Ćwiczenie 1.

$$\lim_{n\to+\infty} \frac{n!\,n^n}{(2n)!}$$

Korzystamy z kryterium D'Alemberta

$$\lim_{n\to+\infty} \frac{\frac{n!\,n^n}{(2n)!}}{\frac{(n-1)!\,(n-1)^{n-1}}{(2n-2)!}} = \lim_{n\to+\infty} \frac{n!\,(2n-2)!\,n^n}{(n-1)!\,(2n)!\,(n-1)^{n-1}}$$

$$= \lim_{n\to+\infty} \frac{n\cdot n^{n-1}\cdot n}{2n(2n-1)(n-1)^{n-1}} = \lim_{n\to+\infty} \frac{n}{2(2n-1)}\left(\frac{n}{n-1}\right)^{n-1}$$

$$= \lim_{n\to+\infty} \left(\frac{n}{n-1}\right)^{n-1}\cdot\frac{1}{4} = \lim_{n\to+\infty}\left(1+\frac{n}{n-1}-1\right)^{n-1}\frac{1}{4}$$

nioz. 1^∞

$$= \lim_{n\to+\infty}\left(1+\frac{1}{n-1}\right)^{n-1}\frac{1}{4} = \frac{e}{4} < 1 \Rightarrow \lim_{n\to+\infty}\frac{n!\,n^n}{(2n)!} = 0$$

Ćwiczenie 2

Stolz.

$$\lim_{n\to+\infty} \frac{\sqrt{2}+\sqrt{4}+\ldots+\sqrt{2n}}{n\sqrt{n}} = \lim_{n\to+\infty}\frac{(\sqrt{2}+\sqrt{4}+\ldots+\sqrt{2n}) - (\sqrt{2}+\sqrt{4}+\ldots+\sqrt{2n-2})}{n\sqrt{n}-(n-1)\sqrt{n-1}}$$

↑ rosnący i zbieżny do $+\infty$.

$$= \lim_{n\to+\infty}\frac{\sqrt{2n}}{n\sqrt{n}-(n-1)\sqrt{n-1}} = \lim_{n\to+\infty}\frac{\sqrt{2n}}{n^{3/2}-(n-1)^{3/2}} = \lim_{n\to+\infty}\frac{\sqrt{2n}\,(n^{3/2}+(n-1)^{3/2})}{n^3-(n-1)^3}$$

$\infty-\infty$

$$= \lim_{n\to+\infty}\frac{\sqrt{2n}\,(n^{3/2}+(n-1)^{3/2})}{+3n^2-3n+1} = \lim_{n\to\infty}\frac{\frac{\sqrt{2n}}{\sqrt{n}}\left(\frac{n^{3/2}}{n^{3/2}}+\left(\frac{n-1}{n}\right)^{3/2}\right)}{\frac{3n^2}{n^2}-\frac{3n}{n^2}+\frac{1}{n^2}} = \frac{\sqrt{2}\cdot 2}{3}.$$

(the last two terms of the denominator → 0, 0)

$$\lim_{n\to+\infty}\frac{\sqrt[3]{1}+\sqrt[3]{3}+\ldots+\sqrt[3]{2n+1}}{n\sqrt[3]{n}} \overset{\text{Stolz}}{=} \lim_{n\to+\infty}\frac{\left(\sqrt[3]{1}+\sqrt[3]{3}+\ldots+\sqrt[3]{2n+1}\right)-\left(\sqrt[3]{1}+\ldots+\sqrt[3]{2n-1}\right)}{n^{4/3}-(n-1)^{4/3}}$$

↑ ściśle rosnący dąży do +∞

$$= \lim_{n\to+\infty}\frac{\sqrt[3]{2n+1}}{n^{4/3}-(n-1)^{4/3}} = \lim_{n\to+\infty}\frac{\sqrt[3]{2n+1}\left((n^{4/3})^2+n^{4/3}(n-1)^{4/3}+\left[(n-1)^{4/3}\right]^2\right)}{n^4-(n-1)^4}$$

$$= \lim_{n\to+\infty}\frac{\sqrt[3]{2n+1}\left(n^{\frac{8}{3}}+n^{4/3}(n-1)^{4/3}+(n-1)^{\frac{8}{3}}\right)}{n^4-n^4+4n^3-6n^2+4n-1} =$$

$$= \lim_{n\to\infty}\frac{\sqrt[3]{\frac{2n+1}{n}}\cdot\frac{n^{8/3}+n^{4/3}(n-1)^{4/3}+(n-1)^{\frac{8}{3}}}{n^{8/3}}}{\frac{4n^3}{n^3}-\frac{6n^2}{n^3}+\frac{4n}{n^3}-\frac{1}{n^3}} = \frac{\sqrt[3]{2}\cdot 3}{4}.$$

($\frac{6n^2}{n^3}\to 0$, $\frac{4n}{n^3}\to 0$, $\frac{1}{n^3}\to 0$)

Zadanie 3

$$\lim_{x\to 2}\frac{2-\sqrt{x+2}}{x-2} \overset{\text{L'Hôpital}}{=} \lim_{x\to 2}\frac{-\frac{1}{2\sqrt{x+2}}}{1} = -\frac{1}{4}$$

$$\lim_{x\to 2}\frac{2-\sqrt{x+2}}{x-2} = \lim_{x\to 2}\frac{2^2-\sqrt{x+2}^2}{(x-2)(2+\sqrt{x+2})} = \lim_{x\to 2}\frac{4-x-2}{(x-2)(2+\sqrt{x+2})} = \frac{-1}{4}.$$

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} \overset{\text{l'Hôpital}}{=} \lim_{x\to 0}\frac{\frac{1}{2\sqrt{x+1}}}{1} = \frac{1}{2}.$$

$$= \lim_{x\to 0}\frac{\sqrt{x+1}^2-1^2}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)} = \frac{1}{2}.$$

$$\lim_{x\to 2}\frac{\sqrt{x+14}-4}{x-2} \overset{\text{l'Hôpital}}{=} \lim_{x\to 2}\frac{\frac{1}{2\sqrt{x+14}}}{1} = \frac{1}{8}$$