State estimation on correlated copies

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quant-ph/0412155 (to appear in PRA)

Abstract

State estimation is usually analyzed in the situation when copies are in a product state, either mixed or pure. We investigate here the concept of state estimation on correlated copies. We analyze state estimation on correlated N qubit states, which are permutationally invariant.Using a correlated state we try to estimate as good as possible the direction of the Bloch vector of a single particle reduced density matrix. We derive the optimal fidelity for all permutation invariant states. We find the optimal state, which yields the highest estimation fidelity among the states with the same reduced density matrix. Interestingly this state is not a product state. We also point out that states produced by optimal universal cloning machines are the worst from the point of view of estimating the reduced density matrix.

1. What is state estimation?

• Given N copies of an unknown quantum state, one performs measurements and judging on measurement results tries to guess what was the state.

$$|\psi\rangle^{\otimes N} = |\psi\rangle \otimes |\psi\rangle \otimes \ldots \otimes |\psi\rangle \xrightarrow{\text{Measurement}} \mu \quad p_\mu \xrightarrow{\text{Guessing}} \rho_\mu$$

N copies of an unknown state — possible result — probability of obtaining result μ — guessed state

• Quality of estimation

fidelity of a single guess:

$$F_\mu = \langle\psi|\rho_\mu|\psi\rangle$$

average fidelity of state estimation strategy:

$$F = \sum_\mu p_\mu \langle\psi|\rho_\mu|\psi\rangle = \langle\psi|\sum_\mu p_\mu \rho_\mu|\psi\rangle$$

If state $|\psi\rangle$ is totally unknown we require fidelity F not to depend on $|\psi\rangle$

2. Generalized measurement

• Standard measurement

$$A = \sum_i \lambda_i P_i$$

observable — measurement outcomes — projection operators

$P_i P_j = \delta_{ij} P_i$ orthogonality

$\sum_i P_i = \mathbf{1}$ completeness

$p_i = Tr(\rho P_i)$ probability of the result i

• Generalized measurement

coupling the system with an external ancillary system (e.g. measuring device), allowing for an evolution and performing a standard measurement on system+ancilla. This results in an effective generalized measurement on a system, described by operators P_μ (POVM – positive operator valued measure) :

probability of the result μ: $p_\mu = Tr(\rho P_\mu)$ $\quad P_\mu \geq 0 \quad \sum_\mu P_\mu = \mathbf{1}$

3. Optimal estimation on product states

• Optimal fidelity of estimating N copies of unknown qudit state [1,2]: $F = \dfrac{N+1}{N+d}$

Optimal estimation strategy requires the use of generalized measurement, performed on all copies simultaneously

4. Cloning and Estimation – the motivation

• $N \rightarrow M$ cloning:

$$|\psi\rangle^{\otimes N} \otimes |0\rangle^{M-N} \otimes |A\rangle \xrightarrow{U} |\Psi\rangle \in \mathcal{H}^{\otimes M} \otimes \mathcal{H}_A$$

input states — blank states — ancilla

output state of M clones: $\tilde{\rho} = Tr_A(|\Psi\rangle\langle\Psi|)$

single particle density matrix of each clone is identical and given by: $\rho = Tr_{2,3,\ldots,M}\tilde{\rho}$

fidelity of cloning: $F = \langle\psi|\rho|\psi\rangle$

optimal fidelity [3]: $F = \dfrac{M(N+1)+N(d-1)}{M(N+d)}$

Clones are correlated!: $\tilde{\rho} \neq \rho^{\otimes M}$

• Estimation on clones

$|\psi\rangle^{\otimes N} \xrightarrow[N \rightarrow M \text{ cloning}]{\text{optimal}} F = \dfrac{M(N+1)+N(d-1)}{M(N+d)}$

estimation on N copies: $F = \dfrac{N+1}{N+d}$

estimation on M correlated copies: $F = \dfrac{N+1}{N+d}$

Clones have to be correlated!

In particular, if $M=\infty$, and clones were not correlated, then performing state estimation on $\rho^{\otimes N}$ we would be able to estimate $|\psi\rangle$ perfectly

In this case correlations worsen state estimation capabilities. Is it general?

5. Estimation on correlated copies

• Given a correlated state of N copies, $\tilde{\rho} \neq \rho^{\otimes M}$ with the same single particle reduced density matrices ρ, we try to find the optimal strategy to estimate ρ

• N qubits

$$\rho = \eta|\psi\rangle\langle\psi| + \frac{1-\eta}{2}\mathbf{1}$$

single particle reduced density matrix.

assuming mixedness parameter η is fixed, we try to estimate $|\psi\rangle$

• What is the optimal fidelity of estimating $|\psi\rangle$ given $\tilde{\rho}$?

• How correlations influence estimation fidelity?

• What is the optimal state $\tilde{\rho}$ for estimating $|\psi\rangle$ among states with fixed mixedness (η) of single particle reduced density matrix? Is it a product state $\tilde{\rho} = \rho^{\otimes M}$

6. Permutationally invariant states

• $\tilde{\rho}$ is permutationally invariant iff for every permutation operation Π , permuting N copies we have:

$$\Pi\tilde{\rho}\Pi^\dagger = \tilde{\rho}$$

• Basis in the space of N qubits (N spins 1/2), in terms of irreducible representations of $SU(2)^{\otimes N}$

$$|j,m,\alpha\rangle, \quad j = 0,\ldots,N/2 \quad m = -j,\ldots,j \quad \alpha = 1,\ldots,d_j$$

total spin (N even) — projection on a 'z' axis — indexes equivalent representations

For fixed j,α, vectors $|j,m,\alpha\rangle$ support an irreducible representation of SU(2)

For fixed j,m, vectors $|j,m,\alpha\rangle$ support an irreducible representation of permutation group S_N

• Permutationally invariant state of N qubits:

$$\tilde{\rho} = \sum_{j=0}^{N/2} p_j \frac{1}{d_j}\sum_{\alpha=1}^{d_j}\tilde{\rho}_{j,\alpha} \quad \text{where} \quad \sum_{j=0}^{N/2} p_j = 1$$

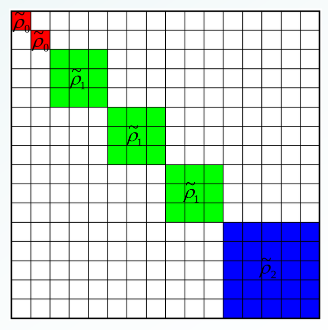
$$\tilde{\rho}_{j,1} = \tilde{\rho}_j \otimes \underbrace{|\Psi_-\rangle\langle\Psi_-|}_{\text{singlet state}}{}^{\otimes N/2-j}$$

$\tilde{\rho}_{j,\alpha}$ is the same as $\tilde{\rho}_{j,1}$ but is supported on subspace with $\alpha \neq 1$

$$\tilde{\rho}_j = \sum_{m,m'=-j}^{j} \lambda_{j,m,m'}|j,m'\rangle\langle j,m|$$

arbitrary state supported on the symmetric subspace of first $2j$ qubits

Example: 4 qubits

$\tilde{\rho} =$ 

7. Optimal estimation on correlated copies

• The optimal estimation scheme on product mixed states $\rho^{\otimes N}$ [4], is also optimal in the more general case of estimation using permutationally invariant states

1. Project on a space with given j,α
2. If $\alpha \neq 1$ apply an operation transforming the state to the subspace with $\alpha=1$
3. Perform optimal state estimation on first $2j$ qubits, which are now supported on symmetric subspace

• Optimal fidelity

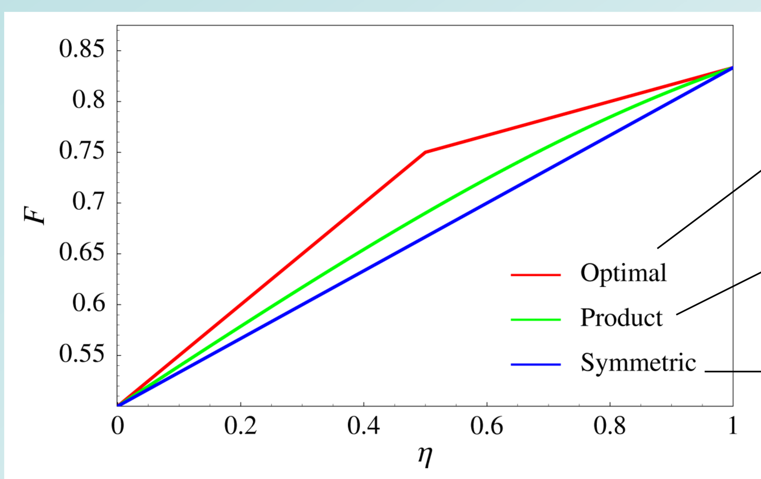
$$F = \frac{1}{2}\left(1 + \sum_{j=0}^{N/2}\frac{p_j\lambda_j}{j+1}\right) \quad \text{where} \quad \lambda_j = \sum_{m=-j}^{j} m\lambda_{j,m,m} \quad -j \leq \lambda_j \leq j$$

8. Correlated state optimal for state estimation

• Fixing mixedness of single particle density matrix η, we look for the highest fidelity F

maximize: $F = \dfrac{1}{2}\left(1 + \sum_{j=0}^{N/2}\dfrac{p_j\lambda_j}{j+1}\right)$ under constraints: $\dfrac{2}{N}\sum_{j=0}^{N/2} p_j\lambda_j = \eta \quad \sum_{j=0}^{N/2} p_j = 1$

• Subspaces with high j are bad for state estimation, yet necessary to yield high values of η

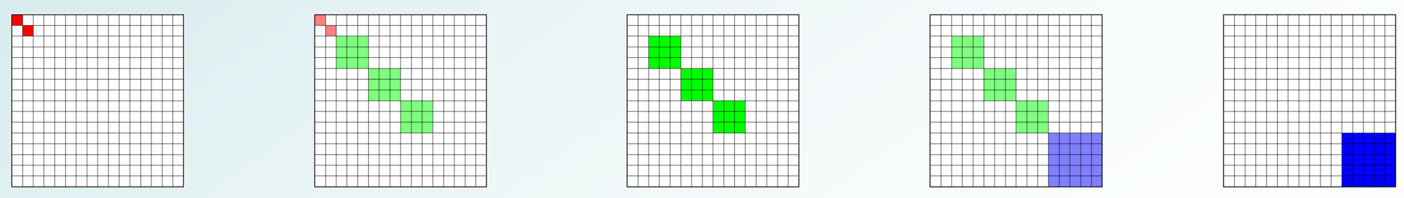


Optimal state is supported on at most two subspace with different j

product state: $\tilde{\rho} = \rho^{\otimes N}$ product state is not optimal!

state supported on symmetric subspace ($j=N/2$). Such states emerge from optimal cloning machines

• Structure of the optimal state in the case of 4 qubits:



$\eta = 0$ $\quad \eta = 1/2 \quad$ $\eta = 1$

Conclusions

• Optimal estimation fidelity found in the case of N qubit permutationally invariant states
• Correlations between copies influence estimation fidelities of single particle density matrix
• Product state is not the optimal one for state estimation
• States supported on symmetric subspace (e.g. states coming out from optimal cloning machines) are the worst from the point of view of state estimation.
• States optimal for state estimation are correlated and supported at most on two subspaces with different j

References

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