Effects of imperfect noise correlations on decoherence-free subsystems: SU(2) diffusion model

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Depolarizing channel

- **Random unitary rotation of a qubit:**

$$|\psi\rangle = \cos\theta|\leftrightarrow\rangle + \sin\theta e^{i\phi}|\updownarrow\rangle \qquad \mathcal{E}(|\psi\rangle\langle\psi|) = \int \mathrm{d}U\, U|\psi\rangle\langle\psi|U^\dagger = \mathbb{1}/2$$

- **In long fibers the output polarization of a photon is completely random**

Collectively depolarizing channel

- N qubit depolarizing channel, where each qubit experience the same disturbance

$$SU(2) \text{ Haar measure}$$

$$\mathcal{E}(\rho_N) = \int \mathrm{d}U \, U^{\otimes N} \rho_N U^{\dagger \otimes N}$$

N qubit state

- The model applies e.g. to:

    - photons transmitted through a long fiber
    - spins ½ being sent through a slowly varying magnetic field
    - communication in the absence of reference frames

Stucture of the output state

- **Irreducible subspaces under the action of** $U^{\otimes N}$:

$$\mathcal{H}^{\otimes N} = \bigoplus_{j=0}^{N/2} \underbrace{\mathcal{H}_j \oplus \ldots \oplus \mathcal{H}_j}_{d_j \ \text{times}} = \bigoplus_{j=0}^{N/2} \mathcal{H}_j \otimes \mathbb{C}_{d_j}$$

multiplicity subspace
(decoherence free subsystem)

$$\mathcal{T}(\rho) := \mathcal{E}(\rho) = \int \mathrm{d}U \, U^{\otimes N} \rho U^{\dagger \otimes N} = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j$$
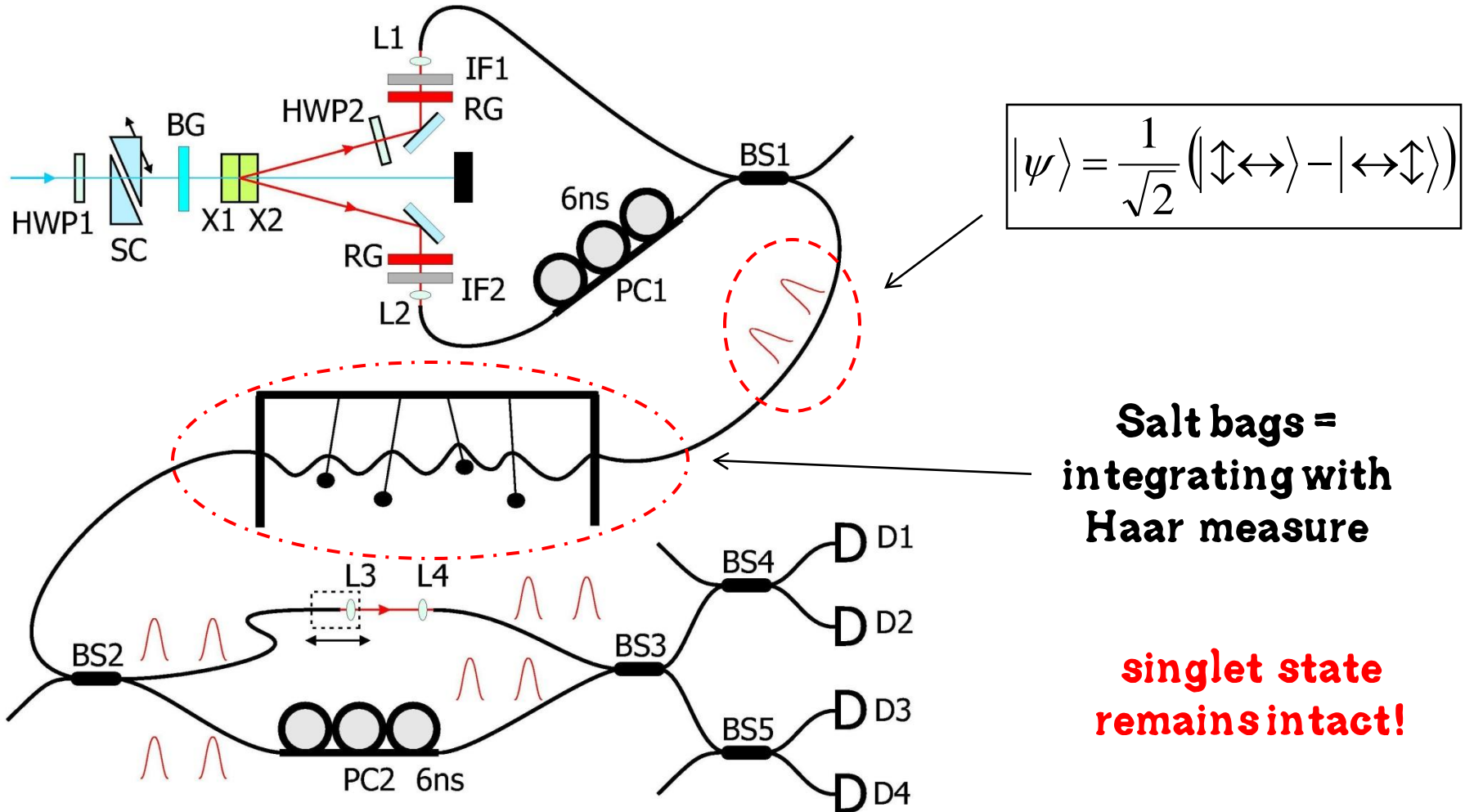
$$p_j = \mathrm{Tr}(P_j \rho) \qquad \rho_j = \frac{1}{p_j} \mathrm{Tr}_{\mathcal{H}_j}(P_j \rho P_j) \qquad P_j \text{ - projection on } \mathcal{H}_j \otimes \mathbb{C}_{d_j}$$

- **Faithfully transmitted states - allow for noiseless classical and quantum communication**

$$\rho = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j \qquad \text{twirled structure}$$

Photons and salt bags

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\updownarrow\leftrightarrow\right\rangle - \left|\leftrightarrow\updownarrow\right\rangle\right)$$

Salt bags =
integrating with
Haar measure

singlet state
remains intact!

What happens if noise is not perfectly correlated?

In other words:

what happens if oscilation time of salt bags is comparable with photon separation time

Imperfectly correlated noise model

- **Consecutive qubits experience slightly different rotations**

$$\mathcal{E}(\rho) = \int \mathrm{d}U_1 \ldots \mathrm{d}U_N \, p(U_1, \ldots, U_N) U_1 \otimes \cdots \otimes U_N \; \rho \; U_1^\dagger \cdots \otimes U_N^\dagger$$

- **The action described via a stationary Markov process**

$$p(U_1, \ldots, U_N) = p(U_N | U_{N-1}) \ldots p(U_2 | U_1)$$

What is the natural choice for conditional probability?

$$p(U_i | U_{i-1}) = ?$$

Diffusion on the SU(2) group

- **Isotropic diffusion on SU(2)**

$$\partial_t p(U;t) = \frac{1}{2} D \; \hat{\Delta} p(U;t)$$

Laplace operator on the SU(2) group

- **Solution, with the initial condition:** $p(U;0) = \delta(U - \mathbb{1})$

$$p(U;t) = \sum_{j=0}^{\infty} (2j+1) \exp\left(-\frac{1}{2}j(j+1) \, t\right) \sum_{m=-j}^{j} \mathfrak{D}^j(U)^m_m$$

diffusion strength   rotation matrices

- **Conditional probability**

$$\boxed{p(U_i|U_{i-1}) = p(U_i U_{i-1}^{\dagger}; t)}$$

$t \to 0$ perfect noise correlation $t \to \infty$ no correlation

Action of the channel

- **Probability distribution for unitaries**

$$p(U_1, \ldots, U_N) = p(U_N | U_{N-1}) \cdots p(U_2 | U_1) = p(U_N U_{N-1}^\dagger; t) \ldots p(U_2^\dagger U_1; t)$$

- **The channel action**

$$\mathcal{E}(\rho) = \int \mathrm{d}U_1 \ldots \int \mathrm{d}U_N \, p(U_1, \ldots, U_N) U_1 \otimes \cdots \otimes U_N \, \rho \, U_1^\dagger \otimes \cdots \otimes U_N^\dagger$$

$$\boxed{\mathcal{E}(\rho) = \mathcal{I}_{N-1} \left(\mathcal{I}_{N-2} \left(\ldots \mathcal{I}_1(\mathcal{T}(\rho)) \ldots \right) \right)}$$

$$\mathcal{T}(\rho) = \int \mathrm{d}U \, U^{\otimes N} \rho U^{\dagger \otimes N}$$

$$\mathcal{I}_i(\rho) = \int \mathrm{d}U \, p(U; t) \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i} \otimes \underbrace{U \otimes \cdots \otimes U}_{N-i} \, \rho \, \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i} \otimes \underbrace{U^\dagger \otimes \cdots \otimes U^\dagger}_{N-i}$$

Action of the channel

• **Output states have a twirled structure (** $\mathcal{T}, \mathcal{I}_i$ **commute)**

$$\mathcal{E}(\rho) = \mathcal{I}_{N-1}\left(\mathcal{I}_{N-2}\left(\ldots\mathcal{I}_1(\mathcal{T}(\rho))\ldots\right)\right)$$

• **Input states can be restricted to have the twirled structure, so the channel action can be described as**

$$\bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j \xrightarrow{\mathcal{E}} \bigoplus_{j=0}^{N/2} \frac{p'_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho'_j$$

• **Using properties of rotation matrices** $\mathfrak{D}^j(U)^m_m$**, it is possible to derive analytical expression for the action of the channel**

$$\mathcal{E}(\rho) = \text{lengthy expression involving } e^{-\frac{1}{2}j(j+1)t} \text{ and Wigner } 6j \text{ symbols}$$

Example: Three qubit channel

- **Structure of a three qubit twirled state**

$$\rho = \frac{p}{2}(\underbrace{\mathbb{1}_{\mathcal{H}_{1/2}} \otimes \rho_{1/2}}) \oplus \frac{1-p}{4}\mathbb{1}_{\mathcal{H}_{3/2}}$$

effectively two dimensional subspace

one dimensional subspace

$|0\rangle \quad (j_{12} = 0, j_{123} = 1/2)$
$|1\rangle \quad (j_{12} = 1, j_{123} = 1/2)$

$|2\rangle \quad (j_{12} = 1, j_{123} = 3/2)$

- **We have a qutrit channel, with no coherence between** $|0\rangle, |1\rangle$ **subspace and** $|2\rangle$

- **If correlations of noise were perfect (no diffusion), the channel would allow for** $\log_2 3$ **bits of classical communication and 1 qubit of quantum communication**

Fidelity of transmitting a qubit

- **Transmitting a qubit**

$$|\psi\rangle = \cos(\theta/2)|e_1\rangle + \sin(\theta/2)e^{i\phi}|e_2\rangle$$

$$|e_1\rangle = (|0\rangle + \sqrt{3}|1\rangle)/2$$
$$|e_2\rangle = (\sqrt{3}|0\rangle - |1\rangle)/2$$

substituting the output state $|2\rangle$, with a maximally mixed state of a qubit we can write the effective qubit channel in terms of evolution of the Bloch vector

$$\mathbf{r}_{\text{out}} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & \frac{2e^{-2t}+e^{-t}}{3} \end{pmatrix} \mathbf{r}_{\text{in}} + \begin{pmatrix} 0 \\ 0 \\ \frac{e^{-2t}-e^{-t}}{3} \end{pmatrix}$$

anisotropic shrinknig with displacement, states with $\phi = 0, \pi$ will tend to have high fidelity (weakest shrinking)

Fidelity of transmitting a qubit

for $t = 1$
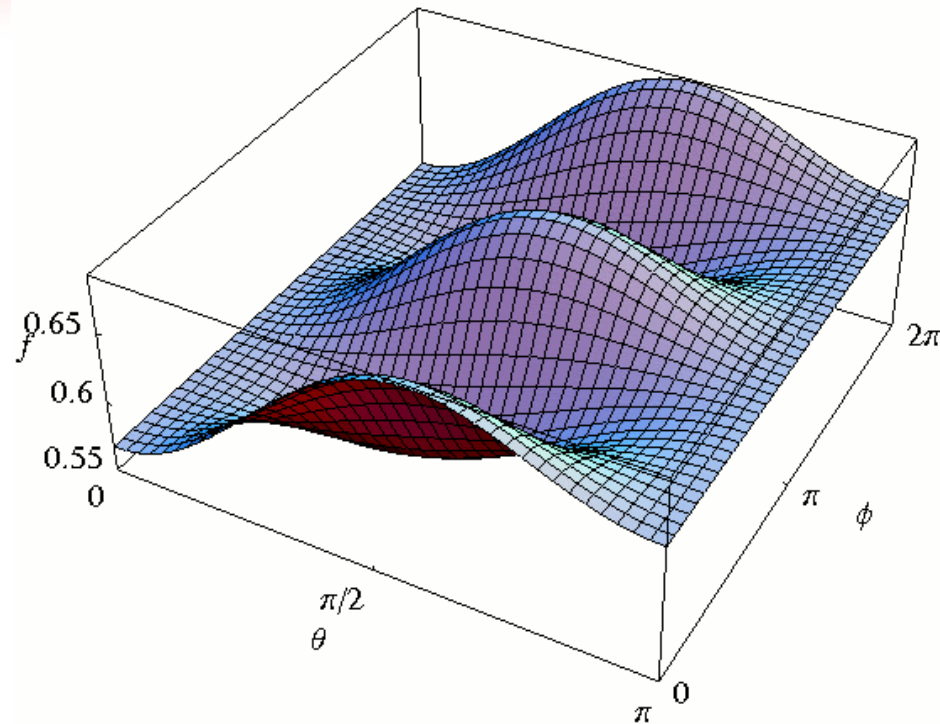
$$\mathbf{r}_{\text{out}} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & \frac{2e^{-2t}+e^{-t}}{3} \end{pmatrix} \mathbf{r}_{\text{in}} + \begin{pmatrix} 0 \\ 0 \\ \frac{e^{-2t}-e^{-t}}{3} \end{pmatrix}$$

anisotropic shrinknig with displacement, states with $\phi = 0, \pi$ will tend to have high fidelity (weakest shrinking)

Summary

• Introduction of a natural model of an N qubit channel with imperfectly correlated random unitary rotations acting on consecutive qubits

• Derivation of an analytic formula for the action of the channel on an arbitrary input state

• Detailed analysis of the case N=3
   - fidelity of the channel
   - optimal classical capacity, and corresponding states
   - orthogonal states that provide almost optimal classical capacity even for non perfect noise correlations
   - threshold of diffusion strength above which coherent information vanishes

• Future work:
   - develop a perturbative approach for weak dffusion for large number of qubits, find optimal capacities and corresponding states
   - analyse within this framework "estimate and correct" startegy for sending information