Sending qubits via a channel with imperfectly correlated noise - natural SU(2) diffusion model

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Abstract

We present a model of an *N*-qubit channel where consecutive qubits experience correlated random rotations. Our model is **an extension to the standard decoherence-free subsystems approach** (DFS) which assumes that all the qubits experience the same disturbance. The variation of rotations acting on consecutive qubits is modeled as diffusion on the SU(2) group. The model may be applied to **spins traveling in a varying magnetic field**, or to **photons passing through a fiber whose birefringence fluctuates** over the time separation between photons**.** We derive an explicit formula describing the action of the channel on an arbitrary *N*-qubit state. For *N*=3 we investigate the effects of diffusion on both classical and quantum capacity of the channel. In particular we observe that nonorthogonal states are necessary to achieve the optimal classical capacity. Furthermore we find the threshold for the diffusion parameter above which coherent information of the channel vanishes.

1. Depolarizing channel with perfectly correlated noise

• *N* qubit depolarizing channel, where each qubit experience the same and completely random disturbance:

$$\boxed{\mathcal{E}(\rho_N) = \int \mathrm{d}U U^{\otimes N} \rho_N U^{\dagger\otimes N}}$$

N qubit state $SU(2)$ Haar measure

• Applications

1. photons transmitted through a long fiber (each photon experience the same random rotation of polarization).
2. spins ½ being sent through a slowly varying magnetic field
3. communication in the absence of reference frames

2. Structure of the output state with perfectly correlated noise

• Irreducibe subspaces under the action of $U^{\otimes N}$

$$\mathcal{H}^{\otimes N} = \bigoplus_{j=0}^{N/2} \underbrace{\mathcal{H}_j \oplus \ldots \oplus \mathcal{H}_j}_{d_j \text{ times}} = \bigoplus_{j=0}^{N/2} \mathcal{H}_j \otimes \mathbb{C}_{d_j}$$

multiplicity subspace (decoherence free subsystem)

twirling operation
$$\boxed{\mathcal{T}(\rho) := \mathcal{E}(\rho) = \int \mathrm{d}U U^{\otimes N} \rho U^{\dagger\otimes N} = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j}$$

$$p_j = \mathrm{Tr}(P_j \rho) \qquad \rho_j = \frac{1}{p_j} \mathrm{Tr}_{\mathcal{H}_j}(P_j \rho P_j) \qquad P_j \text{ - projection on } \mathcal{H}_j \otimes \mathbb{C}_{d_j}$$

• Faithfully transmitted states – allow for noiseless classical and quantum communication (see e.g. [2])

$$\rho = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j \qquad \text{twirled structure}$$

4. What happens if noise is not perfectly correlated?

5. Imperfectly correlated noise model

• Consecutive qubits experience slightly different rotations

$$\mathcal{E}(\rho) = \int \mathrm{d}U_1 \ldots \mathrm{d}U_N p(U_1, \ldots, U_N) U_1 \otimes \cdots \otimes U_N \; \rho \; U_1^\dagger \cdots \otimes U_N^\dagger$$

• The actions described via a stationary Markov process

$$p(U_1, \ldots, U_N) = p(U_N | U_{N-1}) \ldots p(U_2 | U_1)$$

• **Possible applications:**

1. photons transmitted through a long fiber if photon separation is comparable to birefringence fluctuation time
2. spins travelling through relatively quickly varying magnetic field

6. What is the natural choice for conditional probability $p(U_i|U_{i-1})$=?

7. Natural choice for conditional probability Diffusion on the SU(2) group

• Isotropic diffusion on SU(2)

$$\partial_t p(U;t) = \frac{1}{2} \hat{\Delta} p(U;t)$$

Laplace operator on the SU(2) group

• Solution with the initial conditions $p(U;0) = \delta(U - \mathbb{1})$ [3]

$$p(U;t) = \sum_{j=0}^{\infty} (2j+1) \exp\left(-\frac{1}{2} j(j+1)\, t\right) \sum_{m=-j}^{j} \mathfrak{D}^j(U)^m_m$$

diffusion time (strength) rotation matrices

• Conditional probability

$$\boxed{p(U_i | U_{i-1}) = p(U_i U_{i-1}^\dagger; t)}$$

$t \to 0$ perfect noise correlation $t \to \infty$ no correlation

7. Action of the channel with imperfectly correlated noise

• Joint probablity distribution formunitaries

$$p(U_1, \ldots, U_N) = p(U_N | U_{N-1}) \cdots p(U_2 | U_1) = p(U_N U_{N-1}^\dagger; t) \ldots p(U_2^\dagger U_1; t)$$

• The channel action

$$\boxed{\mathcal{E}(\rho) = \mathcal{I}_{N-1}\left(\mathcal{I}_{N-2}\left(\ldots \mathcal{I}_1(\mathcal{T}(\rho)) \ldots\right)\right)}$$

$$\mathcal{I}_i(\rho) = \int \mathrm{d}U p(U;t) \mathbb{1}^{\otimes i} \otimes U^{\otimes N-i} \; \rho \; \mathbb{1}^{\otimes i} \otimes U^{\dagger\otimes N-i} \qquad \mathcal{T}(\rho) = \int \mathrm{d}U U^{\otimes N} \rho U^{\dagger \otimes N}$$

• Output states have the twirled structure ($\mathcal{T}, \mathcal{I}_i$ commute). So effectively the evolution can be described as:

$$\boxed{\bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j \longrightarrow \bigoplus_{j=0}^{N/2} \frac{p'_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho'_j}$$

• More pricesely.... (see [1] for details)

$$\mathcal{E}(\rho) = \text{lengthy expression involving } e^{-\frac{1}{2}j(j+1)t} \text{ and Wigner } 6j \text{ symbols}$$

8. Example: Capacity of the three qubit channel with imperfectly correlated noise

• Three qubit twirlled state. Effectively: one qubit + one classical state

$$\rho = \frac{p}{2}(\mathbb{1}_{\mathcal{H}_{1/2}} \otimes \rho_{1/2}) \oplus \frac{1-p}{4} \mathbb{1}_{\mathcal{H}_{3/2}}$$

• Classical capacity. Holevo-Schumacher-Westmoreland formula:

$$C = \sup_{\{p_i, \rho_i\}} \left[S\left(\mathcal{E}\left(\sum_i p_i \rho_i\right)\right) - \sum_i p_i S(\mathcal{E}(\rho_i)) \right]$$

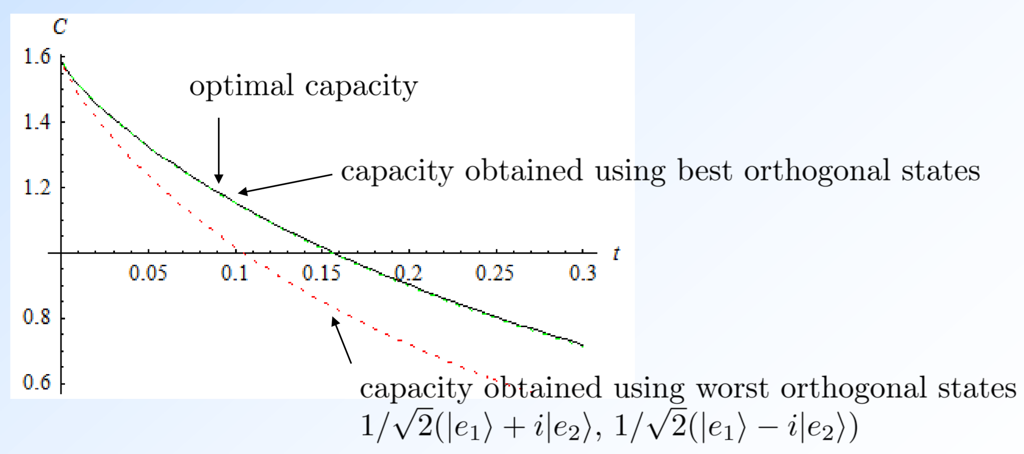
• Ensemble maximizing classical channel capacity *C* contain *non-orthogonal* states:

$$\tfrac{1}{2}\mathbb{1}_{\mathcal{H}_{1/2}} \otimes |\psi_1\rangle\langle\psi_1|, \; \tfrac{1}{2}\mathbb{1}_{\mathcal{H}_{1/2}} \otimes |\psi_2\rangle\langle\psi_2|, \; \tfrac{1}{4}\mathbb{1}_{\mathcal{H}_{3/2}}, \text{ where } \langle\psi_1|\psi_2\rangle \neq 0$$

• Appropriate orthogonal states can, however, achieve almost optimal capacity. Best orthogonal states have the form:

$$|\psi_1\rangle = 1/\sqrt{2}(|e_1\rangle + |e_2\rangle)$$
$$|\psi_2\rangle = 1/\sqrt{2}(|e_1\rangle - |e_2\rangle)$$

$|e_1\rangle$ $(|e_2\rangle)$ corresponds to a subspace obtained by first adding two spins 1/2 and obtaining total angular momentum 0 (1), and then adding the third spin obtaining the total an gular momentum 1/2



• Additionally for diffusion strength t > 0.275 the coherent information is zero. This suggests that quantum communication is impossible in this regime (noise correlation is too weak) [1]

Conclusions

• A natural model of an *N* qubit channel is presented, making use of SU(2) diffusion in order to describe imperfect correlations between random rotations experienced by consecutive qubits.

• In the simplest case of *N*=3, optimal classical capacity of the channel is calculated. States necessary for optimal communication are found to be non-orthogonal , although appropriately chosen orthogonal states perform almost optimal. The strength of diffusion above which coherent information of the channel is zero is found.

References

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