# Quantum Interferometry in the presence of losses


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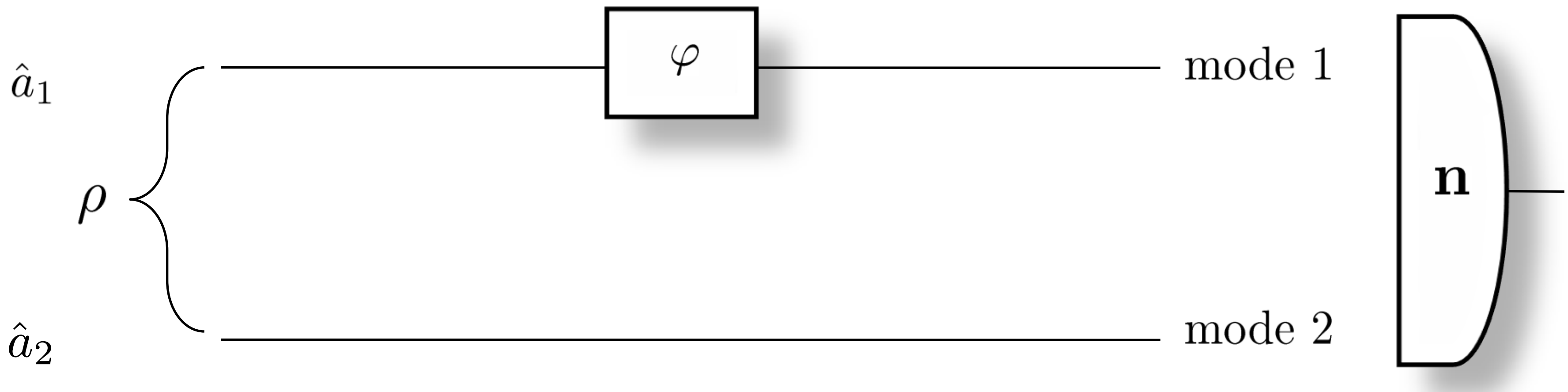
# Interferometry

- **Two optical modes, one delayed with respect to the other**

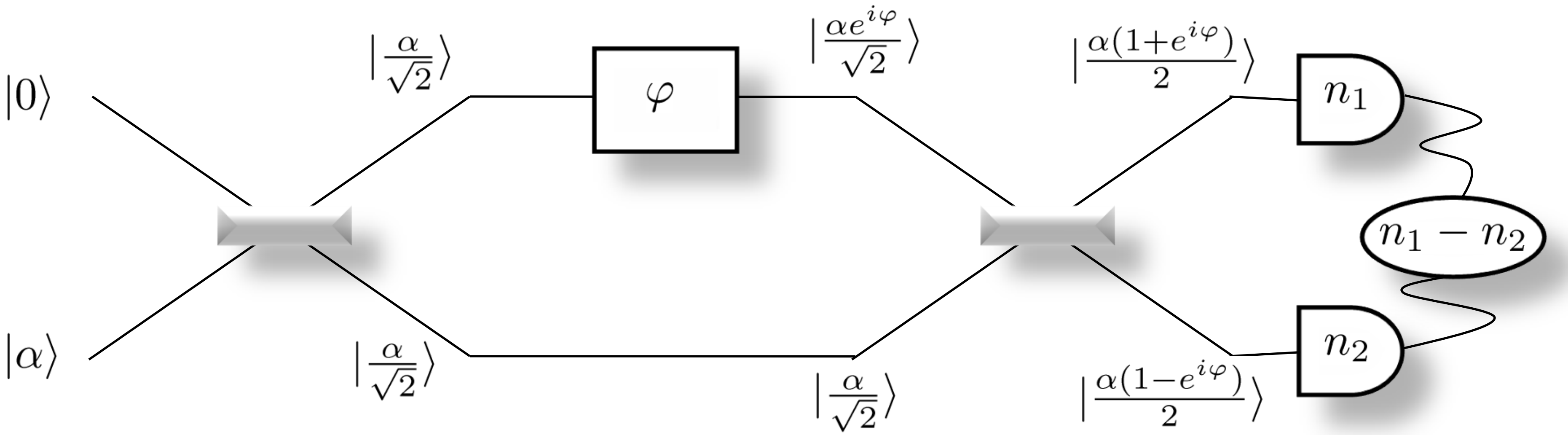



$$U = e^{-i\hat{a}_1^\dagger \hat{a}_1 \varphi}$$

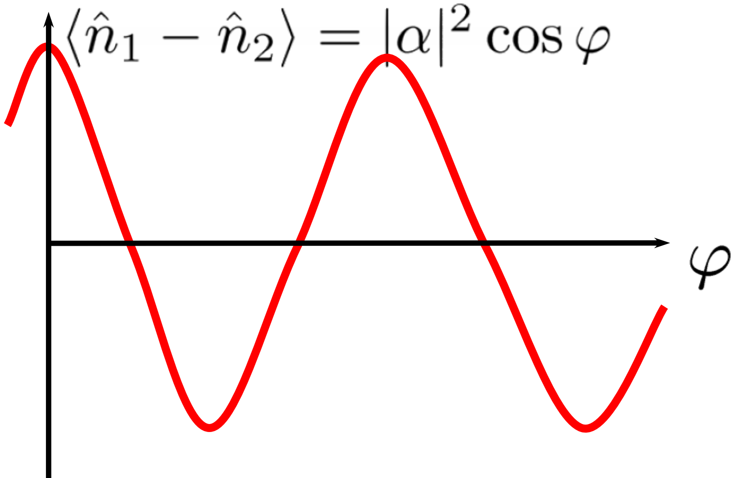
- **We want to find the optimal state and the optimal measurement to estimate** $\varphi$

# First approach to interferometry

- **Coherent state as an input**




$$\langle \hat{n}_1 - \hat{n}_2 \rangle = |\alpha|^2 \cos\varphi$$




estimator $\tilde{\varphi}(n_1, n_2)$

$$\tilde{\varphi}(n_1, n_2) = \arccos\left(\frac{n_1 - n_2}{|\alpha|^2}\right)$$

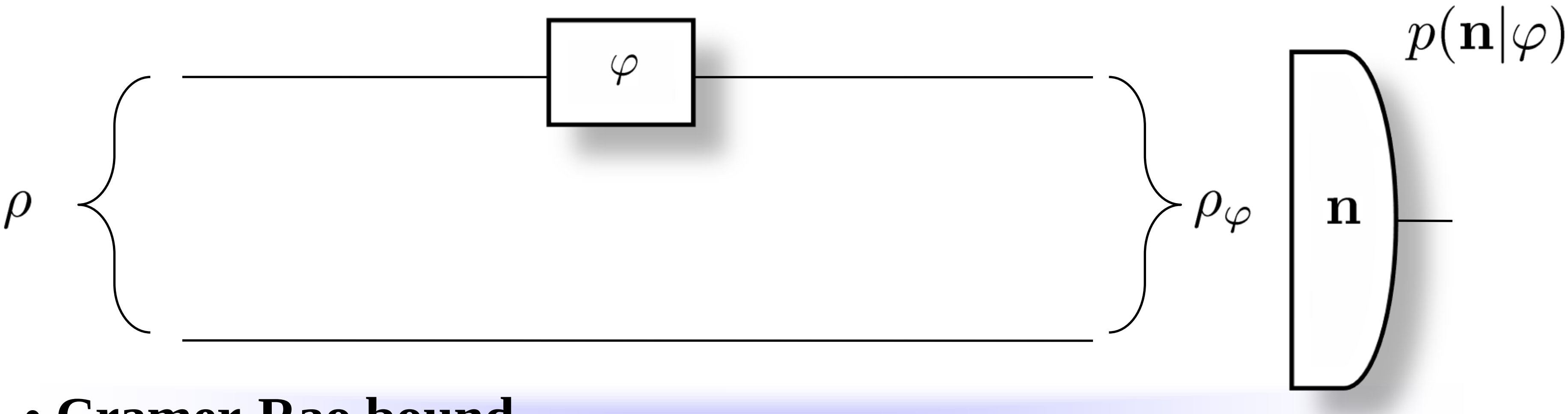
$$\Delta^2\tilde{\varphi} \propto \frac{1}{|\alpha|^2} = \frac{1}{\bar{n}}$$ – "classical" scaling

# What quantum states are optimal for interferometry?

**We would like to avoid looking for the optimal estimator and the optimal measurement**

# Fisher Information

## do not care about the estimator




- **Cramer-Rao bound**

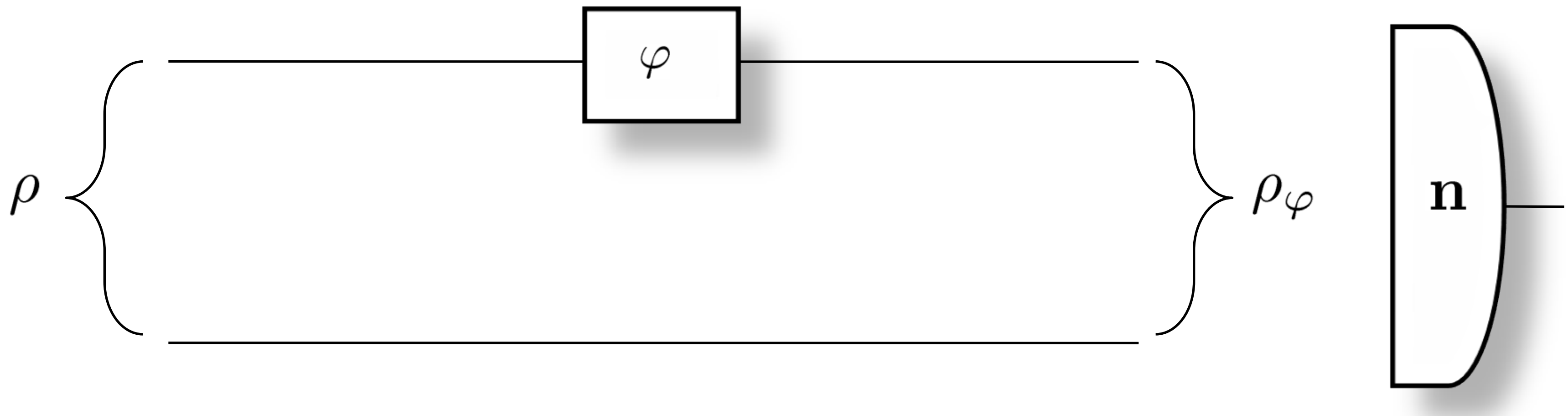
for any unbiased estimator $\tilde{\varphi}$ $\qquad \sum_{\mathbf{n}} \tilde{\varphi}(\mathbf{n}) p(\mathbf{n}|\varphi) = \varphi$

$$\boxed{\Delta^2 \tilde{\varphi} \geq \frac{1}{F}} \qquad F = \sum_{\mathbf{n}} \frac{1}{p(\mathbf{n}|\varphi)} \left( \frac{\partial p(\mathbf{n}|\varphi)}{\partial \varphi} \right)^2$$

if an experiment is repeated $k$ times: $\Delta^2 \tilde{\varphi} \geq \frac{1}{kF}$

# Quantum Fisher Information

**do not care about the measurement**

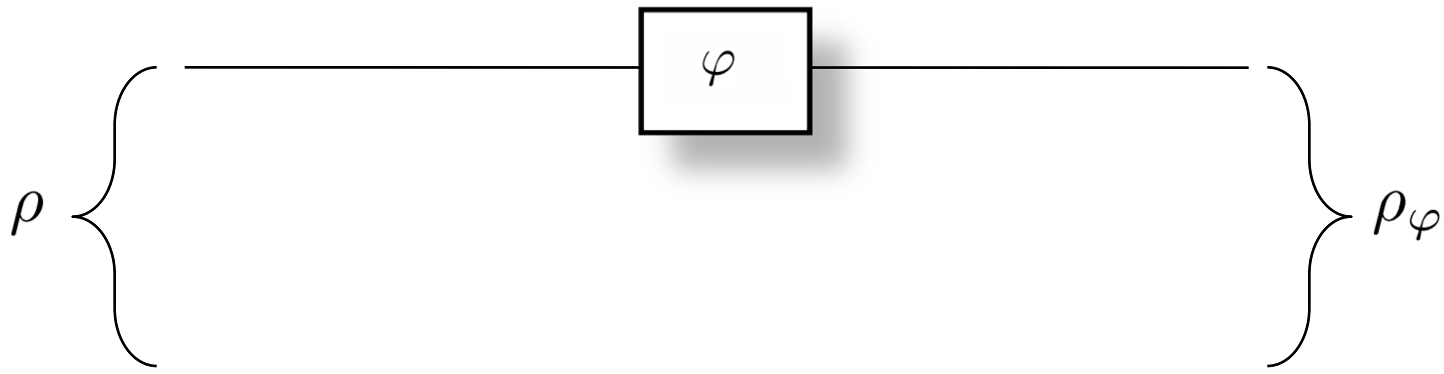



$$p(\mathbf{n}|\varphi) = \mathrm{Tr}(\hat{\Pi}_{\mathbf{n}}\rho_\varphi) \qquad \hat{\Pi}_{\mathbf{n}} \geq 0 \qquad \sum_{\mathbf{n}} \hat{\Pi}_{\mathbf{n}} = \mathbb{1}$$

$$F = \sum_{\mathbf{n}} \frac{1}{\mathrm{Tr}(\hat{\Pi}_{\mathbf{n}}\rho_\varphi)} \left[\frac{\partial}{\partial\varphi}\mathrm{Tr}(\hat{\Pi}_{\mathbf{n}}\rho_\varphi)\right]^2$$

# Quantum Fisher Information

## do not care about the measurement




- **Quantum Cramer-Rao bound**

$$\boxed{\Delta^2\tilde{\varphi} \geq \frac{1}{F_Q}}$$

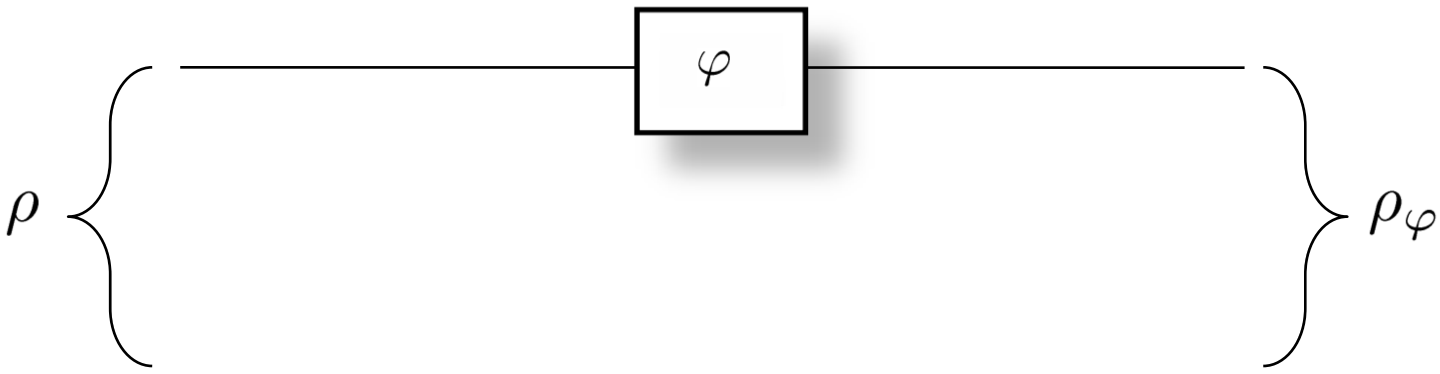
$$(\hat{A})_{ij} = \frac{2}{p_i + p_j}\left(\frac{\partial \rho_\varphi}{\partial \varphi}\right)_{ij}$$

written in $\rho_\varphi$ eigenbasis

$$F_Q = \mathrm{Tr}(\rho_\varphi \hat{A}^2)$$

# Quantum Fisher Information

## do not care about the measurement




- **Quantum Cramer-Rao bound for pure states**

$$\rho_\varphi = |\psi_\varphi\rangle\langle\psi_\varphi| \quad \hat{A} = 2(|\psi_\varphi\rangle\langle\psi'_\varphi| + |\psi'_\varphi\rangle\langle\psi_\varphi|) \qquad |\psi'_\varphi\rangle = \frac{\partial}{\partial\varphi}|\psi_\varphi\rangle$$

$$\boxed{\Delta^2\tilde{\varphi} \geq \frac{1}{F_Q}} \qquad \boxed{F_Q = 4(\langle\psi'_\varphi|\psi'_\varphi\rangle - |\langle\psi_\varphi|\psi'_\varphi\rangle|^2)}$$

# NOON states

$$\boxed{F_Q = 4(\langle\psi'_\varphi|\psi'_\varphi\rangle - |\langle\psi_\varphi|\psi'_\varphi\rangle|^2)}$$

$$|\psi'_\varphi\rangle = \frac{\partial}{\partial\varphi}|\psi_\varphi\rangle$$

**Single photon**

$$\frac{1}{\sqrt{2}}(e^{i\varphi}|10\rangle - |01\rangle)$$

**repeated *N* times:**

$$F_Q = N$$

**classical limit**

**NOON state:**

$$\frac{1}{\sqrt{2}}(e^{iN\phi}|N0\rangle - |0N\rangle)$$

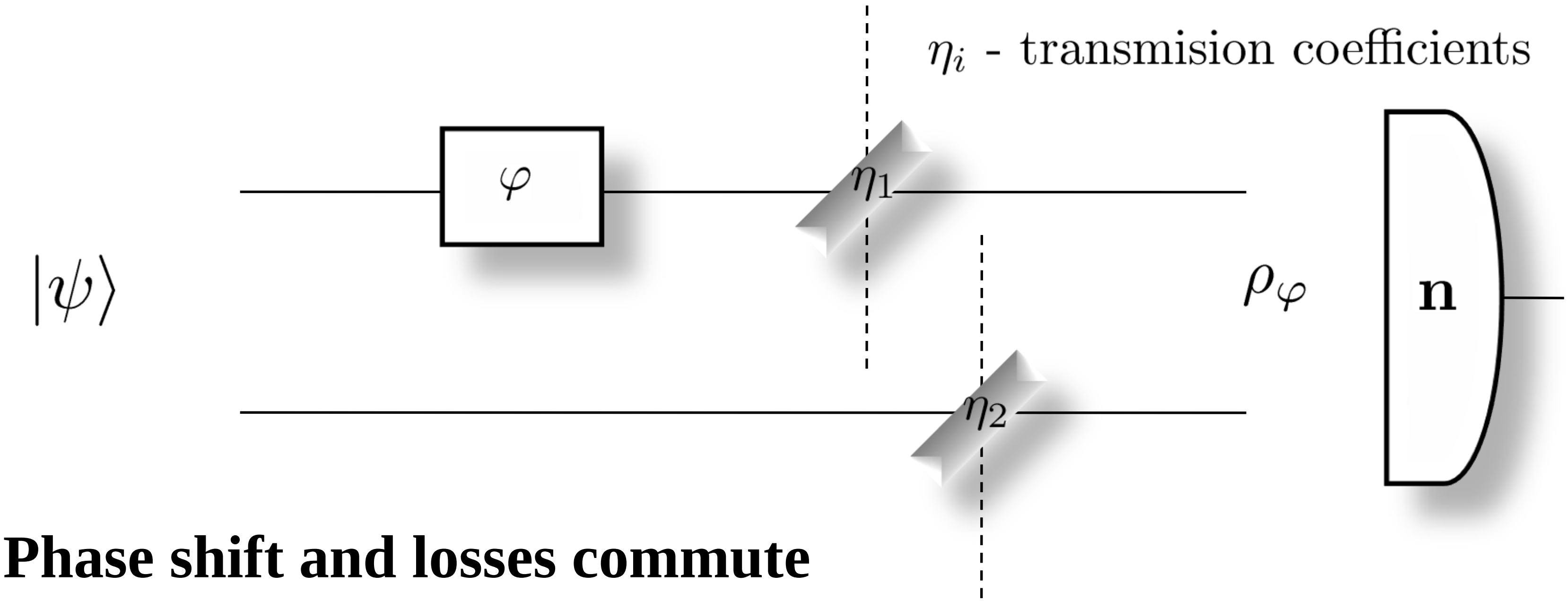
**gives:**

$$F_Q = N^2$$

**Heisenberg limit**

# What are the optimal states in the presence of losses?

# Interferometry with losses in one mode



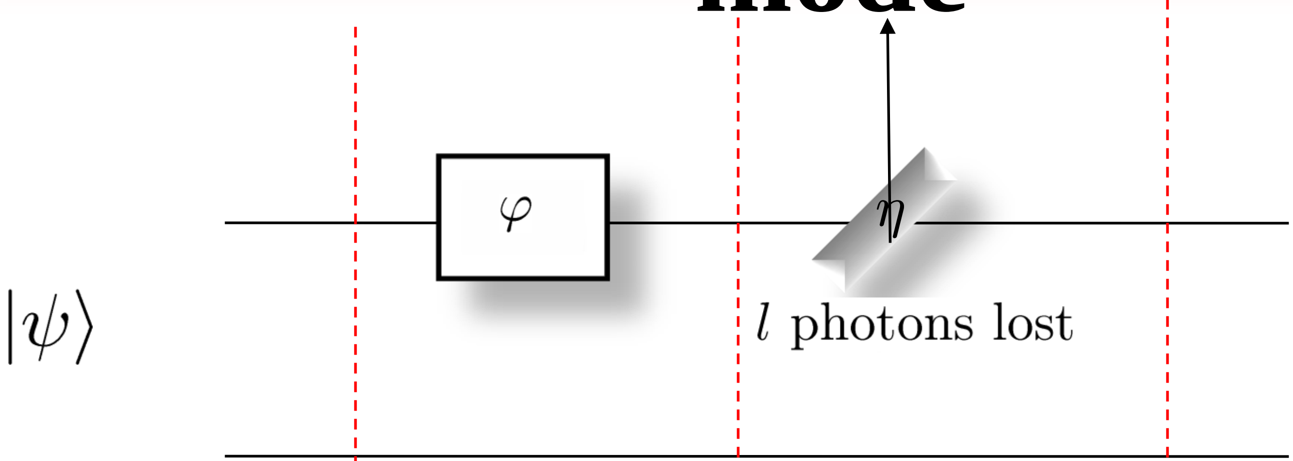

- **Phase shift and losses commute**

- **NOON states are extremely vulnerable to losses**

$$\hat{a}_1 \ \frac{1}{\sqrt{2}}(|N0\rangle + |0N\rangle) \propto \mathrm{e}^{\mathrm{i}N\phi}|N-1,0\rangle$$

- **What is the optimal *N* photon state?** $|\psi\rangle = \sum_{k=0}^{N} \alpha_k |k, N-k\rangle$

# Interferometry with losses in one mode



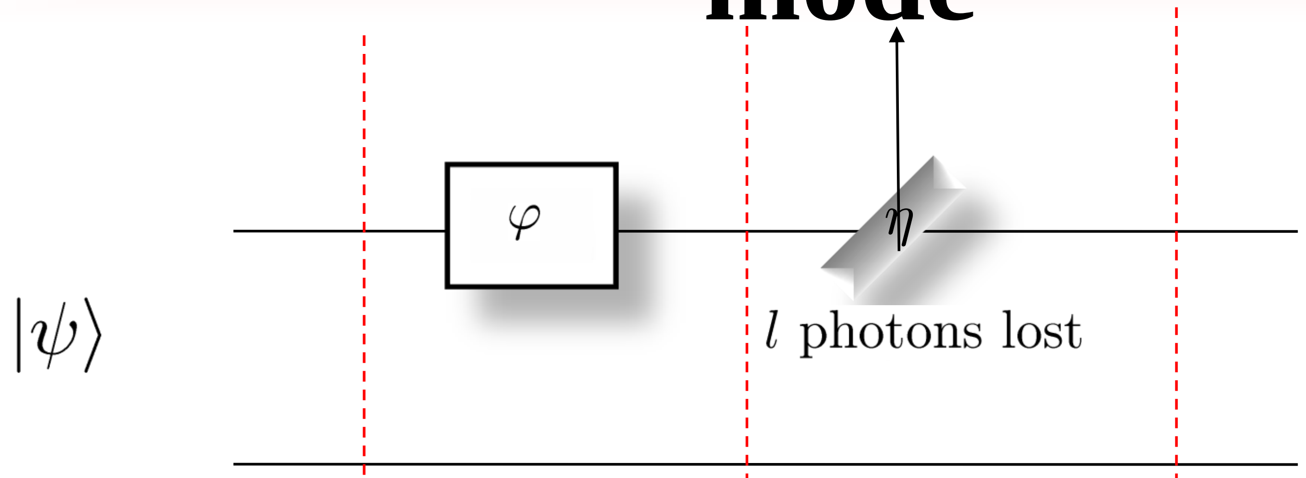

$$|\psi\rangle = \sum_{k=0}^{N} \alpha_k |k, N-k\rangle$$

$$\rho_\varphi = \sum_{l=0}^{N} p_l |\psi_l(\varphi)\rangle\langle\psi_l(\varphi)|$$

$$|\psi(\varphi)\rangle = \sum_{k=0}^{N} \mathrm{e}^{\mathrm{i}k\varphi} \alpha_k |k, N-k\rangle$$

$$|\psi_l(\varphi)\rangle = \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}k\varphi} \alpha_k \sqrt{B_l^k(\eta)} |k-l, N-k\rangle$$

# Interferometry with losses in one mode




$$|\psi\rangle = \sum_{k=0}^{N} \alpha_k |k, N-k\rangle \qquad \rho_\varphi = \sum_{l=0}^{N} p_l |\psi_l(\varphi)\rangle\langle\psi_l(\varphi)|$$

- **Subspaces with different *l* are orthogonal**

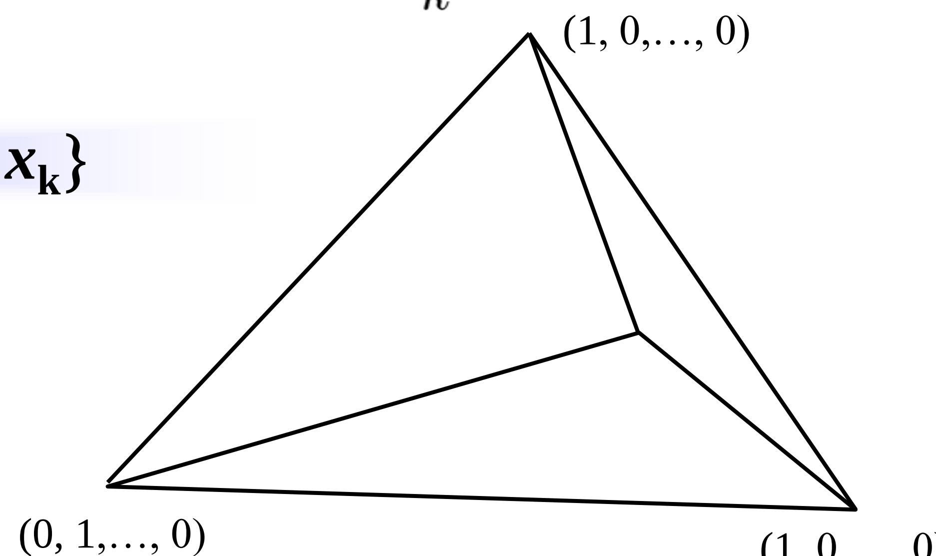
$$F_Q = \sum_{l=0}^{N} p_l F_l \qquad F_l = 4(\langle\psi_l'|\psi_l'\rangle - |\langle\psi_l|\psi_l'\rangle|^2)$$

- **For losses in both modes this is not the case**

# Interferometry with losses in one mode

$$\boxed{F_Q = 4\left(\sum_{k=0}^{N} k^2 x_k - \sum_{l=0}^{N} \frac{\left(\sum_{k=l}^{N} k x_k B_l^k(\eta)\right)^2}{\sum_{k=l}^{N} x_k B_l^k(\eta)}\right)}$$

$$x_k = |\alpha_k|^2, \qquad x_k \geq 0, \qquad \sum_k x_k = 1.$$

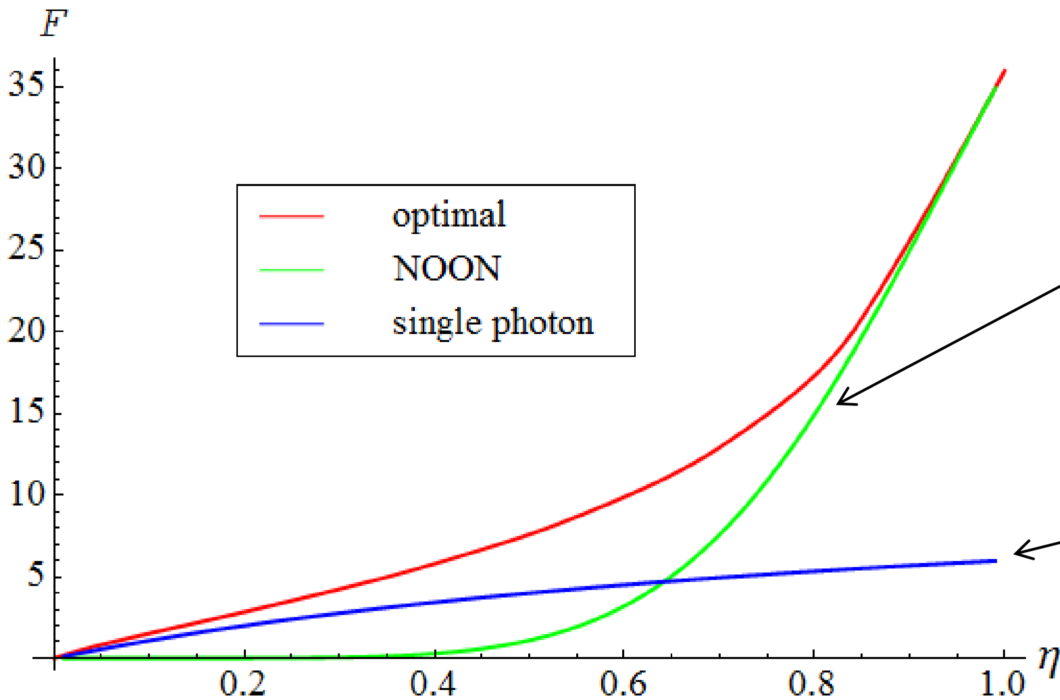



- **$F_Q$ is a concave function in $\{x_k\}$**

$$H_{ij} = \frac{\partial^2 F_Q}{\partial x_i \partial_j} \qquad H \leq 0$$

# Optimal *N*=6 photon state

- **Dependence of Fisher information on transmissivity**



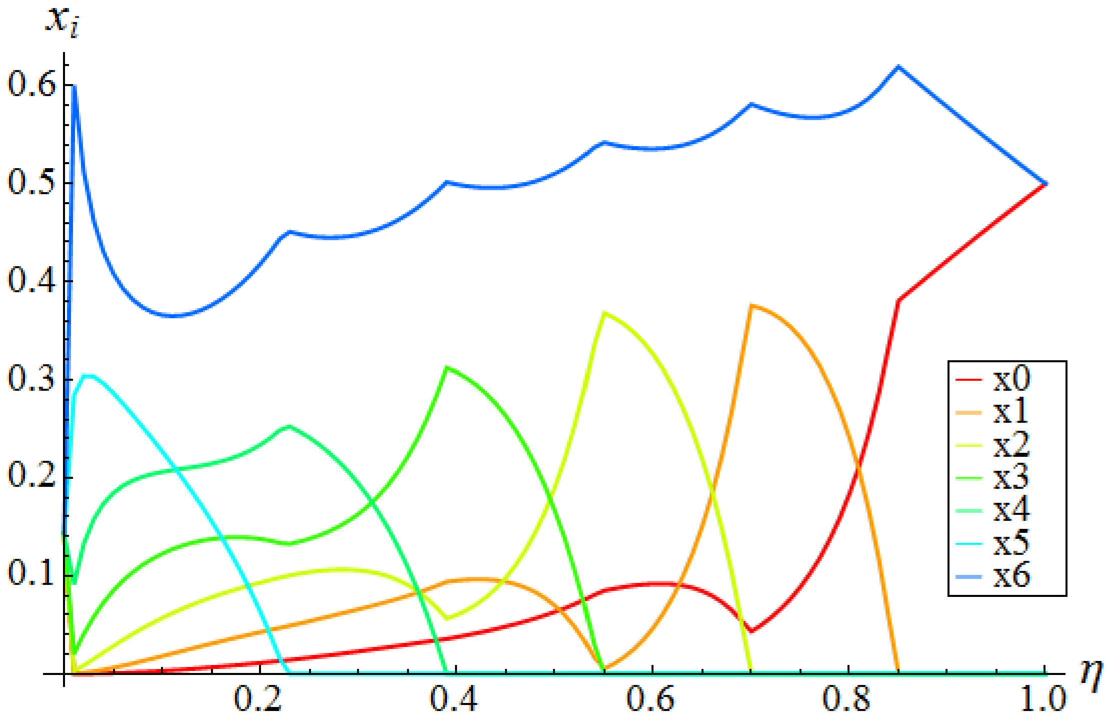

exponential decrease for $\eta < 1$

$$F_{NOON} = \frac{2N^2\eta^N}{1+\eta^N}$$

$$F_{single} = \frac{2N}{1+1/\eta}$$

# Optimal $N$=6 photon state

- **Structure of the optimal state**



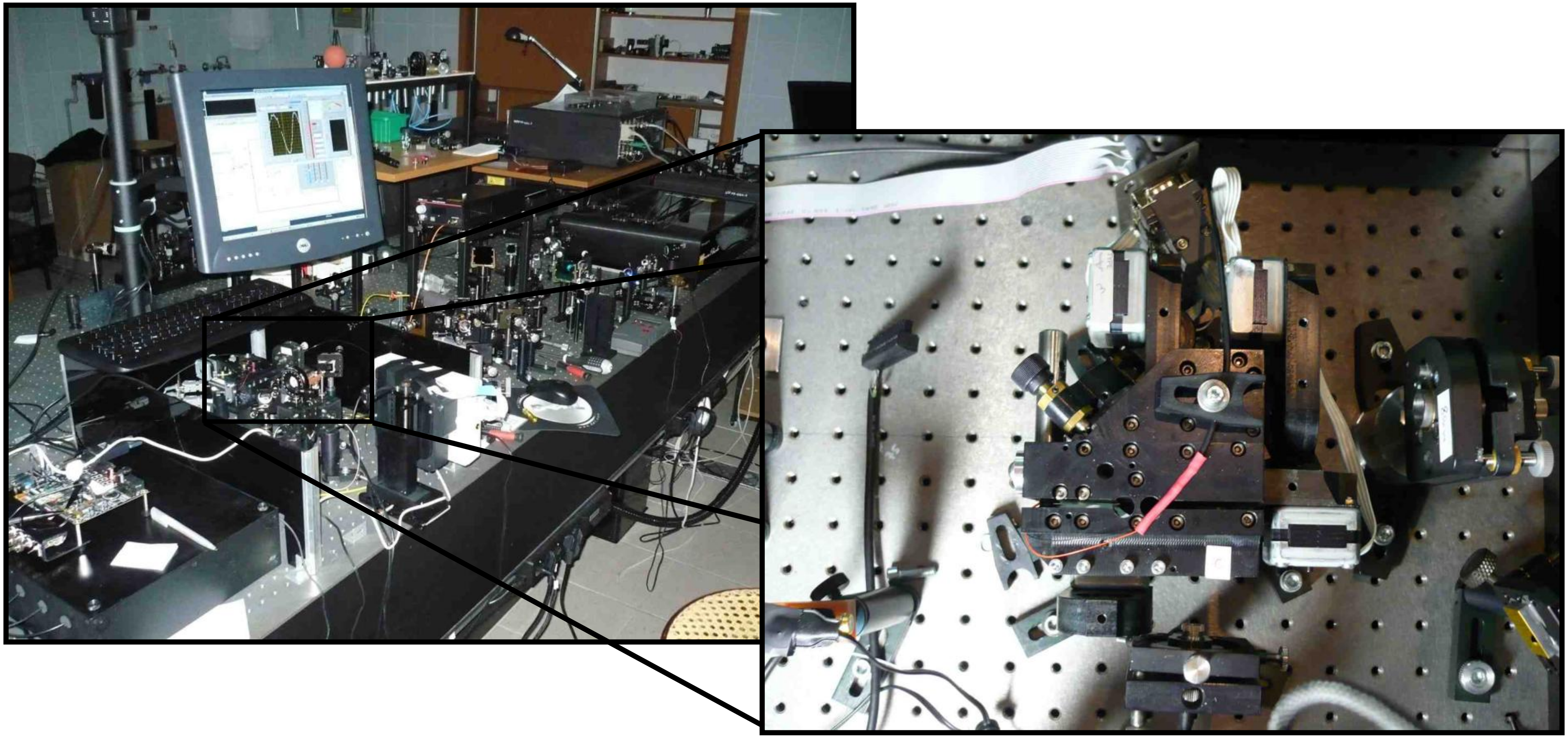

$$|\psi\rangle = \sum_{k=0}^{N} \sqrt{x_k}|k, N-k\rangle$$

- **A pretty good state**

$$|\psi\rangle = \sqrt{x_k}|k, N-k\rangle + \sqrt{x_N}|N, 0\rangle$$

# Experiment is under way...

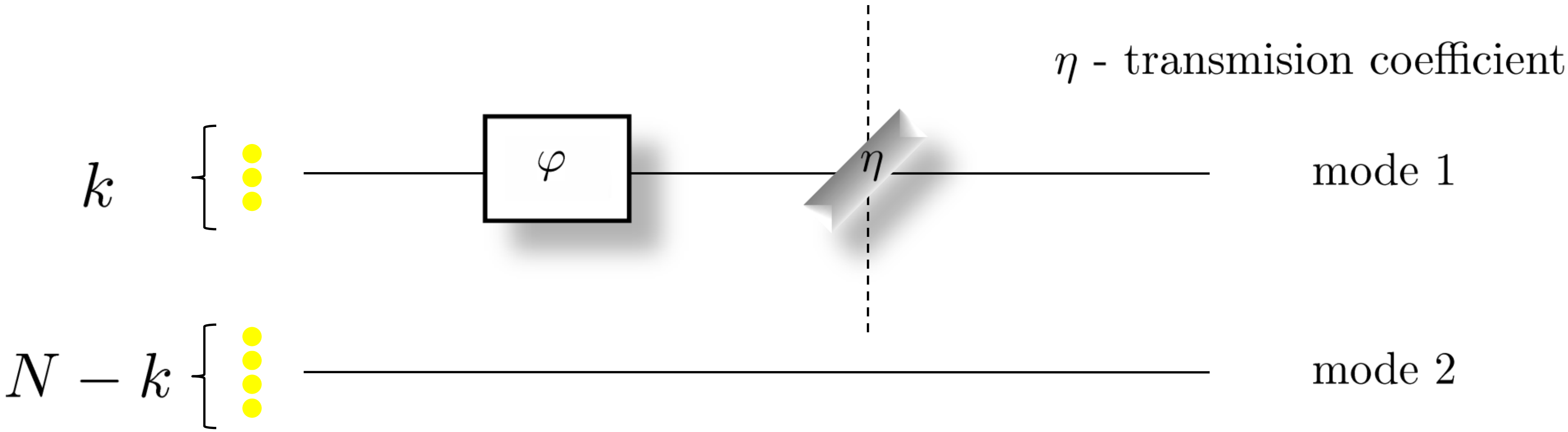
*Wojciech Wasilewski, Marcin Kacprowicz*



$$|\psi\rangle = \alpha_0|02\rangle + \alpha_1|11\rangle + \alpha_2|20\rangle$$

# Does using separate time bins help?

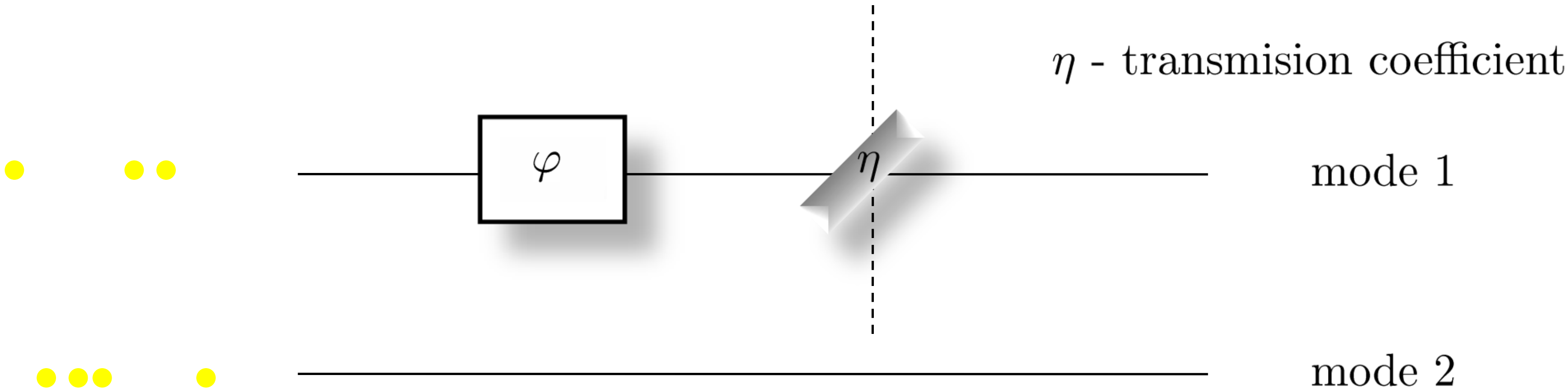
- **Instead of using *N* photons in common modes**




$$|\psi\rangle = \sum_{k=0}^{N} \alpha_k |k, N-k\rangle$$

# Does using separate time bins help?

- **We could send them in different times bins**




$$|1222112\rangle$$

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{2} \alpha_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle$$

**The optimal state is permutationally invariant – distinguishability of photons does not help!**

# Scaling with N

$$F \propto N, \quad \text{classical scaling} \qquad F \propto N^2, \quad \text{Heisenberg scaling}$$

- **Using optimally unbalanced NOON state**

$$|\psi\rangle = \sqrt{x_0}|0, N\rangle + \sqrt{x_N}|N, 0\rangle$$

$$F = \frac{4N^2\eta^N}{(1+\eta^{N/2})^2} \quad \xrightarrow{N\to\infty} \begin{cases} N^2 & \eta = 1 \\ 4N^2\eta^N \to 0 & \eta < 1 \end{cases}$$

# Scaling with N

$$F \propto N, \quad \text{classical scaling} \qquad F \propto N^2, \quad \text{Heisenberg scaling}$$

- **Using optimally chopped  NOON state**

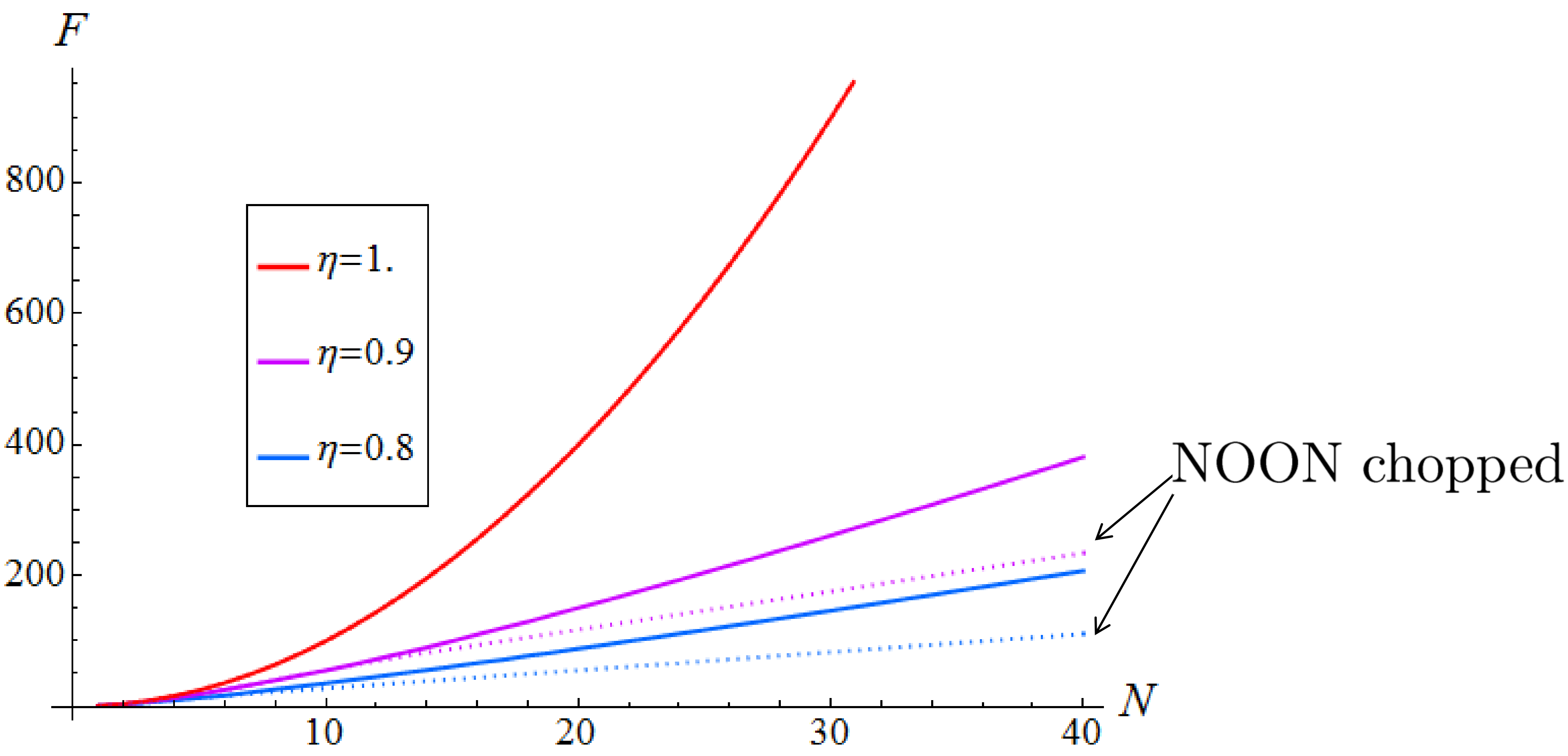
$$|\psi\rangle = (\sqrt{x_0}|0,n\rangle + \sqrt{x_n}|n,0\rangle)^{\otimes N/n}$$

$$F = \begin{cases} \dfrac{4N^2\eta^N}{(1+\eta^{N/2})^2} & \eta \geq \eta_0^{1/N} \\ \dfrac{4N\eta_0}{\ln(1/\eta)(1+\sqrt{\eta_0})} & \eta_0 \leq \eta \leq \eta_0^{1/N} \xrightarrow{N\to\infty} \begin{cases} N^2 & \eta = 1 \\ \propto N & \eta < 1 \end{cases} \\ \dfrac{4N\eta}{(1+\sqrt{\eta})^2} & \eta \leq \eta_0 \end{cases}$$

$$\eta_0 = 0.228169, \quad \text{solution of } 1 + \sqrt{\eta_0} + \ln \eta_0 = 0$$

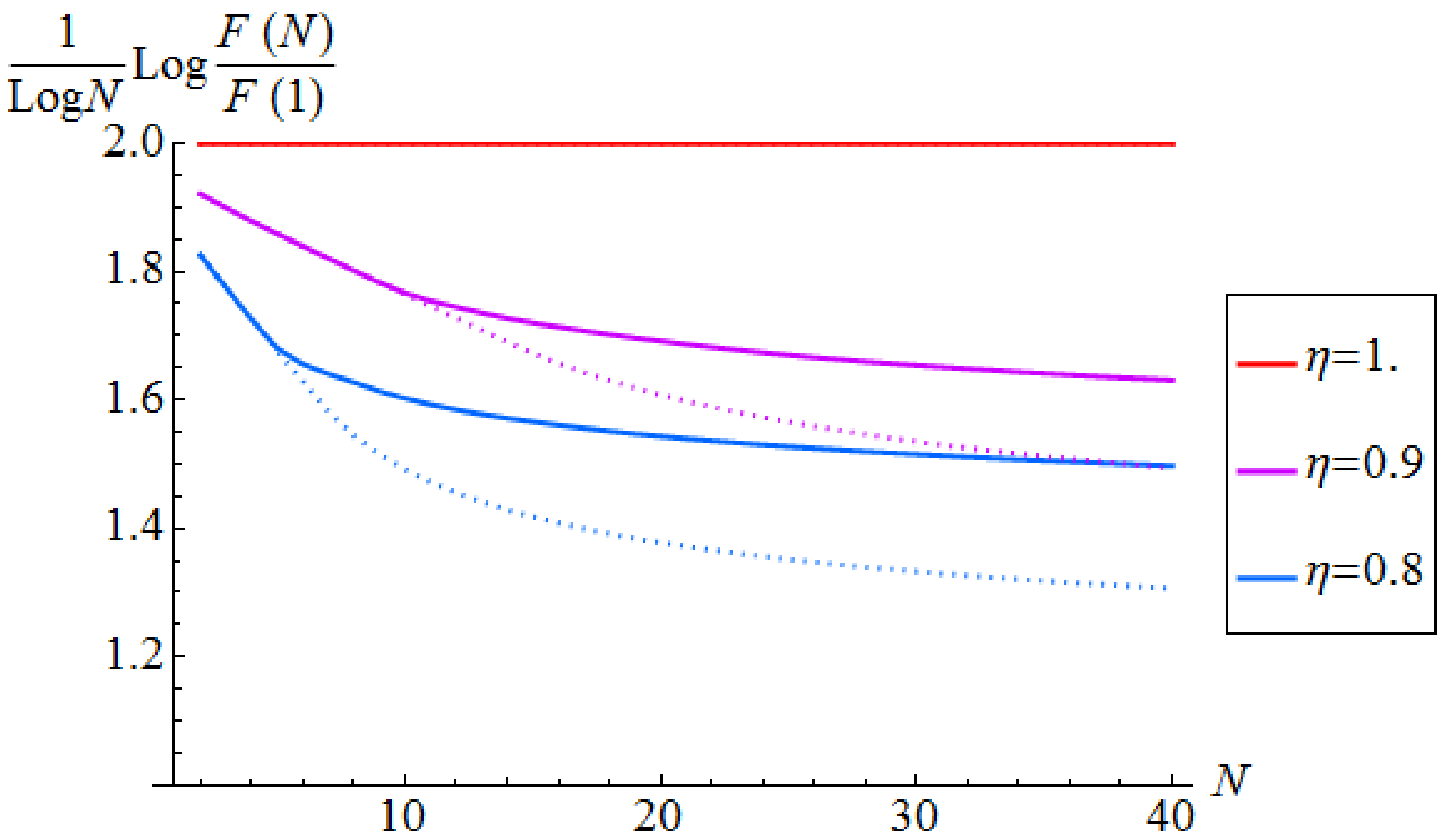
# Scaling with N

$F \propto N$, classical scaling $\qquad$ $F \propto N^2$, Heisenberg scaling



# Scaling with N

$F \propto N,\quad$ classical scaling $\qquad F \propto N^2,\quad$ Heisenberg scaling




- **Open problem: do we have the classical scaling for any $\eta < 1$ ?**

# Summary

- **NOON states are very vulnerable in the presence of losses**

- **Optimal states for interferometry with losses in one mode are well approximated by two component states of the form:**

$$|\psi\rangle = \sqrt{x_k}|k, N-k\rangle + \sqrt{x_N}|N, 0\rangle$$

- **Using multimode approach – each photon sent in a separate time bin can only lower Fisher Information**

**For $\eta < 1$, in the limit $N \to \infty$ do we always get the classical scaling $F \propto N$**