alpha|1\rangle_B - \beta|0\rangle_A\right)$$

Each of these two states can be brought to $|\psi\rangle_B = \alpha|0\rangle_B + \beta|1\rangle_B$ by a unitary transformation. Bob needs to receive two bits of classical information from Alice. Note that:

- all outcomes are equiprobable
- the average state on Bob's side:

$$|\tilde{\Phi}_+\rangle_B\langle\tilde{\Phi}_+| + |\tilde{\Phi}_-\rangle_B\langle\tilde{\Phi}_-| + |\tilde{\Psi}_+\rangle_B\langle\tilde{\Psi}_+| + |\tilde{\Psi}_-\rangle_B\langle\tilde{\Psi}_-| = \frac{1}{2}\hat{\mathbb{1}}_B$$

TELEPORTATION EXPERIMENT



PROBLEMS:

- RANDOM GENERATION OF PHOTON PAIRS
- POOR PHOTON NUMBER CORRELATIONS
  - WE DO NOT KNOW WHETHER PHOTON B ARRIVED IF WE DON'T MEASURE IT.
- BELL STATE MEASUREMENT SUCCESSFUL IN AT MOST 50% CASES.