(I)

Wyznaczyć ekstrema funkcji

Zbadać punkty krytyczne funkcji $z=xy$ przy warunku

$$x^2+y^2=2.$$

Skorzystając z funkcji

$$F(x,y,\lambda) = xy - \lambda(x^2+y^2-2)$$

Obliczymy jej punkty krytyczne

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= y - 2x\lambda = 0\\ \frac{\partial F}{\partial y} &= x - 2y\lambda = 0\end{aligned}\right\} \qquad y - 4\lambda^2 y = 0 \Leftrightarrow y(1-4\lambda^2)=0$$

$$\frac{\partial F}{\partial \lambda} = -(x^2+y^2)+2=0 \Rightarrow \quad x^2+y^2=2 \;!!$$

a) $\lambda = \pm\frac{1}{2} \Rightarrow \quad y = \pm x \Rightarrow \quad 2x^2=2 \Rightarrow x^2=1 \Rightarrow \boxed{x=\pm 1}$

Punkty krytyczne: $\underset{\lambda=\frac{1}{2}}{(1,1)}$, $\underset{\lambda=-\frac{1}{2}}{(1,-1)}$, $\underset{\lambda=-\frac{1}{2}}{(-1,1)}$ $\underset{\lambda=\frac{1}{2}}{(-1,-1)}$.

Ekstrema:

$$\begin{array}{cccc} & 0 & 2k+1.\ 2k+2\ldots & \underline{k+n} \end{array}$$

$$\begin{vmatrix} \frac{\partial^2 F}{\partial \lambda^2} & \frac{\partial^2 F}{\partial x \partial \lambda} & \frac{\partial^2 F}{\partial y \partial \lambda} \\ \frac{\partial^2 F}{\partial \lambda \partial x} & \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial \lambda \partial y} & \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 0 & -2x & -2y \\ -2x & -2\lambda & 1 \\ -2y & 1 & -2\lambda \end{vmatrix}$$

(II)

$$+2x \begin{vmatrix} -2x & -2y \\ 1 & -2\lambda \end{vmatrix} - 2y \begin{vmatrix} -2x & -2y \\ -2\lambda & 1 \end{vmatrix}$$

$$= 2x\,(4x\lambda + 2y) - 2y\,(-2x - 4y\cancel{\lambda}\lambda)$$

$$= 8xy + (8x^2 + 8y^2)\lambda$$

$\boxed{(-1)^k |H|}$!!

ly

$k=1$

$(1,1),\ \lambda = \frac{1}{2} \rightarrow \quad 8 + 16\,\frac{1}{2} = 16 \quad (-1)16 = -16$ max.

$(1,-1),\ \lambda = -\frac{1}{2} \quad -8 - 8 = -16 \quad (-1)(-16) = 16$ min

$(-1,-1)\ \lambda = \frac{1}{2} \quad 8 + \frac{16}{2} = 16 \quad (-1)(16) = -16$ max

$(-1,1) - \lambda = -\frac{1}{2} \quad -16 \quad (-1)(-16) = 16$ min

III

Zbadać przy pomocy mnożników Lagrange'a punkty krytyczne funkcji $f(x,y)=\frac{x}{y}$ w zbiorze $K=\{(x,y):$
$(x-3)^2+(y-1)^2=2\}$.

$g: (x-3)^2+(y-1)^2-2$ <u>zał. dg ≠ 0</u>

<u>y=0 nie istnieje</u>

$$F(x,y,\lambda)=\frac{x}{y}-\lambda\left[(x-3)^2+(y-1)^2-2\right]$$

$$\frac{\partial F}{\partial x}=\frac{1}{y}-2\lambda(x-3)=0 \Rightarrow \boxed{\lambda=\frac{1}{2y(x-3)}}$$

$$\frac{\partial F}{\partial y}=-\frac{x}{y^2}-2\lambda(y-1)=0 \qquad \underline{x=3} \Rightarrow y=?$$

$$\frac{\partial F}{\partial \lambda}=-\left[(x-3)^2+(y-1)^2-2\right]=0$$

$$0=-\frac{x}{y^2}-\frac{y-1}{y(x-3)}=-\frac{1}{y}\left[\frac{x}{y}+\frac{y-1}{x-3}\right]=-\frac{1}{y}\,\frac{x(x-3)+y(y-1)}{y(x-3)}$$

$$x(x-3)+(y-1)y=0 \Rightarrow x^2-3x+y(y-1)=0 \Rightarrow$$

$$x=\frac{3\pm\sqrt{9-4(y-1)y}}{2}$$

$$(x-3)^2+(y-1)^2=2 \Rightarrow \left[\frac{-3\pm\sqrt{9-4y(y-1)}}{2}\right]^2+(y-1)^2=2$$

$$\frac{9+9-4y(y-1)\mp 6\sqrt{9-4y(y-1)}}{4}+(y-1)^2=2.$$

$$\frac{18-4\cancel{y^2}+4y\mp 6\sqrt{9-4y(y-1)}}{4}+\cancel{y^2}-2y-1=0$$

$$\frac{14}{4}\,{}^{\to\frac{7}{2}}-y\mp\frac{3}{2}\sqrt{9-4y(y-1)}=0 \Rightarrow 7\mp3\sqrt{9-4y(y-1)}=2y$$

IV

$$\mp 3\sqrt{9-4y(y-1)} = 2y-7 \Rightarrow \quad 9(9-4y(y-1)) = 4y^2+49-28y$$

$$81-36y^2+36y = 4y^2+49-28y \Rightarrow \quad 40y^2-64y-32=0$$

$$10y^2-16y-8=0 \Rightarrow \quad 5y^2-8y-4=0 \rightarrow \quad y=\frac{8\pm\sqrt{64+4\cdot5\cdot4}}{2\cdot5}$$

$$y=\frac{8\pm\sqrt{144}}{2\cdot5} = \frac{8\pm12}{2\cdot5} \nearrow \begin{matrix} y_1=\frac{10}{5}=2 \\ y_2=-\frac{2}{5}. \end{matrix}$$

$$y_1=2 \Rightarrow x=\frac{3\pm\sqrt{9-4\cdot2}}{2} = \frac{3\pm1}{2} \nearrow \begin{matrix} \underline{\underline{x_1=2}} \\ \cancel{x_2=1} \end{matrix}$$

$$y_2=-\frac{2}{5} \Rightarrow x=\frac{3\pm\sqrt{9-4(-\frac{7}{5})(-\frac{2}{5})}}{2} = \frac{3\pm\sqrt{9-\frac{56}{25}}}{2}$$

$$=\frac{3\pm\sqrt{\frac{225-56}{25}}}{2} = \frac{3\pm\sqrt{\frac{169}{25}}}{2} = \frac{3\pm\frac{13}{5}}{2}$$

$$x \nearrow x_1=\frac{14}{5} \qquad \searrow \cancel{x_2=-\frac{1}{5}}.$$

$F(2,2)=1.$

$\left\{ \begin{array}{l} (2,2) \text{ Punk kr.} \\ \lambda=-\frac{1}{4} \\ \left(\frac{14}{5},-\frac{2}{5}\right) \quad \lambda=\frac{25}{4}. \end{array} \right.$

$$H(x,y,\lambda)=\begin{vmatrix} 0 & -2(x-3) & -2(y-1) \\ -2(x-3) & -2\lambda & -\frac{1}{y^2} \\ -2(y-1) & -\frac{1}{y^2} & \frac{2x}{y^3}-2\lambda \end{vmatrix} \nearrow \searrow \begin{vmatrix} 0 & +2 & -2 \\ +2 & \frac{1}{2} & -\frac{1}{4} \\ -2 & -\frac{1}{4} & \frac{1}{2}+\frac{1}{2} \end{vmatrix} = -4$$

$(-1)^1\cdot(-4)=4$

<u>minimum!!</u>

$f: \mathbb{R}^3 \to \mathbb{R} : (x,y,z) \to ax^2+by^2+cz^2$

$\omega: \mathbb{R}^3 \to \mathbb{R} : (x,y,z) \to x+y+z-1$

Widać, że $\frac{\partial \omega}{\partial z} \neq 0$. Więc zawsze możemy napisać $z=z(x,y)$. Wtedy, znaleźć punkty krytyczne f w ω jest równoważny znalezieniu punktów krytycznych funkcji

$$\bar{f}(x,y) = f(x,y,z(x,y))$$

Więc,

$$0 = \frac{\partial \bar{f}}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \qquad 0 = \frac{\partial \bar{f}}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} \qquad (1)$$

Trzeba znaleźć $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$. Chociaż to możemy znaleźć bezpośrednio bo $x+y+z-1=0 \Rightarrow z = 1-x-y-z$. Zrobimy to ~~w~~ innym sposobem. To pozwala nam sprawdzić ~~to zrob~~ jak obliczyć $\partial z/\partial x$ i $\partial z/\partial y$ w innych przypadkach. Mamy, że $z(x,y)$ spełnia.

$$\omega(x,y,z(x,y))=0 \Rightarrow \frac{\partial \bar{\omega}(x,y,z(x,y))}{\partial x} = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial \bar{\omega}(x,y,z(x,y))}{\partial y} = \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z}\frac{\partial z}{\partial y} = 0.$$

Z tego

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial \omega}{\partial x}}{\frac{\partial \omega}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial \omega}{\partial y}}{\frac{\partial \omega}{\partial z}}.$$

Korzystając z (1)

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial z}\frac{\frac{\partial \omega}{\partial x}}{\frac{\partial \omega}{\partial z}} = 0 \qquad \frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\frac{\frac{\partial \omega}{\partial y}}{\frac{\partial \omega}{\partial z}} = 0.$$

To

$$\frac{\partial f}{\partial x}\frac{\partial \omega}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial \omega}{\partial x} = 0 \qquad \frac{\partial f}{\partial y}\frac{\partial \omega}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial \omega}{\partial y} = 0.$$ (2)

To możemy przedstawić jako dwa wyznaczniki.

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial z} \end{vmatrix} = 0 \qquad \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{vmatrix} = 0$$

To jest równoważny

$$\mathrm{Rg}\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{pmatrix} = 1$$

To oznacza $\nabla f \propto \nabla \omega$.

I problem znalezienia punktów krytycznych jest ~~punktu w~~ problemem rozwiązania

$$\omega(x,y,z) = 0 \qquad \mathrm{Rg}\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{pmatrix} = 1. \quad (1)$$

Niezależnie od tego, czy $\frac{\partial \omega}{\partial z} \neq 0$ albo $\frac{\partial \omega}{\partial y} \neq 0$ albo $\frac{\partial \omega}{\partial x} \neq 0$ zawsze trafimy do czegoś jak (1). (chyba z więcej ~~w~~ warunkami $\omega_1, \omega_2, \omega_2$, np. $\mathrm{Rg}\begin{pmatrix} \nabla f \\ \nabla \omega_1 \\ \nabla \omega_2 \end{pmatrix} = 2$)

Korzystając z (1), w naszym problemie mamy.

$$\mathrm{Rg}\begin{pmatrix} \nabla f \\ \nabla \omega \end{pmatrix} = \begin{pmatrix} 2ax & 2by & 2cz \\ 1 & 1 & 1 \end{pmatrix} = 1 \Rightarrow \begin{vmatrix} 2ax & 2by \\ 1 & 1 \end{vmatrix} = 0$$

$$2(ax - by) = 0$$

$$\begin{vmatrix} 2by & 2cz \\ 1 & 1 \end{vmatrix} = 0$$

$$2(by - cz) = 0.$$

$$\omega(x,y,z) = 0 \iff \boxed{x + y + z = 1}$$

$$\left.\begin{array}{l} ax = by \\ by = cz. \\ x+y+z=1 \end{array}\right\} \quad \begin{pmatrix} a & -b & 0 \\ 0 & b & -c \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{x = \frac{bc}{ab+ac+bc} \quad y = \frac{ac}{ab+bc+bd} \quad z = \frac{ab}{ab+ac+bc}.}$$

Możemy teraz zbadać punkty krytyczne

Obliczymy Hessian.

$$H = \begin{bmatrix} \frac{\partial^2 \bar{f}}{\partial x^2} & \frac{\partial^2 \bar{f}}{\partial x \partial y} \\ \frac{\partial^2 \bar{f}}{\partial y \partial x} & \frac{\partial^2 \bar{f}}{\partial y^2} \end{bmatrix}$$

Trzeba pamiętać, że tutaj $\frac{\partial^2 \bar{f}}{\partial x^2}$ $\quad \bar{f} = f(x,y,z(x,y))$

Więc, $z = z(x,y)$!! Wiemy $\frac{\partial w}{\partial z} = 1 \neq 0$ możemy to zrobić zawsze.

$$\frac{\partial \bar{f}}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \qquad \frac{\partial z}{\partial x} =$$

$$= 2ax + 2zc \underbrace{\frac{\partial z}{\partial x}}_{-1} \qquad \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial z}}$$

$$= 2ax - 2zc \qquad = -1.$$

$$\frac{\partial \bar{f}}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$$

$$= 2by + 2cz(-1) \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial w}{\partial y}}{\frac{\partial w}{\partial z}} = -1$$

$$= 2by - 2cz.$$

(4)

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2ax - 2zc\right) = 2a - 2\frac{\partial z}{\partial x}c = 2a + 2c.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(2ax - 2zc) = -2\frac{\partial z}{\partial y}c = -2c$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(2by - 2cz) = 2b - 2c\frac{\partial z}{\partial y} = 2b + 2c.$$

$$H = \begin{pmatrix} 2a+2c & -2c \\ -2c & 2b+2c \end{pmatrix} \Rightarrow$$

St

$2a + 2c > 0$, $2(a+c)(b+c) - 4c^2 > 0$ — ~~maksimum~~ minimum.

$2a + 2c < 0$, $2(a+c)(b+c) - 4c^2 > 0$ — maksimum.

$2(a+c)(b+c) - 4c^2 < 0$ Punkt siodło.

Reszta → Nie wiadomo...

Skoro $a, b, c > 0$ zawsze, nie może mieć maksimum,

Jeżeli

$2(a+c)(b+c) - 4c^2 > 0$ minimum

$2(a+c)(b+c) - 4c^2 < 0$ punkt siodło.