Effects of imperfect noise correlations on decoherence-free subsystems: SU(2) diffusion model

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joint work with

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Depolarizing channel

- **Random unitary rotation of a qubit:**

$$|\psi\rangle = \cos\theta|\leftrightarrow\rangle + \sin\theta e^{i\phi}|\updownarrow\rangle \qquad\qquad \mathcal{E}(|\psi\rangle\langle\psi|) = \int \mathrm{d}U\, U|\psi\rangle\langle\psi|U^\dagger = \mathbb{1}/2$$

- **In long fibers the output polarization of a photon is completely random**

Collectively depolarizing channel

- **N qubit depolarizing channel, where each qubit experience the same disturbance**

$SU(2)$ Haar measure

$$\mathcal{E}(\rho_N) = \int \mathrm{d}U U^{\otimes N} \rho_N U^{\dagger \otimes N}$$

N qubit state

- **The model applies e.g. to:**

  - photons transmitted through a long fiber
  - spins ½ being sent through a slowly varying magnetic field
  - communication in the absence of reference frames

Stucture of the output state

- Irreducible subspaces under the action of $U^{\otimes N}$:

$$\mathcal{H}^{\otimes N} = \bigoplus_{j=0}^{N/2} \underbrace{\mathcal{H}_j \oplus \ldots \oplus \mathcal{H}_j}_{d_j \text{ times}} = \bigoplus_{j=0}^{N/2} \mathcal{H}_j \otimes \mathbb{C}_{d_j}$$

multiplicity subspace
(decoherence free subsystem)

$$\boxed{\mathcal{T}(\rho) := \mathcal{E}(\rho) = \int \mathrm{d}U U^{\otimes N} \rho U^{\dagger \otimes N} = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j}$$

$$p_j = \mathrm{Tr}(P_j \rho) \qquad \rho_j = \frac{1}{p_j} \mathrm{Tr}_{\mathcal{H}_j}(P_j \rho P_j) \qquad P_j \text{ - projection on } \mathcal{H}_j \otimes \mathbb{C}_{d_j}$$

- Faithfully transmitted states - allow for noiseless classical and quantum communication

$$\rho = \bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j \qquad \text{twirled structure}$$

What happens if noise is not perfectly correlated?

Imperfectly correlated noise model

- Consecutive qubits experience slightly different rotations

$$\mathcal{E}(\rho) = \int \mathrm{d}U_1 \ldots \mathrm{d}U_N p(U_1, \ldots, U_N) U_1 \otimes \cdots \otimes U_N \; \rho \; U_1^\dagger \cdots \otimes U_N^\dagger$$

- The action described via a stationary Markov process

$$p(U_1, \ldots, U_N) = p(U_N | U_{N-1}) \ldots p(U_2 | U_1)$$

What is the natural choice for conditional probability?

$$p(U_i | U_{i-1}) = ?$$

Diffusion on the SU(2) group

- **Isotropic diffusion on SU(2)**

$$\partial_t p(U;t) = \frac{1}{2} D \, \hat{\Delta} p(U;t)$$

Laplace operator on the SU(2) group

- **Solution, with the initial condition:** $p(U;0) = \delta(U - \mathbb{1})$

$$p(U;t) = \sum_{j=0}^{\infty} (2j+1) \exp\left(-\frac{1}{2} j(j+1)\, t \right) \sum_{m=-j}^{j} \mathfrak{D}^{j}(U)^{m}_{m}$$

diffusion strength

rotation matrices

- **Conditional probability**

$$\boxed{p(U_i | U_{i-1}) = p(U_i U_{i-1}^{\dagger}; t)}$$

$t \to 0$ perfect noise correlation

$t \to \infty$ no correlation

Action of the channel

- **Probability distribution for unitaries**

$$p(U_1,\ldots,U_N) \;=\; p(U_N|U_{N-1})\cdots p(U_2|U_1) \;=\; p(U_N U_{N-1}^\dagger;t)\ldots p(U_2^\dagger U_1;t)$$

- **The channel action**

$$\mathcal{E}(\rho) = \int \mathrm{d}U_1 \ldots \int \mathrm{d}U_N p(U_1,\ldots,U_N) U_1\otimes\cdots\otimes U_N \;\rho\; U_1^\dagger\otimes\cdots\otimes U_N^\dagger$$

$$\boxed{\mathcal{E}(\rho) = \mathcal{I}_{N-1}\left(\mathcal{I}_{N-2}\left(\ldots\mathcal{I}_1(\mathcal{T}(\rho))\ldots\right)\right)}$$

$$\mathcal{T}(\rho) = \int \mathrm{d}U U^{\otimes N}\rho U^{\dagger\otimes N}$$

$$\mathcal{I}_i(\rho) = \int \mathrm{d}U p(U;t)\, \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{i}\otimes\underbrace{U\otimes\cdots\otimes U}_{N-i}\;\rho\;\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{i}\otimes\underbrace{U^\dagger\otimes\cdots\otimes U^\dagger}_{N-i}$$

Example: Three qubit channel

- **Structure of a three qubit twirled state**

$$\rho = \frac{p}{2}\underbrace{(\mathbb{1}_{\mathcal{H}_{1/2}} \otimes \rho_{1/2})} \oplus \frac{1-p}{4}\mathbb{1}_{\mathcal{H}_{3/2}}$$

effectively two dimensional subspace

one dimensional subspace

$|0\rangle \quad (j_{12} = 0, j_{123} = 1/2)$

$|1\rangle \quad (j_{12} = 1, j_{123} = 1/2)$

$|2\rangle \quad (j_{12} = 1, j_{123} = 3/2)$

- **We have a qutrit channel, with no coherence between $|0\rangle, |1\rangle$ subspace and $|2\rangle$**

- **If correlations of noise were perfect (no diffusion), the channel would allow for $\log_2 3$ bits of classical communication and 1 qubit of quantum communication**

Action of the channel

- Output states have a twirled structure ($\mathcal{T}, \mathcal{I}_i$ commute)

$$\mathcal{E}(\rho) = \mathcal{I}_{N-1}\left(\mathcal{I}_{N-2}\left(\ldots\mathcal{I}_1(\mathcal{T}(\rho))\ldots\right)\right)$$

- Input states can be restricted to have the twirled structure, so the channel action can be described as

$$\bigoplus_{j=0}^{N/2} \frac{p_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho_j \xrightarrow{\mathcal{E}} \bigoplus_{j=0}^{N/2} \frac{p'_j}{2j+1} \mathbb{1}_{\mathcal{H}_j} \otimes \rho'_j$$

- Using properties of rotation matrices $\mathfrak{D}^j(U)^m_{m'}$, it is possible to derive analytical expression for the action of the channel

$$\mathcal{E}(\rho) = \text{lengthy expression involving } e^{-\frac{1}{2}j(j+1)t} \text{ and Wigner } 6j \text{ symbols}$$

Example: Two qubit channel

- **Structure of a two qubit twirled state (Werner state)**

$$\rho = p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{3}P_1 = \tilde{p}|\psi^-\rangle\langle\psi^-| + \frac{1-\tilde{p}}{4}\mathbb{1}$$

singlet state

projection on the triplet subspace

- **The output state**

$$\mathcal{E}(\rho) = \mathcal{I}_1(\mathcal{T}(\rho)) = \int \mathrm{d}U p(U;t) \mathbb{1}\otimes U \;\rho\; \mathbb{1}\otimes U^\dagger$$

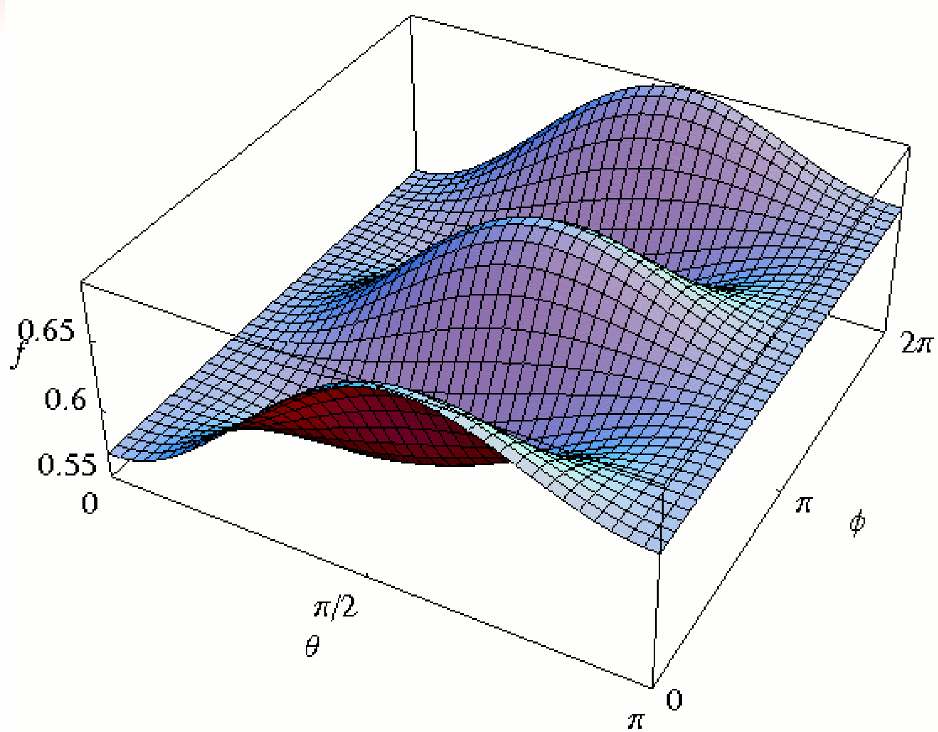
$$\mathcal{E}(\rho) = e^{-t}\tilde{p}|\psi^-\rangle\langle\psi^-| + \frac{1-e^{-t}\tilde{p}}{4}\mathbb{1}$$

quantum capacity $= 0$

classical capacity [Ball, Dragan, Banaszek, Phys. Rev. A, 69, 042324 (2004)]

Fidelity of transmitting a qubit

for $t = 1$



$$\mathbf{r}_{\text{out}} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & \frac{2e^{-2t}+e^{-t}}{3} \end{pmatrix} \mathbf{r}_{\text{in}} + \begin{pmatrix} 0 \\ 0 \\ \frac{e^{-2t}-e^{-t}}{3} \end{pmatrix}$$

anisotropic shrinknig with displacement, states with $\phi = 0, \pi$ will tend to have high fidelity (weakest shrinking)

Fidelity of transmitting a qubit

- **Transmitting a qubit**

$$|\psi\rangle = \cos(\theta/2)|e_1\rangle + \sin(\theta/2)e^{i\phi}|e_2\rangle$$

$$|e_1\rangle = (|0\rangle + \sqrt{3}|1\rangle)/2$$
$$|e_2\rangle = (\sqrt{3}|0\rangle - |1\rangle)/2$$

substituting the output state $|2\rangle$, with a maximally mixed state of a qubit we can write the effective qubit channel in terms of evolution of the Bloch vector

$$\mathbf{r}_{\text{out}} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & \frac{2e^{-2t}+e^{-t}}{3} \end{pmatrix} \mathbf{r}_{\text{in}} + \begin{pmatrix} 0 \\ 0 \\ \frac{e^{-2t}-e^{-t}}{3} \end{pmatrix}$$

anisotropic shrinknig with displacement, states with $\phi = 0, \pi$ will tend to have high fidelity (weakest shrinking)

Classical capacity

- **How many classical bits can be transmitted using 3 qubit states as letters**

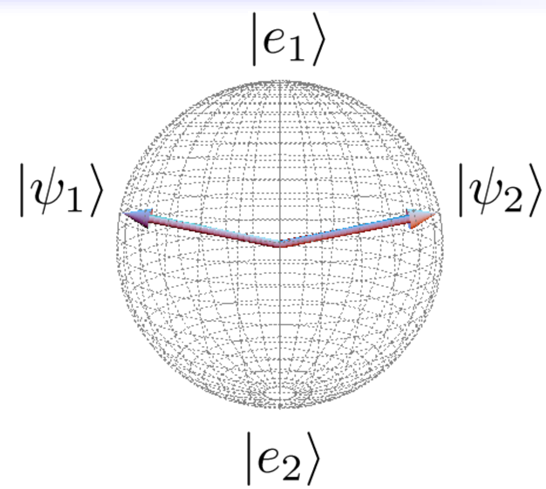
Holevo-Schumacher-Westmoreland formula:

$$C = \sup_{\{p_i,\rho_i\}} \left[S\left(\mathcal{E}\left(\sum_i p_i\rho_i\right)\right) - \sum_i p_i S(\mathcal{E}(\rho_i)) \right]$$

three qubit states   von Neumann entropy

- **States achieving optimal capacity**

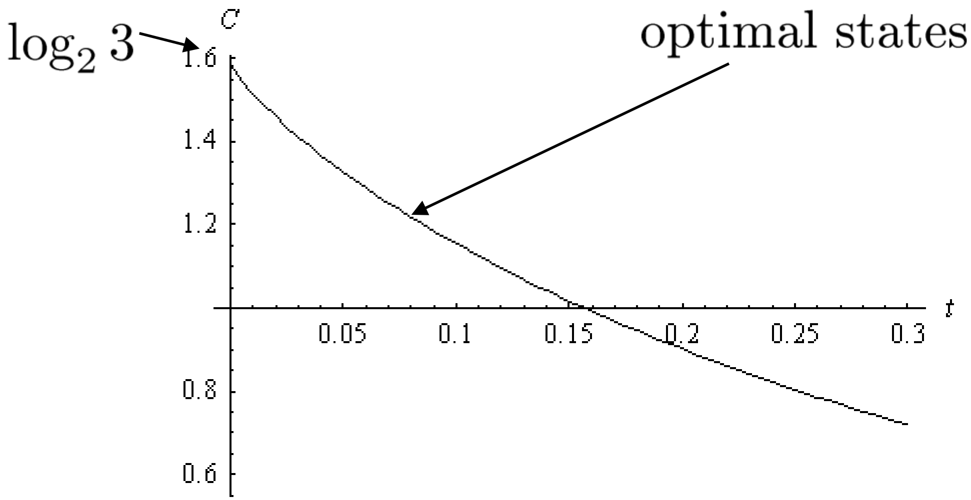
$$|\psi_1\rangle = \cos(\theta/2)|e_1\rangle + \sin(\theta/2)|e_2\rangle$$
$$|\psi_2\rangle = \cos(\theta/2)|e_1\rangle - \sin(\theta/2)|e_2\rangle$$
$$|\psi_3\rangle = |2\rangle$$



$|\psi_1\rangle, |\psi_2\rangle$ are not orthogonal!

Classical capacity

- **Optimal capacity**



- **States achieving optimal capacity**

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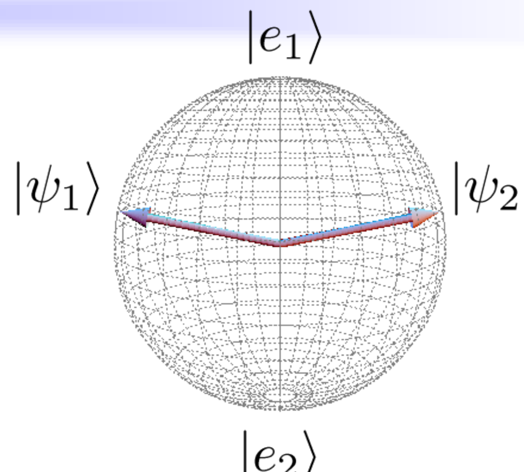
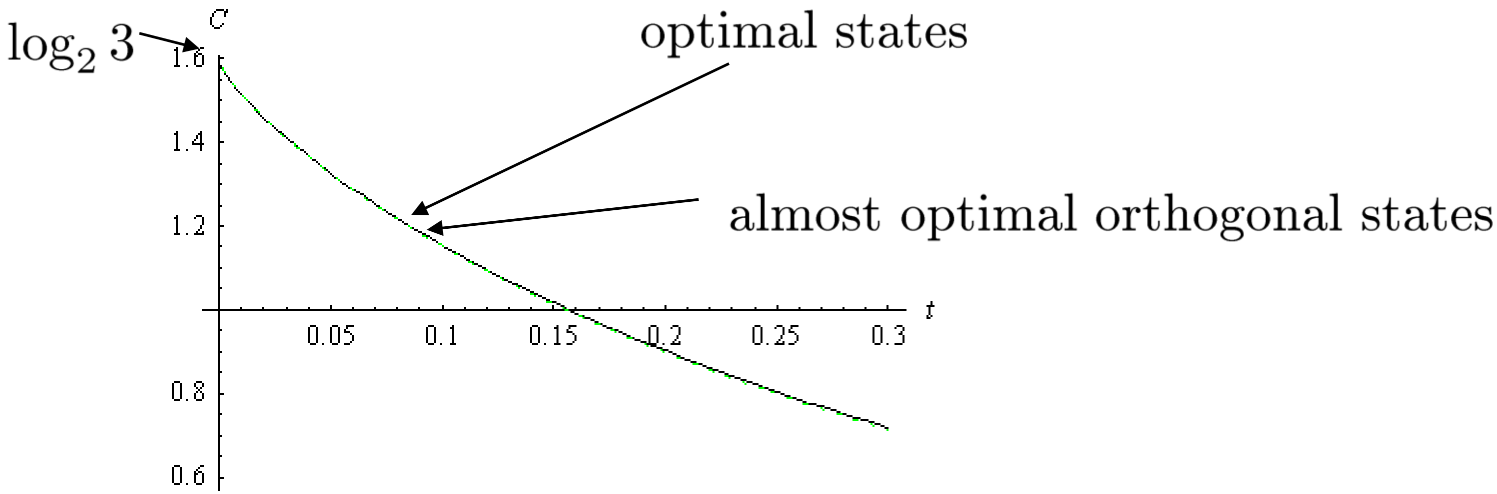


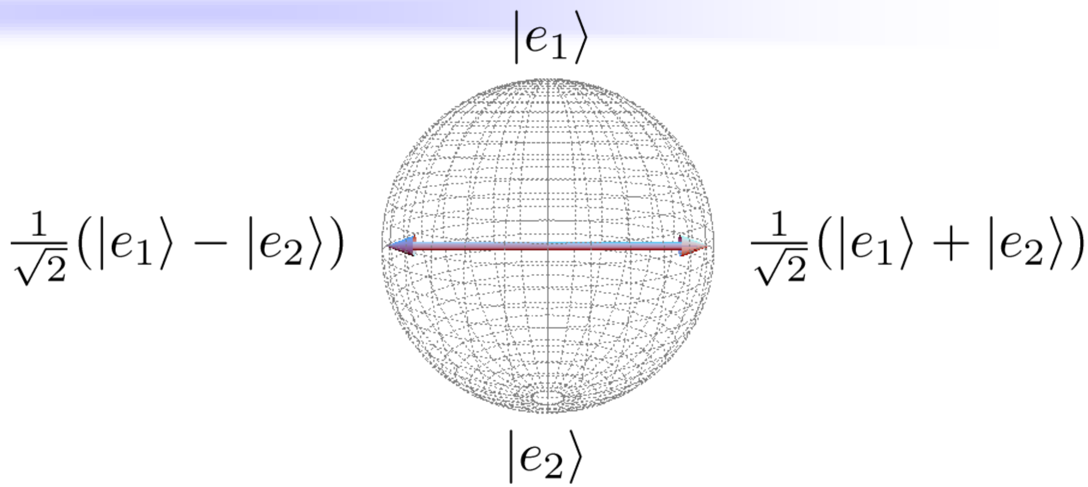
$|\psi_1\rangle, |\psi_2\rangle$ are not orthogonal!

Classical capacity

- **Optimal capacity**

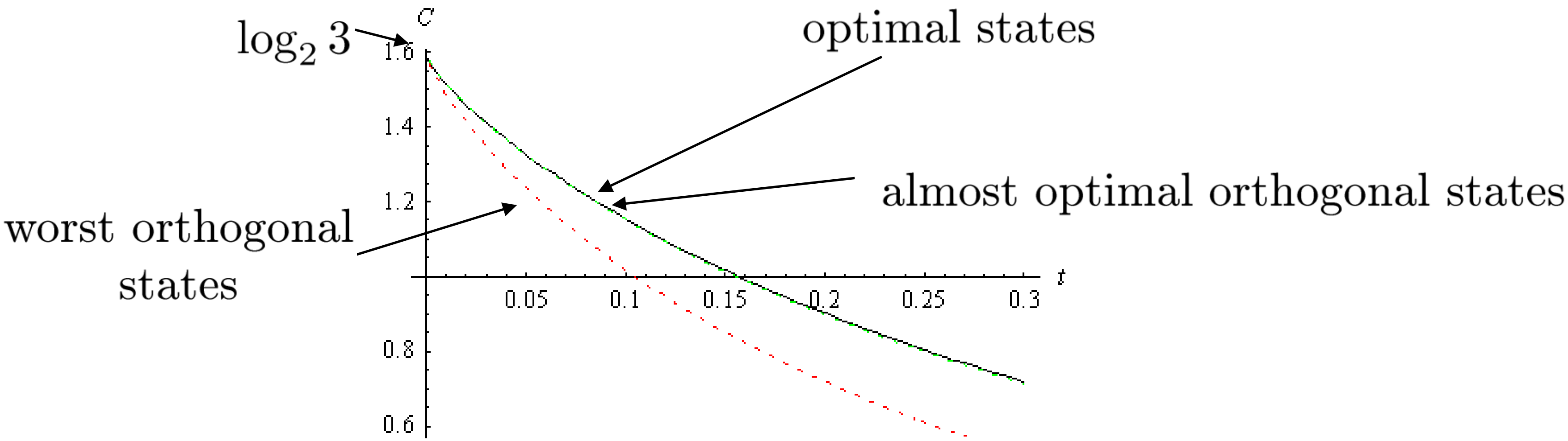


- **Optimal orthogonal states**

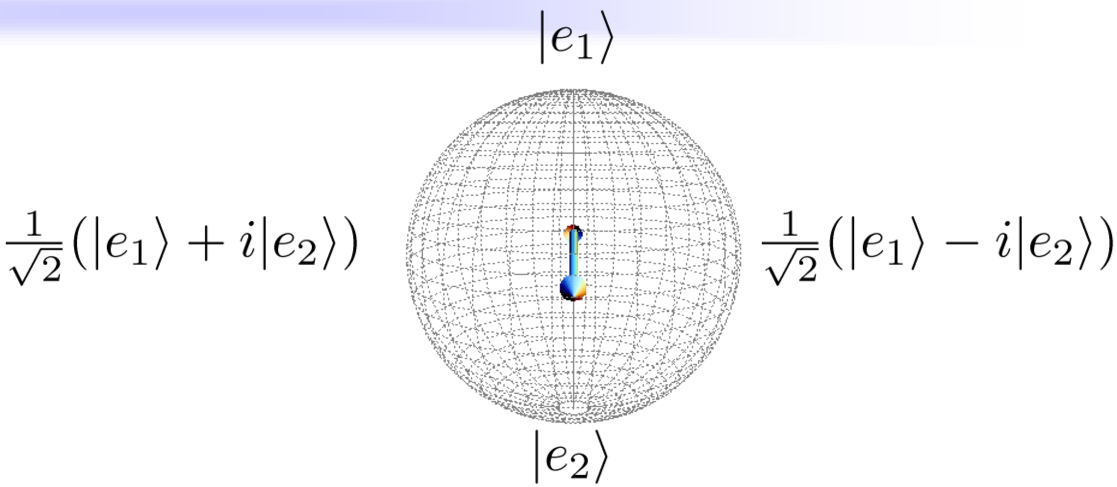


Classical capacity

- **Optimal capacity**



- **Worst orthogonal states**

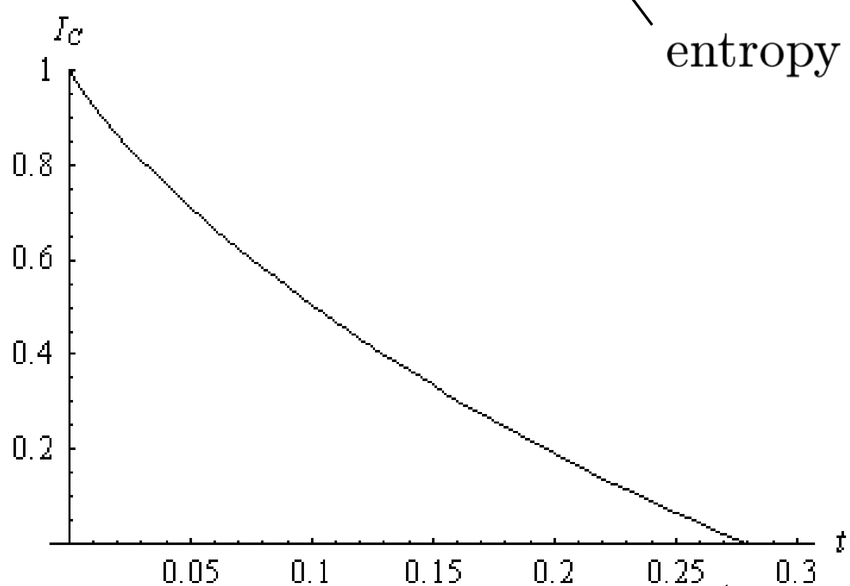


Coherent information

- To assess the quality of quantum information transmission, one can calculate the coherent information

$$I_C = \sup_{\rho} \left(S(\mathcal{E}(\rho)) - S_{\text{env}}(\mathcal{E}, \rho) \right)$$

entropy exchange



above this diffusion threshold $t > 0.275$
coherent information vanishes

Summary

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- Introduction of a natural model of an N qubit channel with imperfectly correlated random unitary rotations acting on consecutive qubits

- Derivation of an analytic formula for the action of the channel on an arbitrary input state

- Detailed analysis of the case N=3
  - fidelity of the channel
  - optimal classical capacity, and corresponding states
  - orthogonal states that provide almost optimal classical capacity even for non perfect noise correlations
  - threshold of diffusion strength above which coherent information vanishes

- Future work:
  - develop a perturbative approach for weak dffusion for large number of qubits, find optimal capacities and corresponding states
  - analyse within this framework "estimate and correct" startegy for sending information