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Estimate, Clone and Eavesdrop!

what you can do with a unknown quantum state

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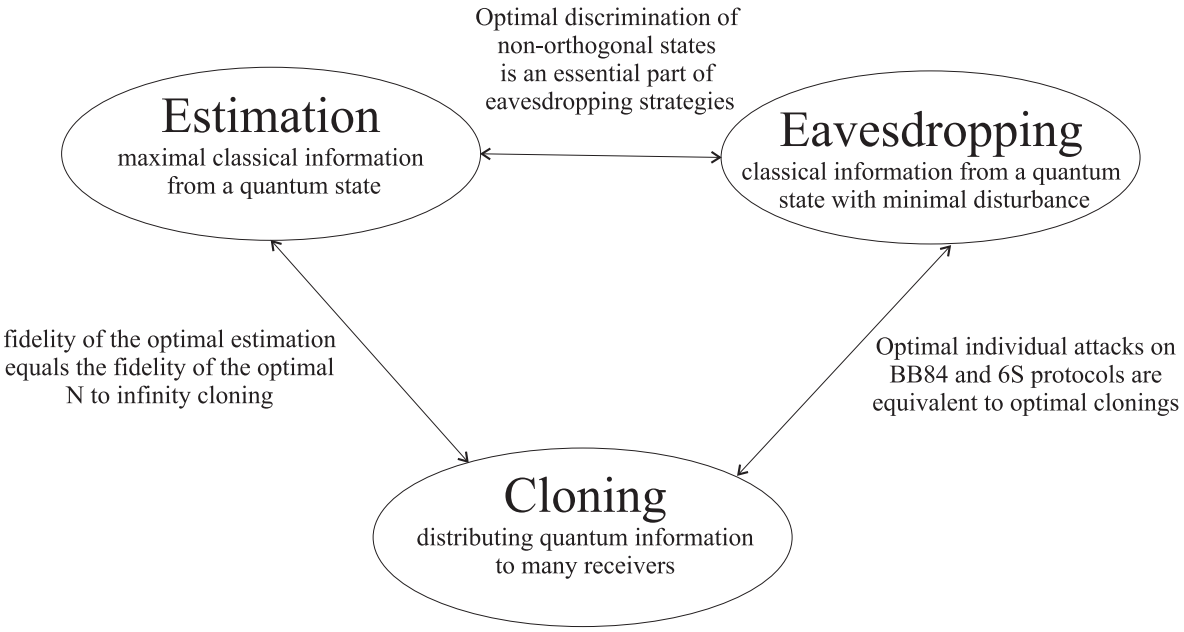


Figure 1: Interplay between cloning, estimation and eavesdropping

Chapter 1

Evolution and measurement in quantum mechanics - quantum channels.

1.1 An isolated quantum system

1.1.1 Pure states

Provided we have a full knowledge on the system we describe its quantum state as a normalized vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$ where global phase has no physical significance (thus it is equivalent to a ray) [1].

1.1.2 Evolution.

Evolution is described by a unitary operator: $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, $U(t)^\dagger U(t) = \mathbb{1}$.

1.1.3 Measurement.

Measurement is described by a set of orthogonal projectors $\{\Pi_i\}$ such that: $\sum_i \Pi_i = \mathbb{1}$, $\Pi_i\Pi_j = \delta_{i,j}\Pi_i$, $\Pi_i \geq 0$. Probability of a measurement result i:

$$p_i = \langle\psi|\Pi_i|\psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|\Pi_i) \tag{1.1}$$

1.1.4 Mixed states

If we only know that a system is being prepared in a state $|\psi_k\rangle$ with probability q_k then we can introduce a density matrix of a state $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ and the correct

[1] But we do not believe in "reality" of $|\psi\rangle$. It just describes our information. Consider an entangled pair, measurements on either particle, and if this is not enough for you move to a different inertial frame – cause and effect will be reversed. At least we do not believe today...

probability rule (i.e. the one which complies with the meaning of mixed state – it denotes pure states prepared with different probabilities) is:

$$p_i = \sum_k p_k \mathrm{Tr}(|\psi_k\rangle\langle\psi_k|\Pi_i) = \mathrm{Tr}(\rho\Pi_i). \tag{1.2}$$

Evolution of a mixed state of an isolated system is obviously $\rho(t) = U(t)\rho(0)U(t)^\dagger$. Set of quantum states is a convex set of density operators.

1.1.5 Why $\mathbf{Tr}(\rho\Pi_i)$ probability rule?

Gleason's theorem Let $\mathcal{P} = \Pi_i$ be a set of projectors acting on a real or complex Hilbert space $\mathcal{H}$, and let $f : \mathcal{P} \mapsto [0,1]$ (by writing such a function and treating it as probability assignment we assume non-contextuality of QM, it does not matter in which ensemble of projectors a given projector is, the probability only depends on it) be a function such that $\sum_i f(\Pi_i) = 1$ whenever $\{\Pi_i\}$ forms an observable. Then there exists an operator ρ such that $f(\Pi_i) = \mathrm{Tr}(\rho\Pi_i)$ (Strictly speaking whenever $\dim\mathcal{H} \geq 3$. For $\dim\mathcal{H} = 2$ it does not work). Amazingly we do not need to assume anything about f it in principle could be any strange function. The structure of observables forces it to be linear. For the proof see e.g. .

Counter example for $\dim\mathcal{H} = 2$ A measurement is either trivial (projection is an identity matrix) or is defined by giving one rank one projector which can be written as: $\Pi_{\vec{s}} = 1/2(\mathbb{1} + \vec{\sigma}\cdot\vec{s})$, where $\vec{s}$ is a 3D real unit vector, and $\vec{\sigma}$ is sigma Pauli matrices vector. Any non trivial measurement corresponds to a pair of projectors $\{\Pi_{\vec{s}}, \Pi_{-\vec{s}}\}$. For simplicity we can write $f(\vec{s})$ instead of $f(\Pi_{\vec{s}})$. The function f satisfies Gleason's theorem assumptions provided $f(\vec{s}) + f(-\vec{s}) = 1$.

For example we can choose $f(\vec{s}) = 1/2(1 + \vec{s}\cdot\vec{n})$ for some $\vec{n}$. This corresponds to a standard quantum mechanics where we have $f(\Pi_{\vec{s}}) = \mathrm{Tr}(\rho\Pi_{\vec{s}})$, with $\rho = 1/2(\mathbb{1} + \vec{\sigma}\cdot\vec{n})$. But we could also take f to be any function of the form $f(\vec{s}) = 1/2(1 + g(\vec{s}))$, where $g(\vec{s}) = -g(-\vec{s})$. For example we could take $g(\vec{s}) = (\vec{s}\cdot\vec{n})^3$ – or any other odd function in $\vec{n}$. Clearly this is not linear in $\vec{s}$ and hence in Π_s, and f cannot have the form $f(\Pi) = \mathrm{Tr}(\rho\Pi)$. As an extreme example we could choose f to be 1 on the northern hemisphere, 0 on the southern hemisphere and 1/2 on the equator. This function is neither linear nor even continuous.

1.2 Composite systems

Consider two quantum systems A and B described using Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ respectively. A state of combined system AB is described using a tensor product: $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $|e_i\rangle$, $|f_i\rangle$ be a basis in $\mathcal{H}_A$, $\mathcal{H}_B$ respectively. Then every state in $\mathcal{H}_{AB}$ can be written as

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij}|e_i\rangle \otimes |f_j\rangle. \tag{1.3}$$

A pure state which can be written as a product:

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\xi\rangle_B \tag{1.4}$$

is called a *product state*, and represents an uncorrelated state of two subsystems. A pure state which cannot be written as a product is called an *entangled state*.

1.2.1 Description of a subsystem

Consider a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, and let us assume that we have access only to one subsystem e.g. A. In particular this implies that we cannot perform nor have any information on results of measuring subsystem B.

Imagine that you perform a measurement $\{\Pi_i^A\}$ on subsystem A, while somebody else performs a measurement $\{\Pi_j^B\}$ on subsystem B. Probability of combined result (i, j) is given by:

$$p_{i,j} = \langle\psi|\Pi_i^A \otimes \Pi_j^B|\psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|\,\Pi_i^A \otimes \Pi_j^B). \tag{1.5}$$

Since you do not have access to subsystem B, what you effectively measure is the marginal probability distribution:

$$p_i = \sum_j \mathrm{Tr}(|\psi\rangle\langle\psi|\,\Pi_i^A \otimes \Pi_j^B) = \mathrm{Tr}(|\psi\rangle\langle\psi|\,\Pi_i^A \otimes \sum_j \Pi_j^B). \tag{1.6}$$

Making use of completeness property of a measurement $\sum_j \Pi_j^B = \mathbb{1}_B$ we get:

$$p_i = \mathrm{Tr}(|\psi\rangle\langle\psi|\ \Pi_i^A \otimes \mathbb{1}_B). \tag{1.7}$$

This can be equivalently written as:

$$p_i = \mathrm{Tr}(\rho_A \Pi_i^A), \tag{1.8}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ is called a reduced density matrix of the state $|\psi\rangle\langle\psi|$ to subsystem A. The Tr_B notion means a partial trace over indices corresponding to B subsystem.

In general, any operator X_{AB} acting on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, can be written in a basis:

$$X_{AB} = \sum_{i,i',j,j'} (X_{AB})_{i,j}^{i',j'} |e_{i'}\rangle\langle e_i| \otimes |f_{j'}\rangle\langle f_j|. \tag{1.9}$$

We define a partial trace over subsystem B as follows:

$$X_A = \mathrm{Tr}_B(X_{AB}) \Leftrightarrow (X_A)_i^{i'} = \sum_j (X_{AB})_{i,j}^{i',j} \tag{1.10}$$

Partial trace over subsystem A is defined analogously.

As seen from Eq. (1.8) all information available for an observer restricted to performing measurements on subsystem A, is described by the reduced density matrix ρ_A.

Notice that this is a a different way in which mixed states appear in quantum mechanics. In Sec. 1.1.4 mixed states were introduced in order to describe the effect of random preparation of different pure states. Here we state that mixed state description naturally arises when we restrict ourselves to a part of a larger quantum system.

If the state of the composite system is itself mixed ρ_{AB}, the reduced density matrix of subsystem A is calculated in the same way and reads: $\rho_A = \mathrm{Tr}_B(\rho_{AB})$.

1.3 Interaction with the environment

1.3.1 Generalized measurement

Consider system + environment and let the initial state be $\rho^{SE} = \rho^S \otimes \rho^E$, i.e. the system is decoupled from the environment. Let ρ^{SE} evolve under a unitary evolution (which in general will couple the system with the environment), after which a standard projection measurement is performed on the environment part (this is no loss of generality, if we also insited on performing measurement on the system part, we can equivalently perform a swapping operation between the system and part of environment space before the measurement, and measure only the environment), and can be denoted as $\{\mathbb{1}^S \otimes \Pi_i^E\}$. The probability of obtaining the result i reads:

$$p_i = \mathrm{Tr}\left(U\rho^S \otimes \rho^E U^\dagger \ \mathbb{1}^S \otimes \Pi_i^E\right) = \mathrm{Tr}_S(\rho_S M_i) \tag{1.11}$$

where

$$M_i = \mathrm{Tr}_E(\mathbb{1}^S \otimes \rho^E \ U^\dagger \ \mathbb{1}^S \otimes \Pi_i^E U). \tag{1.12}$$

The set $\{M_i\}$ describes a generalized measurement and is referred to as POVM (Positive Operator Valued Measure). Hence, every generalized measurement (i.e. the one where we allow interaction with environment and perform standard measurement on the environment) can be described by a set $\{M_i\}$ of positive operators $M_i \geq 0$, which sum up to one $\sum_i M_i = \mathbb{1}$. Observe that they are not necessarily orthogonal nor have to be projectors.

Conversely for every set $\{M_i\}$ such that $M_i \geq 0$, $\sum_i M_i = \mathbb{1}$, there exist ρ^E, U and Π_i^E such that Eq. (1.12) is satisfied (A consequence of Steinspring's dilation theorem). Hence there is a one to one correspondence between generalized measurements and POVMs.

Is it really more natural to postulate standard projection measurement then POVM measurement? (Christopher Fuchs) Judge by yourself:

Standard measurement	Generalized measurement
$\{\Pi_i\}$	$\{M_i\}$
$\Pi_i \geq 0$	$M_i \geq 0$
$\sum_i \Pi_i = \mathbb{1}$	$\sum_i M_i = \mathbb{1}$
$p_i = \mathrm{Tr}(\rho\Pi_i)$	$p_i = \mathrm{Tr}(\rho M_i)$
$\Pi_i\Pi_j = \delta_{i,j}\Pi_i$	–

What about POVM version of Gleason's theorem Consider a function $f : \mathcal{M} \mapsto [0,1]$ (non-contextuality), where $\mathcal{M}$ denote a set of positive operators $0 \leq M_i \leq \mathbb{1}$. Assuming that $\sum_i f(M_i) = 1$ whenever $\sum_i M_i = \mathbb{1}$ there exist ρ such that $f(M_i) = \mathrm{Tr}(\rho M_i)$. No problem with $\dim\mathcal{H} = 2$ here and the proof is much simpler.

1.3.2 CP maps

A general evolution of a system + environment (initially decoupled, without loosing generality we can assume the enivironment is initially in a pure state $|0\rangle$) will result in an effective evolution of the system in the form

$$\rho^{S\prime} = \mathrm{Tr}_E(U\ \rho^S \otimes |0\rangle\langle 0|U^\dagger) = \sum_i \langle e_i|U\ \rho^S \otimes |0\rangle\langle 0|U^\dagger|e_i\rangle = \sum_i K_i \rho^S K_i^\dagger \tag{1.13}$$

where

$$K_i = \langle e_i|U|0\rangle \tag{1.14}$$

Trace preservation condition implies that $\sum K_i^\dagger K_i = \mathbb{1}_S$.

Conversely any transformation of the form:

$$\rho^{S\prime} = \sum_i K_i \rho_S K_i^\dagger \tag{1.15}$$

where $\sum K_i^\dagger K_i = \mathbb{1}$ can be realized via extension + unitary + partial trace (Stinespring theorem). Any map which can be written in the form (1.15) is a Completely Positive map. A map $\mathcal{E}$ is called CP if it is positive and when extended to bigger system using $\mathcal{E} \otimes \mathcal{I}$ remains positive. Moreover every CP map (not necessarily trace preserving, we simply do not have condition $\sum K_i^\dagger K_i = \mathbb{1}$) can be written in the form (1.15)

1.4 Linearity of quantum mechanics

Standard quantum mechanics is linear by which we usually mean that the evolution of the wave function is linear (even unitary):

$$\mathcal{E}(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha\mathcal{E}(|\psi_1\rangle) + \beta\mathcal{E}(|\psi_2\rangle) \tag{1.16}$$

it can be written as $|\psi'\rangle = A|\psi\rangle$. We introduce the density matrix $\rho = |\psi_i\rangle\langle\psi_i|$, its evolution is obviously also linear: $\rho' = A|\psi\rangle\langle\psi|A^\dagger = A\rho A^\dagger$. But there is something more in it!. We have also convex combinations of pure density matrices why there should also be linearity. Because of probabilistic interpretation of density matrix.

Why the evolution of density matrix must be linear? Evolution of probability distributions is linear by definition of probability! Consider a set of events $\{1, \ldots, n\}$ - can represent for example position of a ball . Dynamics of the ball is described by a conditional probability $q(j|i)$ of given at the initial time the ball was at the position i

it will finally end at position j. Now let $p_{\text{in}}(i)$ be the initial probability for the ball to be found at position i. What is the output probability? It is simply the rule for the total probability

$$p_{\text{out}}(j) = \sum_i q(j|i)p_{\text{in}}(i). \tag{1.17}$$

Or written concisely $p_{\text{out}} = \mathcal{E}(p_{\text{in}})$, where $\mathcal{E}$ is obviously linear. Simply put: since $sp_{1\text{in}} + (1-s)p_{2\text{in}}$ (for $0 \leq s \leq 1$) simply denotes a probability distribution which corresponds to a situation in which with probability s we were given $p_{1\text{in}}$ and with probability $(1-s)$ we were given $p_{2\text{in}}$ the output probability distribution $\mathcal{E}(sp_{1\text{in}}+(1-s)p_{2\text{in}})$ must be equal to the probability distribution of a situation in which with probability s we obtained $\mathcal{E}(p_{1\text{in}}$ while with probability $(1-s)$ we obtained $\mathcal{E}(p_{2\text{in}}$. Hence $\mathcal{E}$ must be linear. Even in extremely non-linear, chaotic etc. classical systems evolution of probability distribution is linear!

Isomorphic argument now can be adapted for the case of evolution of density matrix. The only feature we need to remind is that density matrix is used in probability assignment rule $p_i = \text{Tr}(\rho M_i)$ and that the density matrix $\rho = s\rho_1 + (1-s)\rho_2$ describes the situation in which we are given with probability s the state ρ_1 and with probability $1-s$ we are given the state ρ_2. Hence its evolution *is* linear.

So what about so called non linear quantum mechanics? Bear in mind the difference between the wave function and the density matrix! In non linear quantum mechanics one means the non linear evolution of the wave function and not the density matrix. So what about the density matrix? It simply does not exist, or at least we cannot introduce it in the same way and ascribe the same sense to it as in standard theory i.e. as an operator over the Hilbert space and use it in the Born rule. For example two decompositions of the identity matrix $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1| = |+\rangle\langle +| + |-\rangle\langle -|$ could in principle be distinguishable! So standard density matrix description would not be sufficient.

If we want to describe the situation in which we have imperfect knowledge about the system we simply have to say: with probability s we have a wave function $|\psi_1\rangle$ and with probability $1-s$ we have $|\psi_2\rangle$, and then track this two (or more) options during the evolution, and that is all we can do .... If you really insist we can build a probability "supervector", where each entry corresponds to a probability that a system has a give wave function (we need an infinite dimensional vector for this). Then the evolution of such an object would indeed be linear, but most probably completely useless.

1.5 Multipartite systems and operations

1.5.1 Multipartite states

We describe an N partite system using a density matrix $\rho_{1,\ldots,N} \in \mathcal{L}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$. We call a state uncorrelated iff it has the form:

$$\rho_{1,\ldots,N} = \rho_1 \otimes \cdots \otimes \rho_N \tag{1.18}$$

We call a state separable (or classically correlated) iff

$$\rho_{1,\ldots,N} = \sum_j p_j \rho_1^{(j)} \otimes \cdots \otimes \rho_N^{(j)}, \quad p_j \geq 0 \tag{1.19}$$

Otherwise the state is entangled.

1.5.2 Multipartite operations

If a CP map $\mathcal{E}$ can be written in the form:

$$\mathcal{E}(\rho) = \sum_{i_N} \mathbb{1}_{1,\ldots,N} \otimes K_N^{(i_N)} (\ldots (\sum_{i_1} K_1^{(i_1)} \otimes \mathbb{1}_{2,\ldots,N} \rho K_1^{\dagger(i_1)} \otimes \mathbb{1}_{2,\ldots,N}) \ldots) \mathbb{1}_{1,\ldots,N} \otimes K_N^{\dagger(i_N)} \tag{1.20}$$

$$= \sum_{i_1,\ldots,i_N} K_1^{(i_1)} \otimes \cdots \otimes K_N^{(i_N)} \rho K_1^{\dagger(i_1)} \otimes \cdots \otimes K_N^{\dagger(i_N)} \tag{1.21}$$

where for every $k \in \{1, \ldots, N\}$ $\sum_{i_k} K_k^{\dagger(i_k)} K_k^{(i_k)} = \mathbb{1}_k$ we call it a *local operation*. Which means it can be realized by acting independently on each party without any quantum interaction nor classical communication between the parties.

LOCC operations (local operation assisted with classical communication) have the form:

$$\mathcal{E}(\rho) = \sum_{i_N} \mathbb{1}_{1,\ldots,N} \otimes K_N^{(i_N)} (\ldots (\sum_{i_1} K_1^{(i_1,\ldots,i_N)} \otimes \mathbb{1}_{2,\ldots,N} \rho K_1^{\dagger(i_1,\ldots,i_N)} \otimes \mathbb{1}_{2,\ldots,N}) \ldots) \mathbb{1}_{1,\ldots,N} \otimes K_N^{\dagger(i_N)} \tag{1.22}$$

where for simplicity we have allowed only for one-way communication from N to $N-1$ to $N-2$ etc. This is reflected by dependence on more indices i_k for parties with lower index k. A crucial condition that this is really an LOCC *deterministic* operation is that: for every k, for every $i_{k+1}, \ldots, i_N$ $\sum_{i_k} K_k^{\dagger(i_k,\ldots i_N)} K_k^{(i_k,\ldots i_N)} = \mathbb{1}_k$. Since index of a Kraus operator can be interpreted as a measurement result, it means that when parties communicate their measurement results to the party k it has to perform local trace preserving (deterministic) CP map which can depend on the data transmitted by other parties. An example would be the process of teleportation. Taking the input state: $\rho_{\text{in}} = |\Psi^-\rangle_{12}\langle\Psi^-| \otimes |\psi\rangle_3\langle\psi|$ we have:

$$\mathcal{E}(\rho_{\text{in}}) = \sum_{i=0}^{3} \sigma_1^{(i)} \otimes K_{23}^{(i)} \rho \sigma_1^{(i)\dagger} \otimes K_{23}^{(i)\dagger} = \mathbb{1}_{12} \otimes |\psi\rangle\langle\psi| \tag{1.23}$$

where $K^{(0)} = |\Psi^-\rangle\langle\Psi^-|$, $K^{(1)} = |\Phi^-\rangle\langle\Phi^-|$, $K^{(2)} = |\Phi^+\rangle\langle\Phi^+|$, $K^{(3)} = |\Psi^+\rangle\langle\Psi^+|$, while $\sigma^{(i)}$ are Pauli matrices ($\sigma^{(0)} = \mathbb{1}$).

If we allow communication in all directions we should now allow for a repetition of this procedure from party 1 to N and so on. Structure of LOCC transformation is extremely hard to characterize. That is why one introduces...

Separable operations are operations that can be written in the form:

$$\mathcal{E}(\rho) = \sum_i K_1^{(i)} \otimes \cdots \otimes K_N^{(i)} \rho K_1^{\dagger(i)} \otimes \cdots \otimes K_N^{\dagger(i)}, \tag{1.24}$$

where $\sum_i K_1^{\dagger(i)} K_1^{(i)} \otimes \cdots \otimes K_N^{\dagger(i)} K_N^{(i)}$. Of course every LOCC operation is also a separable operation but not vice versa . Not every separable operation can be realized deterministically via LOCC. Note however that every separable operation can be realized via LOCC probabilistically (i.e. when neglecting trace preservation condition at every step of performing LOCC operation). Though separable operation have no physical significance they are often used in many proofs replacing LOCC operations since they are easier to handle. And of course they give useful bound if something cannot be done by separable operations then it cannot be done by LOCC. Or if we optimize something using the separability condition on the optimized transformation (weaker then LOCC), and afterwards find that the optimal solution is LOCC we have found the solution of LOCC optimization problem

Remaining operations we call *global operations* since they require quantum interaction between parties.

1.6 Choi-Jamiołkowski isomorphism

Choi-Jamiołkowski isomorphism establishes a one to one correspondence between completely positive maps $\mathcal{E} : \mathcal{L}(\mathcal{H}_{\text{in}}) \mapsto \mathcal{L}(\mathcal{H}_{\text{out}})$ and positive operators $P_{\mathcal{E}} \in \mathcal{L}(\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}})$ as follows:

$$P_{\mathcal{E}} = \mathcal{E} \otimes \mathcal{I}(|\Psi\rangle\langle\Psi|), \tag{1.25}$$

where $|\Psi\rangle = \sum_i |i\rangle \otimes |i\rangle$ is an unnormalized maximally entangled state in $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{in}}$. Trace preservation condition of $\mathcal{E}$ is equivalent to $\text{Tr}_{\text{out}} P_{\mathcal{E}} = \mathbb{1}_{\text{in}}$ The evolution can be expressed using $P_{\mathcal{E}}$ as follows:

$$\mathcal{E}(\rho) = \text{Tr}_{\text{in}}(P_{\mathcal{E}}\ \mathbb{1} \otimes \rho^T). \tag{1.26}$$

The following notation will prove useful. Given a matrix $A = \sum_{ij} A_{ij} |i\rangle\langle j|$, we define a vector $|A\rangle\!\rangle = \sum A_{ij} |i\rangle \otimes |j\rangle$. Given a CP map in the Kraus form $\mathcal{E} = \sum_i K_i \rho K_i^\dagger$ then $P_= \sum_i |K_i\rangle\!\rangle\langle\!\langle K_i|$.

As a result the positive operator corresponding to: a local map is proportional to an uncorrelated state, a separable map is proportional to a separable state, a global map is proportional to an entangled state.

This allows using all techniques know from investigating separability of states for investigation of separability of maps, such as e.g. PPT criterion.

Chapter 2

Elements of classical information

2.1 Shannon entropy – Data compression

2.1.1 Motivation

Imagine you want to transmit a message where each letter can be one of 4 symbols a, b, c, d via a binary channel – encoding the message as a sequence $01111010\ldots$. How to encode the symbols to encode the message using the smallest number of bits?

If symbols a, b, c, d appear with equal frequency $p(a) = p(b) = p(c) = p(d) = 1/4$ you will probably assign two digit encodings to them:

symbols	a	b	c	d
probability	1/4	1/4	1/4	1/4
encoding	00	01	10	11

(2.1)

You need two bits per letter transmitted. Imagine now that in the message symbols appear with not equal frequencies e.g. $p(a) = 1/2$, $p(b) = 1/4$, $p(c) = 1/8$, $p(d) = 1/8$. What is the most economical way to encode the symbols? You may try this:

symbols	a	b	c	d
probability	1/2	1/4	1/8	1/8
encoding	0	10	110	111

(2.2)

On average you use $1/2 \cdot 1 + 1/4 \cdot 2 + 1/8 \cdot 3 + 1/8 \cdot 3 = 1.75$bits per letter transmitted!. The optimal transmission rate is quantified by Shannon entropy. Notice that these are instantaneous codes, we know when a given codeword ends and can decode it without a reference to future codewords.

2.1.2 Definition

Let X be a random variable, with possible outcomes $x \in \underbrace{\{0, 1, 2, \ldots, \}}_{\mathcal{X}}$. Let $p(x)$ be a probability of outcome x. Shannon entropy is defined as:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x). \tag{2.3}$$

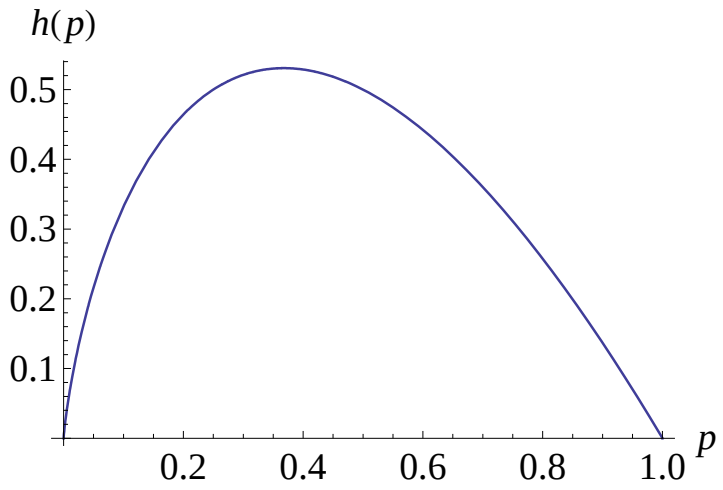


Figure 2.1: The function $h(p) = -p \log_2 p$. It is visible it is concave

$H(X)$ is given in bits. Notice that $H(X) \geq 0$. $H(X)$ should intuitively be understood as a measure of randomness of random variable X. Alternatively one can regard it as an amount of information one gets once learning the exact outcome of the variable. Some simple examples if we consider only binary outcomes $\mathcal{X} = \{0, 1\}$:

- $p(0) = 1/2$, $p(1) = 1/2$ – complete randomness: $H(X) = 1$
- $p(0) = 0$, $p(1) = 1$ – no randomness: $H(X) = -0 \log_2 0 = 0$
- $p(0) = p$, $p(1) = 1 - p$, $H(X) = -p \log_2 p - (1 - p) \log_2 (1 - p)$

Let us calculate Shannon entropy for examples presented in previous subsection. In the case of equal frequencies we have $H(X) = -4 \cdot 1/4 \log_2 1/4 = 2$, in the second example we have

$$H(X) = -1/2 \log_2 1/2 - 1/4 \log 1/4 - 2 \cdot 1/8 \log 1/8 = 1.75. \tag{2.4}$$

Let us denote $h(p) = -p \log_2 p$. The function is plotted in Fig. **??** This function is concave which means that for any weights $w_i \geq 0$, which sum up to one we have: $\sum_i w_i h(p_i) \leq h(\sum_i w_i p_i)$. Since $H(X) = \sum_{x_i \in \mathcal{X}} h(p(x_i))$, and the sum of concave functions is concave we have:

$$\frac{1}{\bar{\mathcal{X}}} H(X) = \frac{1}{\bar{\mathcal{X}}} \sum_{x_i \in \mathcal{X}} h(p(x_i)) \leq \mathcal{H}\left(\frac{1}{\bar{\mathcal{X}}} \sum_i p(x_i) \right) \leq h(1/\bar{\mathcal{X}}), \tag{2.5}$$

where $\bar{\mathcal{X}}$ denotes the number of elements in the set $\mathcal{X}$. Hence we have $H(X) \leq \log_2 \bar{\mathcal{X}}$.

Given many variables $X_1, \ldots, X_N$ which joint probability distribution is $p(x_1, \ldots, x_N)$ their entropy is defined:

$$H(X_1, \ldots, X_N) = \sum_{x_1, \ldots, x_N} -p(x_1, \ldots, x_N) \log_2 p(x_1, \ldots, x_N). \tag{2.6}$$

Notice the inequality:

$$H(X_i) \leq H(X_1, \ldots, X_N) \leq \sum_i H(X_i), \tag{2.7}$$

which means that the entropy of the full system is larger then each of subsystems, and that correlations decrease the entropy.

In particular given N independent realization of random variable X, we have:

$$H(X^N) = NH(X). \tag{2.8}$$

2.1.3 Relative entropy

For future use we introduce here the concept of relative entropy. The relative entropy $D(p \parallel q)$ of probability distribution p with respect to q is defined:

$$D(p \parallel q) = \sum_i p_i \log_2 \frac{p_i}{q_i} \tag{2.9}$$

It is a measure of distinguishability between two probability distributions. Notice, however, that this is not a symmetric function and hence cannot be regarded as a proper measure of a distance between probability distributions. The relative entropy is always positive:

$$D(p \parallel q) = -\sum_i p_i \log_2 \frac{q_i}{p_i} \leq \log_2 \left(\sum_i p_i \frac{q_i}{p_i} \right) = 0, \tag{2.10}$$

where we have used concavity of $-\log_2 t$ function.

2.1.4 Shannon source coding theorem

Given the random variable X with Shannon entropy $H(X)$ the minimal average length of a codeword L we need to use is bounded by:

$$H(X) \leq L \leq H(X) + 1 \tag{2.11}$$

Lemma. Kraft inequality We first proof the Kraft inequality. Suppose we want to have a code with codewords of length l_i. If the code is to be instantaneous we have a constraint on the lengths of the codewords. Consider a tree, where at each node a branch splits into two. . Let $l_{\max}$ be the maximum length of a codeword, which corresponds to the number of levels in the tree. We have total $2^{\max}$ leaves and hence this many different codewords. If a given codewords has length l_i which is shorter then $l_{\max}$, however, it automatically exclude $2^{l_{\max}-l_i}$ codewords from being used (all leaves having steming from this branch). Since the total number of codewords is $2^{l_{\max}}$ we have an inequality: $\sum_i 2^{l_{\max}-l_i} \leq 2^{l_{\max}}$, which leads to:

$$\sum_i 2^{-l_i} \leq 1 \tag{2.12}$$

called Kraft inequality. A codeword of given lengths exist iff the Kraft inequality is satisfied.

Proof. Given probabilities p_i, we construct codewords of length:

$$log_2 \frac{1}{p_i} \leq l_i \leq log_2 \frac{1}{p_i} + 1. \tag{2.13}$$

Note that the first inequality is equivalent to $p_i \geq 2^{-l_i}$, and the Kraft inequality is satisfied. The average codeword length reads:

$$H(X) \leq \sum_i p_i l_i \leq H(X) + 1. \tag{2.14}$$

What remains to be proven is that one can do no better then $H(X)$. Let us define probability distribution $q_i = 2^{-l_i} / \sum_i 2^{-l_i}$. Positivity of relative entropy $D(p \parallel q)$ leads to:

$$-\sum_i p_i (\log_2 p_i - \log_2 q_i) = -\sum_i p_i (\log_2 p_i - l_i - \log_2 \sum_j 2^{-l_j}) \geq 0 \tag{2.15}$$

$$\sum_i p_i l_i \geq H(X) + \sum_i p_i \log_2 \left(\sum_j 2^{-l_j} \right) \tag{2.16}$$

Thanks to the Kraft inequality the second term on the right hand side above is non-negative. Finally we have

$$\sum_i p_i l_i \geq H(X). \tag{2.17}$$

Which ends the proof ■. Actually this inequality holds also for non-instantaneous codes the proof is a bit more complicated then.

Instead of single letter encoding consider block encoding when we use codewords encoding N letter words. If L_N denotes average codeword length for N-letter block encoding, using Eqs (2.8,2.11) we have:

$$NH(X) \leq L_N \leq NH(X) + 1 \tag{2.18}$$

hence

$$H(X) \leq \frac{L_N}{N} \leq H(X) + \frac{1}{N}. \tag{2.19}$$

As a result asymptotically $L_N/N \lim_{N\to\infty} = H(X)$, so the rate of transmission given by Shannon entropy can be saturated in the limit of large N.

2.1.5 Typical sequences

How intuitively understand that optimal data compression of N bits can be done using $NH(X)$ bits. Let X be binary random variable, taking value 0 with probability q and 1 with probability $1 - q$. Let us take a long sequence of N realizations of X.

If sequence is long we will usually have sequence with approximately qN bits 0 and $(1-q)N$ bits 1. Probability of a given sequence is

$$p(x_1, \ldots, x_N) = q^{qN}(1-q)^{(1-q)N} \tag{2.20}$$

Hence:

$$\log_2 p(x_1, \ldots, x_N) = -NH(X) \tag{2.21}$$

$$p(x_1, \ldots, x_N) = 2^{-NH(X)}. \tag{2.22}$$

Since these are approximately only sequences that happen , we have $2^{NH(X)}$ typical sequences. In compression when encoding large blocks in order to transmit 2^N sequences we need only use $2^{NH(X)}$ codewords. This is an intuitive understanding of the result from previous section.

2.2 Shannon mutual information – Communication over noisy channel

Consider a channel which is noisy and can flip transmitted bits with some probability. Let X be input random variable and Y be a random variable describing the output of the channel. Conditional probability $p(y_j|x_i)$ describes the action of the channel. One would like to know how to protect transmitted information against errors and what is maximal number of logical bits that can be transmitted per one physical bit sent (channel capacity). We start by quantify correlations between two random variables X and Y.

2.2.1 Conditional entropy

Let joint probability of X and Y be $p(x_i, y_j)$. One quantifies the amount of randomness of random variable Y provided one knows the value x_i of random variable X using conditional entropy:

$$H(Y|x_i) = -\sum_j p(y_j|x_i) \log_2 p(y_j|x_i) \tag{2.23}$$

On average the conditional entropy reads:

$$H(Y|X) = -\sum_{ij} p(x_i)p(y_j|x_i) \log_2 p(y_j|x_i) = -\sum_{ij} p(x_i, y_j) \log_2 p(y_j|x_i). \tag{2.24}$$

This characterizes randomness of random variable Y provided variable X is known.

2.2.2 Mutual Information

Let us introduce the measure of correlation between random variables X and Y which will represent: How much do we learn about variable Y once we learn the value of variable X:

$$I(X:Y) = H(Y) - H(Y|X) \tag{2.25}$$

and is called *mutual information*. It is symmetric in X, Y since:

$$I(X:Y) = H(X) + H(Y) - H(X,Y). \tag{2.26}$$

It is zero iff X and Y are uncorrelated. Notice that if $p(x,y)$ is the joint probability distribution, and by $p_x(x)$, $p_y(y)$ we denote its marginal distributions we have:

$$I(X:Y) = D(p(x,y) \parallel p_x(x)p_y(y)), \tag{2.27}$$

which reflect the fact that mutual information measures in some sense the distance between $p(x,y)$ and uncorrelated probability distribution with the same marginals.

2.2.3 Channel capacity

Let us consider a channel, which action is described by $p(y_j|x_i)$. Let the input random variable be X. Let us transmit N bits via the channel using typical sequences. Transmitting a sequence of N bits, there are on average $2^{H(Y|X)}$ typical error sequences (similar argument as in Sec. 2.1.5). . At the output we have $2^{NH(Y)}$ typical sequence. For reliable transmission we can use at most $2^{NH(Y)}/2^{NH(Y|X)} = 2^{NI(X:Y)}$ different inputs – we cannot use all typical sequences. Hence we can transmit at most $I(X:Y)$ logical bits per one physical bit transmitted. Moreover this rate can be achieved. .

Channel capacity is defined (Shannon noisy channel theorem):

$$C = \max_{p(x)} I(X:Y). \tag{2.28}$$

Where we maximize over input probability distribution. For binary symmetric channel optimal choice for input probability distribution is $p(0) = p(1) = 1/2$.

Chapter 3

Group theory in quantum information

The most important fact from the group theory for a physicist is decomposition of tensor product of representation into direct product of irreducible ones:

$$V^{\otimes n} = \bigoplus_i V_i^{\oplus k_i} \tag{3.1}$$

where V_i denote different irreducible representations and k_i their multiplicities. In quantum information most commonly V will be a defining representation of $SU(2)$ group, since we usually deal with qubits, occasionally $SU(d)$.

3.1 Symmetric group

Symmetric group S_N is the group of all permutations of N elements.

permutations, representations (Young diagrams etc...)

2.Representations of SU(2), [moze SU(d)] group, tensor representation decomposition into irreducible representations, Schur Lemma a) wspomniec o decoherence free subspaces, subsystems (Capacity of locally depolarizing N -qubit channesl) 3.Shur-Weyl theorem

4. Haar measure

Chapter 4

Covariant operations

4.1 Covariant maps and measurements

4.1.1 Covariant maps

Consider a quantum map $\mathcal{E}$ mapping density matrices on the Hilbert space $\mathcal{H}_1$ onto density matrices acting on a Hilbert space $\mathcal{H}_2$. Let G be a group, whereas U_g, V_g its unitary representations acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We say that $\mathcal{E}$ is covariant with respect to representations U_g, V_g of the group G (or simply covariant with respect to G, if it is clear from context what representations we have in mind) if for all g and all ρ:

$$\mathcal{E}(U_g \rho U_g^\dagger) = V_g \mathcal{E}(\rho) V_g^\dagger \tag{4.1}$$

Example 1 (Qubit depolarization)**.** *Let $dim(\mathcal{H}_1) = dim(\mathcal{H}_2) = 2$, and let V_g and U_g be the defining representations of $SU(2)$. Consider the following map:*

$$\mathcal{E}(\rho) = \frac{1+3\eta}{4}\rho + \frac{1-\eta}{4}\sum_{i=1}^{3}\sigma_i\rho\sigma_i \tag{4.2}$$

which shrinks the Bloch vector of a qubit by η. This transformation can be rewritten in the form:

$$\mathcal{E}(\rho) = \eta\rho + (1-\eta)/2\mathbb{1} \tag{4.3}$$

which is clearly covariant, since:

$$\mathcal{E}(U\rho U^\dagger) = \eta U\rho U^\dagger + (1-\eta)/2\mathbb{1} = U\mathcal{E}(\rho)U^\dagger. \tag{4.4}$$

4.1.2 Covariant measurements

Given a POVM measurement $\{\Pi_g\}$ acting on Hilbert space $\mathcal{H}$, parameterized by group elements $g \in G$, we say that it is covariant with respect to the representation U_g of the group G (or simply with respect to the group G) iff, an operator Π_0 exists that for every g we have:

$$\Pi_g = U_g \Pi_0 U_g^\dagger. \tag{4.5}$$

Notice that completeness property of POVM imposes a constraint:

$$\int \mathrm{d}g \Pi_g = \int \mathrm{d}g U_g \Pi_0 U_g^\dagger = \mathbb{1}. \tag{4.6}$$

Example 2 (Single qubit covariant measurement)**.** *POVM measurement of a qubit defined as:*

$$\Pi_g = 2U_g|0\rangle\langle 0|U_g^\dagger \tag{4.7}$$

where U_g is the defining representation of $SU(2)$ is obviously covariant. In order to check the completeness property notice that the integral over normalized Haar measure:

$$\mathcal{A} = 2\int dg U_g|0\rangle\langle 0|U_g^\dagger \tag{4.8}$$

commutes with U_h for all $h \in SU(2)$. Because the U_g representation is irreducible, it implies by the Schur Lemma that $\mathcal{A} \propto \mathbb{1}$. Moreover:

$$Tr(\mathcal{A}) = \int dg Tr(U_g 2|0\rangle\langle 0|U_g^\dagger) = 2\int dg = 2. \tag{4.9}$$

Therefore $\mathcal{A} = \mathbb{1}$ which proves Π_g is indeed the POVM.

4.2 Covariant optimization problems

Covariant operations and measurements are useful, since in many problems where the figure of merit enjoys certain group symmetry the optimal maps or measurements can always be found in the subset of covariant operations. This usually simplifies the problem dramatically and allows for analytical solutions.

4.2.1 Covariant map problems

Let $\mathcal{E}$ be a quantum map. Let ρ_g be input states parameterized by group elements $g \in G$ using some unitary representation U_g: $\rho_g = U_g \rho_0 U_g^\dagger$. Let $\sigma_g = V_g \sigma_0 V_g^\dagger$ be the state we ideally would like to obtain if we are given ρ_g, where V_g is again some unitary representation of G. Moreover we assume that the input state ρ_g is unknown and the a priori probability distribution of g is uniform with respect to the Haar measure on G. Last, we need to define figure of merit $F[\mathcal{E}(\rho_g), \sigma_g]$ which measures how close is the actual to the expected ideal output state (fidelity). The optimal solution for $\mathcal{E}$ is the one that maximizes the average fidelity:

$$\bar{F} = \int \mathrm{d}g F[\mathcal{E}(\rho_g), \sigma_g] \tag{4.10}$$

The last condition we impose is that the figure of merit is invariant under joint action of V_h representation: $F[V_h \mathcal{E}(\rho_g) V_h^\dagger, V_h \sigma_g V_h^\dagger] = F[\mathcal{E}(\rho_g), \sigma_g]$.

In the above setting we may prove the following:

Theorem 1. *The optimal operation can always be found as a covariant one.*

Proof. Let $\mathcal{E}$ be the optimal operation yielding average fidelity $\bar{F}$ Let us define a new operation $\mathcal{E}_c$ by:

$$\mathcal{E}_c(\rho_g) = \int \mathrm{d}h V_h^\dagger \mathcal{E}(U_h \rho_g U_h^\dagger) V_h. \tag{4.11}$$

This operation yields the average fidelity:

$$\bar{F}_c = \int \mathrm{d}g\mathrm{d}h \; F[V_h^\dagger \mathcal{E}(U_h \rho_g U_h^\dagger) V_h, \sigma_g] = \int \mathrm{d}g\mathrm{d}h \; F[\mathcal{E}(U_h \rho_g U_h^\dagger), V_h \sigma_g V_h^\dagger] = \\ = \int \mathrm{d}g\mathrm{d}h \; F[\mathcal{E}(\rho_{hg}), \sigma_{hg}] \tag{4.12}$$

Introducing new variable $h' = hg$ and making use of of invariance of the Haar measure we get:

$$\bar{F}_c = \int \mathrm{d}g\mathrm{d}h' \; F[\mathcal{E}(\rho_{h'}), \sigma_{h'}] = \bar{F}. \tag{4.13}$$

Therefore operation $\mathcal{E}_c$ is also optimal. What remains to be shown is its covariance property:

$$\mathcal{E}_c(U_{g'} \rho_g U_{g'}^\dagger) \quad = \quad \int \mathrm{d}h V_h^\dagger \mathcal{E}(U_h U_{g'} \rho_g U_{g'}^\dagger U_h^\dagger) V_h \quad = \quad \int \mathrm{d}h V_h^\dagger \mathcal{E}(U_{hg'} \rho_g U_{hg'}^\dagger) V_h \tag{4.14}$$

Introducing new variable $h' = hg'$ we get:

$$\int \mathrm{d}h' V_{h'g'^{-1}}^\dagger \mathcal{E}(U_{h'} \rho_g U_{h'}^\dagger) V_{h'g'^{-1}} = V_{g'} \int \mathrm{d}h' V_{h'}^\dagger \mathcal{E}(U_{h'} \rho_g U_{h'}^\dagger) V_{h'} \; V_{g'}^\dagger = V_{g'} \mathcal{E}_c(\rho_g) V_{g'}^\dagger \tag{4.15}$$

□

4.2.2 Covariant estimation problems

1. covariant operations (moze juz tu...) 2. Jamiolkowski isomorphism, Covariance condition on Jamiolkowski isomorphism 3. Examples: 1->1 qubit covariant operations, 1->2 qubits covariant operations (moze ale raczej nie)

Chapter 5

Quantum state estimation - Quantum->Classical channel

5.1 State discrimination - two states

Imagine you are given one of two states $|\psi_1\rangle$, $|\psi_2\rangle$. Your goal is to perform a measurement in order to determine which state we received.

The task is simple provided states are orthogonal $\langle\psi_1|\psi_2\rangle = 0$. One simply performs projective measurement in $\{|\psi_1\rangle, |\psi_2\rangle\}$ basis. Distinguishability is perfect. Let us now take nonorthogonal, nonidentical states $0 < |\langle\psi_1|\psi_2\rangle| < 1$. Assume for simplicity that both states are equiprobable.

5.1.1 Minimizing probability of error

We want to minimize probability of error i discrimination. Let $p(j|i)$ be the probability that we guess the state j when the actual state is i. We need to find the optimal measurement that will minimize the error. Since this is a two outcome measurement we model our measurement with two POVM: M_1, M_2, ($\sum_i M_i = \mathbb{1}$). Probability distribution is given by $p(j|i) = \mathrm{Tr}\left(M_j|\psi_i\rangle\langle\psi_i|\right) = \langle\psi_i|M_j|\psi_i\rangle$. Error we want to minimize is given by

$$E = \frac{1}{2}\langle\psi_1|M_2|\psi_1\rangle + \frac{1}{2}\langle\psi_2|M_1|\psi_2\rangle. \tag{5.1}$$

We want to minimize E over $\{M_i\}$. Substituting $M_2 = \mathbb{1} - M_1$ we have:

$$E = \frac{1}{2} \ + \frac{1}{2}\mathrm{Tr}\left[M_1(|\psi_2\rangle\langle\psi_2| - |\psi_1\rangle\langle\psi_1|)\right]. \tag{5.2}$$

Without loosing generality we may take:

$$|\psi_1\rangle = \sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{5.3}$$

$$|\psi_2\rangle = -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{5.4}$$

Introducing

$$W = |\psi_2\rangle\langle\psi_2| - |\psi_1\rangle\langle\psi_1| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |+\rangle\langle+| - |-\rangle\langle-|, \tag{5.5}$$

where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. We see that the problem amounts to finding M_1 such that $\mathrm{Tr}WM_1$ is maximal. Keeping constraints $M_1 \geq 0$, and $M_2 \geq 0$ (which means $M_1 \leq 1$ - has to have all eigenvalues no bigger than 1). Notice that $\mathrm{Tr}WM_1 \in [-1, 1]$ since for every $|\psi\rangle$ we have $\mathrm{Tr}M_i|\psi\rangle\langle\psi| \in [0, 1]$. Hence the optimal choice is to take: $M_1 = |+\rangle\langle+|$ since it gives $\mathrm{Tr}WM_1 = 1$. The optimal POVMs and minimal discrimination error thus read:

$$E = \frac{1}{2}(1 + |\langle\psi_2|+\rangle|^2 - |\langle\psi_1|+\rangle|^2) = \frac{1}{2}(1 - \sin\theta) \tag{5.6}$$

Notice that the optimal measurement is an von Neumann projection measurement (this is also true for discrimination of N linearly independent states). Only for $\theta = \pi/2$ we have $E = 0$, hence perfect discrimination is possible only when states are orthogonal. This has profound consequences and leads to Quantum Cryptography. Replacing $\sin\theta$ with a function of scalar product between two states we get:

$$E = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}\right). \tag{5.7}$$

This formula can be used easily for arbitrary states.

Many copies Imagine that as above you are given with probability 1/2 either $|\psi_1\rangle$, or $|\psi_2\rangle$, but this time not a single copy but N copies. So in fact you are given either $|\psi_1\rangle^{\otimes N}$, or $|\psi_2\rangle^{\otimes N}$. Using Eq. 5.7 we get:

$$E_N = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^{2N}}\right). \tag{5.8}$$

Notice that if $\langle\psi_1|\psi_2\rangle < 1$, $E_{N+1} < E_N$ so more copies we have the better is distinguishability. I particular

$$\lim_{N\to\infty} E_N = 0, \tag{5.9}$$

which means one can distinguish quantum states perfectly once one has an arbitrary large number of copies.

5.1.2 Unambiguous discrimination

We again face the problem of discriminating between $|\psi_1\rangle$, $|\psi_2\rangle$, but this time we only tell which state we received when we are sure. Otherwise we say we do not know. The goal is to find a measurement strategy that will minimize the probability of cases when we do not know.

The measurement will be described by three POVMs (it is clear that von Neuman measurement cannot descirbe this): M_1, M_2, $M_?$, corresponding to the result that leads

us to guess correctly that the state was $|\psi_1\rangle$, result that leads us to guess correctly that the state was $|\psi_2\rangle$, and the result when we say we do not know.

Unambiguity conditions read:

$$\langle\psi_2|M_1|\psi_2\rangle = 0 \quad \langle\psi_1|M_2|\psi_1\rangle = 0, \tag{5.10}$$

since $M_1 \geq 0$, $M_2 \geq 0$, this leads to $M_1 = \xi_1|\psi_2^\perp\rangle\langle\psi_2^\perp|$, $M_2 = \xi_2|\psi_1^\perp\rangle\langle\psi_1^\perp|$. Symmetry between $|\psi_1\rangle$, $|\psi_2\rangle$ allows us to take $\xi_1 = \xi_2 =: \xi$. Positivity of M_i requires $\xi \geq 0$. However, we have additional constraint, namely $M_? = \mathbb{1} - M_1 - M_2 \geq 0$. Using parametrization of states given in Eq. 5.3 this condition can be written as:

$$M_? = \mathbb{1} - \xi(|\psi_2^\perp\rangle\langle\psi_2^\perp| + \xi|\psi_1^\perp\rangle\langle\psi_1^\perp|) = \begin{pmatrix} 1-\xi(1+\cos\theta) & 0 \\ 0 & 1-\xi(1-\cos\theta) \end{pmatrix} \tag{5.11}$$

This means that ξ is limited by $\xi \leq 1/(1+\cos\theta)$. Taking $\xi = 1/(1+\cos\theta)$ we find that the optimal probability of successful discrimination reads:

$$1 - \frac{1}{2}\mathrm{Tr}\left[M_?(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)\right] = 1 - |\langle\psi_1|\psi_2\rangle|. \tag{5.12}$$

5.1.3 Mutual information in ambiguous and unambiguous discrimination

What is better strategy ambiguous or unambiguous strategy if one aims at optimizing mutual information between classical values $X = 0, 1$ encoded in states $|\psi_1\rangle$, $|\psi_2\rangle$ and values Y obtained in the measurement.

For optimal ambiguous discrimination we have random variables $X = \{0, 1\}$, $Y = \{0, 1\}$:

$$p(1|1) = p(0|0) = p_a \tag{5.13}$$

$$p(1|0) = p(0|1) = 1 - p_a, \quad \text{where} \tag{5.14}$$

$$p_a = \frac{1}{2}\left(1 + \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}\right) = \frac{1}{2}(1 + \sin\theta) \tag{5.15}$$

Thus mutual information $I_a(X:Y)$ reads:

$$I_a(X:Y) = 1 + p_a \log_2 p_a + (1-p_a)\log_2(1-p_a) \tag{5.16}$$

In unambiguous discrimination we have $X = \{0, 1\}$, $Y = \{0, 1, ?\}$

$$p(1|1) = p(0|0) = p_u \tag{5.17}$$

$$p(?|0) = p(?|1) = 1 - p_u \quad \text{where} \tag{5.18}$$

$$p_u = 1 - |\langle\psi_1|\psi_2\rangle| = 1 - \sqrt{|\cos\theta|} \tag{5.19}$$

And the mutual information reads:

$$I_u(X:Y) = p_u \tag{5.20}$$

Fig. 5.1 presents mutual information in both ambiguous and unambiguous discrimination.

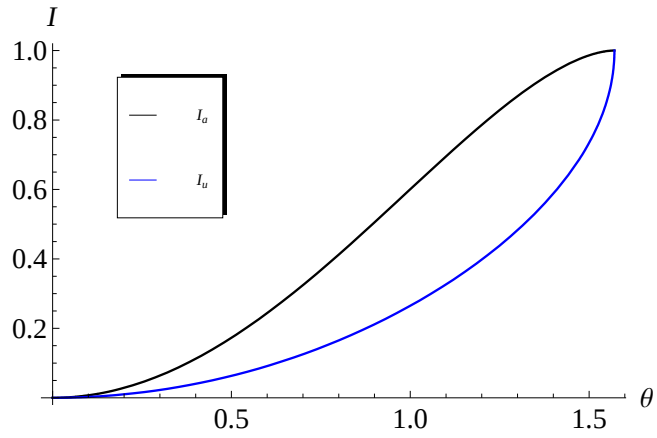


Figure 5.1: Comparison of mutual information obtained in the optimal ambiguous I_a and unambiguous I_u discrimination of two states

Chapter 6

Quantum cloning - Quantum->Quantum channel

6.1 No-cloning theorems

Classical information in its digital form can be copied perfectly. Can you copy a unknown quantum state that is given to you. First of all you may grow suspicious since you have learned before that one can not discriminate nonorthogonal quantum states perfectly and hence one can not just measure and then reprepare more copies of the state, as this will induce unavoidable errors. But can you just copy an unknown state without measuring it? The answer is again no!

The general framework for cloning is the following. Consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_X$, where $\mathcal{H}_1$ is the space supporting the state of a system to be copied, $\mathcal{H}_2$ supports the states of the system which is our "blank page", and $\mathcal{H}_X$ supports all other degrees of freedom including the copying machine and the rest of the universe. We say that the operation U (according to quantum theory should be unitary) performs cloning of a state $|\psi\rangle$ iff:

$$U|\psi\rangle \otimes |0\rangle \otimes |X_0\rangle = |\psi\rangle \otimes |\psi\rangle \otimes |X_\psi\rangle. \tag{6.1}$$

In other words should produce a state $|\psi\rangle$ in both systems 1 and 2 while the remaining degrees of freedom can change depending on the cloned state. Notice that the output state is a product state – there is no entanglement between subsystems. It has to be so, otherwise clones when inspected independently would be in mixed states.

6.1.1 Linearity → No-cloning of linearly dependent states

Theorem. If there is a quantum machine that can copy two quantum states $|\psi_1\rangle$, $|\psi_2\rangle$ perfectly, then it cannot copy the state which is their linear superposition $|\phi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$.

Proof. If an operation U clones two states $|\psi_1\rangle$, $|\psi_2\rangle$ we have:

$$U|\psi_1\rangle \otimes |0\rangle \otimes |X_0\rangle = |\psi_1\rangle \otimes |\psi_1\rangle \otimes |X_{\psi_1}\rangle \quad (6.2)$$
$$U|\psi_2\rangle \otimes |0\rangle \otimes |X_0\rangle = |\psi_2\rangle \otimes |\psi_2\rangle \otimes |X_{\psi_2}\rangle. \quad (6.3)$$

Thanks to linearity of Quantum Mechanics:

$$U|\phi\rangle \otimes |0\rangle \otimes |X_0\rangle = aU|\psi_1\rangle \otimes |0\rangle \otimes |X_0\rangle + bU|\psi_2\rangle \otimes |0\rangle \otimes |X_0\rangle = a|\psi_1\rangle \otimes |\psi_1\rangle \otimes |X_{\psi_1}\rangle + |\psi_2\rangle \otimes |\psi_2\rangle \otimes |X_{\psi_2}\rangle \quad (6.4)$$

whereas for cloning of $|\phi\rangle$ we would like to have at the output:

$$|\phi\rangle \otimes |\phi\rangle \otimes |X_\phi\rangle = (a|\psi_1\rangle + b|\psi_2\rangle) \otimes (a|\psi_1\rangle + b|\psi_2\rangle) \otimes |X_\phi\rangle \quad (6.5)$$

Clearly cloning of $|\phi\rangle$ is impossible.

The above proof only made use of linearity of transformation and not unitarity. In particular the above proof does not forbids cloning of two nonorthogonal states. But this will come... Nevertheless the proof basing on linearity is useful since it also forbids probabilistic cloning of linearly dependent states – probabilistic transformation need not be unitary bur are always linear.

6.1.2 Unitarity → No-cloning of non-orthogonal states

Theorem There is no deterministic cloning transformation (unitary) performing cloning for two nonorthogonal state

Proof. Let $|\psi_1\rangle$, $|\psi_2\rangle$ be two different nonorthogonal states: $0 < |\langle\psi_1|\psi_2\rangle| < 1$. Assume the cloning is possible:

$$U|\psi_1\rangle \otimes |0\rangle \otimes |X_0\rangle = |\psi_1\rangle \otimes |\psi_1\rangle \otimes |X_{\psi_1}\rangle \quad (6.6)$$
$$U|\psi_2\rangle \otimes |0\rangle \otimes |X_0\rangle = |\psi_2\rangle \otimes |\psi_2\rangle \otimes |X_{\psi_2}\rangle. \quad (6.7)$$

Thanks to unitarity scalar product of input states should be equal to scalar product of output states:

$$\langle\psi_1|\psi_2\rangle\langle 0|0\rangle\langle X_0|X_0\rangle = \langle\psi_1|\psi_2\rangle\langle\psi_1|\psi_2\rangle\langle X_{\psi_1}|X_{\psi_2}\rangle \quad (6.8)$$

this leads to:

$$\langle\psi_1|\psi_2\rangle(1 - \langle\psi_1|\psi_2\rangle\langle X_{\psi_1}|X_{\psi_2}\rangle) = 0 \quad (6.9)$$

which is only possible for $\langle\psi_1|\psi_2\rangle = 0$ or $\langle\psi_1|\psi_2\rangle = 0$, hence we arrive at contradiction and conclude that cloning of nonorthogonal states is impossible.

6.2 Optimal cloning

Since the perfect cloning is impossible except for very limited cases, we would like to investigate what is the best quality of copies that can be obtained. Let us consider a general problem of producing M imperfect copies out of N perfect originals $|\psi\rangle^{\otimes N}$. Let

us consider unitary transformation U acting on the Hilbert space $\mathcal{H}^{\otimes M} \otimes \mathcal{H}_X$, where $\mathcal{H}_X$ represents the space of the cloning machine:

$$U: \ |\psi\rangle^{\otimes N} \otimes |0\rangle^{\otimes (M-N)} \otimes |X\rangle \mapsto |\Psi\rangle_{M,X}. \tag{6.10}$$

The output $|\Psi\rangle_{M,X}$ is in general an entangled states of all M copies and the cloning machine. Tracing out the space X we obtain the state of M copies:

$$\rho_M = \mathrm{Tr}_X(|\Psi\rangle\langle\Psi|) \tag{6.11}$$

It should be remembered that this state typically will contain correlations between copies. Comparing the obtained state with the perfect M copies state $|\psi\rangle^{\otimes}$ can be done using *global fidelity* figure of merit:

$$F_G = \langle\psi|^{\otimes M} \rho_M |\psi\rangle^{\otimes M}. \tag{6.12}$$

If instead we are only interested in *single copy fidelity*, we can calculate single copy fidelity for the i-th copy:

$$F_i = \langle\psi|\rho_i|\psi\rangle, \tag{6.13}$$

where

$$\rho_i = \mathrm{Tr}_{1,\ldots,i-1,i+1,\ldots,M}(\rho_M) \tag{6.14}$$

is obtained after tracing out all the copies instead of the i-th one. Provided all F_i are equal we call the cloning *symmetric*, otherwise we call it *asymmetric*.

6.2.1 Optimal $1 \to 2$ qubit asymmetric cloning cloning

We present below an intuitive construction of optimal $1 \to 2$ universal asymmetric cloning, without the proof for its optimality. Consider the following unitary operation acting on three qubits, denoted A, B, and X:

$$V: \ |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_X \mapsto |i \oplus j \oplus k\rangle_A \otimes |i \oplus j\rangle_B \otimes |i \oplus k\rangle_X \tag{6.15}$$

Notice the following properties of operation V. If the system A is prepared in an unknown state $|\psi\rangle_A$, while subsystems BX are prepared in $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2})$ we get:

$$V : |\psi\rangle_A \otimes |\Phi^+\rangle_{BX} \mapsto |\psi\rangle_A \otimes |\Phi^+\rangle_{BX}, \tag{6.16}$$

and hence the state $|\psi\rangle_A$ remains where it was. On the other hand if at the input we take subsystems BX in the state $|0\rangle_B \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_X$ the transformation read:

$$V : |\psi\rangle_A \otimes |\Phi^+\rangle_{BX} \mapsto |\psi\rangle_B \otimes |\Phi^+\rangle_{AX}, \tag{6.17}$$

and as a result the state is "teleported" to subsystem B.

Since we want to have imperfect copies in both A and B subsystems it is natural to consider the transformation where initially we prepare systems BX in a superposition of $|\Phi^+\rangle_{BX}$ and $|0\rangle_B \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_X$. The transformation then reads:

$$V : |\psi\rangle_A \otimes \left(a|\Phi^+\rangle_{BX} + b|0\rangle_B \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_X \right) \mapsto a|\psi\rangle_A \otimes |\Phi^+\rangle_{BX} + b|\psi\rangle_B \otimes |\Phi^+\rangle_{AX}. \tag{6.18}$$

Calculating the output single copy reduced density matrices we get:

$$\rho_A = \left[\frac{(a+b)^2}{2} + \frac{a^2}{2} \right] |\psi\rangle\langle\psi| + \frac{b^2}{2} |\psi_\perp\rangle\langle\psi_\perp| \tag{6.19}$$

$$\rho_B = \left[\frac{(a+b)^2}{2} + \frac{b^2}{2} \right] |\psi\rangle\langle\psi| + \frac{a^2}{2} |\psi_\perp\rangle\langle\psi_\perp| \tag{6.20}$$

where $|\psi_\perp\rangle$ is the orthogonal state to $|\psi\rangle$. The corresponding fidelities read:

$$F_A = 1 - \frac{b^2}{2} \tag{6.21}$$

$$F_B = 1 - \frac{a^2}{2} \tag{6.22}$$

The *symmetric* cloning corresponds to the choice $a = b$, which together with normalization constraint implies $a = b = 1/\sqrt{3}$ and leads to the optimal cloning fidelity $F = 5/6$.

Chapter 7

Entangled states and no-signaling

7.1 Entangled states

Consider a bipartite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $|i\rangle_A$, $|j\rangle_B$ be some basis in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The most general pure state of the system can be written as:

$$|\psi\rangle = \sum_{ij} c_{ij} |i\rangle_A \otimes |j\rangle_B. \tag{7.1}$$

States which can be written as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ are called *product states*, and describe a situation in which there is no correlation between subsystems. All other pure states are called *entangled*.

Imagine a local measurements are performed by both parties described by a two set of POVM: Π_i^A, $\sum_i \Pi_i^A = \mathbb{1}_A$, Π_j^B, $\sum_j \Pi_j^B = \mathbb{1}_B$, This two local measurements together constitute a measurement on the whole system $\sum_{ij} \Pi_i^A \otimes \Pi_j^B = \mathbb{1}$. Joined probability distribution of obtained results reads:

$$p(i,j) = \mathrm{Tr}(|\psi\rangle\langle\psi| \; \Pi_i^A \otimes \Pi_j^B). \tag{7.2}$$

If the state is product, it is clear that $p(i,j) = p_A(i) p_B(j)$, and hence there are no correlations between subsystems for any measurement. For entangled states there always exist local measurements for which probability distribution is correlated.

Example: polarization measurement on the singlet state. Consider two photons in the singlet state:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\leftrightarrow\rangle \otimes |\updownarrow\rangle - |\updownarrow\rangle \otimes |\leftrightarrow\rangle). \tag{7.3}$$

Let two parties perform local von Neumann projection measurements in $|\leftrightarrow\rangle, |\updownarrow\rangle$ basis. Joint probability of measurement results reads:

$p_{\|\Psi^-\rangle}(\ ,\ )$	$\leftrightarrow$	$\updownarrow$
$\leftrightarrow$	0	1/2
$\updownarrow$	1/2	0

(7.4)

Clearly this is a correlated state.

From this example one could think that entanglement is nothing more that correlations known from the classical world i.e. photons were prepared with probability 1/2 in $|\leftrightarrow\rangle \otimes |\leftrightarrow\rangle$ and with probability 1/2 in $|\updownarrow\rangle \otimes |\updownarrow\rangle$, and the measurement only *reveals* which of this alternatives were realized. In this case we would describe the state using density matrix:

$$\rho = \frac{1}{2}|\leftrightarrow\rangle\langle\leftrightarrow| \otimes |\updownarrow\rangle\langle\updownarrow| + \frac{1}{2}|\updownarrow\rangle\langle\updownarrow| \otimes |\leftrightarrow\rangle\langle\leftrightarrow| \tag{7.5}$$

This state will give us the same probability distribution when local measurement in $|\leftrightarrow\rangle, |\updownarrow\rangle$ basis are performed $p_\rho(i,j) = \mathrm{Tr}(\rho\ \Pi_i^A \otimes \Pi_i^B)$.

Consider now, however, local polarization measurement in $|\nearrow\rangle = 1/\sqrt{2}(|\leftrightarrow\rangle + |\updownarrow\rangle)$, $|\nwarrow\rangle = 1/\sqrt{2}(|\leftrightarrow\rangle - |\updownarrow\rangle)$ basis. Probability distributions for $|\Psi^-\rangle$ and ρ state respectively reads:

$p_{\lvert\Psi^-\rangle}(\ ,\ )$	$\nearrow$	$\nwarrow$
$\nearrow$	0	1/2
$\nwarrow$	1/2	0

$p_\rho(\ ,\ )$	$\nearrow$	$\nwarrow$
$\nearrow$	1/4	1/4
$\nwarrow$	1/4	1/4

(7.6)

Clearly probability distribution differ and hence we cannot think of $|\Psi^-\rangle$ as photons being prepared with probability 1/2 in $|\leftrightarrow\rangle \otimes |\leftrightarrow\rangle$ and with probability 1/2 in $|\updownarrow\rangle \otimes |\updownarrow\rangle$. Notice also the interesting property of the singlet state $|\Psi^-\rangle$, that probability distribution in case of $|\nearrow\rangle, |\nwarrow\rangle$ is the same as for $|\leftrightarrow\rangle, |\updownarrow\rangle$ measurement. This is a general property of $|\Psi^-\rangle$, namely the state does not change under identical unitary operations on two subsystems i.e. $U \otimes U|\Psi^-\rangle = |\Psi^-\rangle$ and hence will give the same probabilities for outcomes of measurement in all basis provided they are identical for both parties.

This can not be mimic by a separable state and moreover it can not be mimic by any local, realistic theory due to violation of Bell inequalities by Quantum mechanics.

> Correlations in Quantum theory are present also in the case of full knowledge about the system – entanglement, and measurements on such systems cannot be treated as merely revealing previously encoded values.

7.2 No signaling

Presence of entangled states in quantum theory prompted some ideas of superluminal communication. Consider again the singlet state:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\leftrightarrow\rangle \otimes |\updownarrow\rangle - |\updownarrow\rangle \otimes |\leftrightarrow\rangle). \tag{7.7}$$

and imagine A performs the measurement in $|\leftrightarrow\rangle, |\updownarrow\rangle$ basis. If A gets the result $\leftrightarrow$ the state is projected to $|\leftrightarrow\rangle \otimes |\updownarrow\rangle$ if A gets the result $\updownarrow$ the state is projected to $|\updownarrow\rangle \otimes |\leftrightarrow\rangle$. So clearly the measurement of A *changes* the state of B. Moreover, the change is

instantaneous. So does it allow for super-luminal communication? If A could influence which measurement result will occur he could clearly send information, but results are probabilistic and he does not have power to choose which result should happen. But A could attempt the following strategy: I will consider two set of measurements Π_i^A, $\Pi_i'^A$, when I want to send 0 I will choose the measurement Π_i^A, when I want to send 1 I will perform the measurement $\Pi_i'^A$, since different measurements sets project on different states B may have chance to recognize which measurement I have performed.

Specifically, when wanting to transmit 0, A performs measurement in $|\leftrightarrow\rangle, |\updownarrow\rangle$ basis, if and in order to transmit 1, A performs measurement in $|\nwarrow\!\!\searrow\rangle, |\swarrow\!\!\nearrow\rangle$ basis. We get the following communication channel:

$$A \text{ sends } 0, \quad B \text{ gets } |\leftrightarrow\rangle \text{ or } |\updownarrow\rangle \text{ with probabilities } 1/2 \tag{7.8}$$
$$A \text{ sends } 1, \quad B \text{ gets } |\nwarrow\!\!\searrow\rangle \text{ or } |\swarrow\!\!\nearrow\rangle \text{ with probabilities } 1/2 \tag{7.9}$$
$$\tag{7.10}$$

Can B distinguish these two cases? It would be enough if B could distinguish them with probability higher then 1/2 (pure guessing), it would allow super-luminal communication. Unfortunately (or fortunately) he cannot.

7.2.1 No signaling theorem

B measurement probabilities are independent of whether A chose to perform Π_i^A or $\Pi_i'^A$ measurement. Let ρ be a general state of the whole system. Joint probability of outcomes of local measurements reads:

$$p(i,j) = \mathrm{Tr}(\rho \Pi_i^A \otimes \Pi_i^B). \tag{7.11}$$

Probability distribution of results of B read:

$$p_B(j) = \sum_i \mathrm{Tr}(\rho \Pi_i^A \otimes \Pi_i^B) = \mathrm{Tr}(\rho (\sum_i \Pi_i^A) \otimes \Pi_i^B) = \mathrm{Tr}(\rho \; \mathbb{1}_A \otimes \Pi_i^B). \tag{7.12}$$

So clearly B probabilities does not depend on the measurement of A.

7.2.2 Reduced density matrix

Since B probabilities does not depend on what is happening in A subsystem, one can introduce an object that will represent the state of B when the state of A is ignored. Looking at Eq. (7.12) one can rewrite it as follows:

$$p_B(j) = \mathrm{Tr}(\rho \; \mathbb{1}_A \otimes \Pi_i^B) = \mathrm{Tr}(\rho_B \; \Pi_i^B), \tag{7.13}$$

where $\rho_B = \mathrm{Tr}_A(\rho)$ is a partial trace of ρ over subsystem A. ρ_B is called the reduced density matrix for subsystem B. It carries all information necessary to predict probabilities of local measurements of B.

7.2.3 If discrimination of nonorthogonal states or cloning was possible super-luminal signaling would be possible

Notice that what prevents super-luminal communication is the impossibility for B to distinguish between two situations: (i): $|\leftrightarrow\rangle$, $|\updownarrow\rangle$ with probabilities 1/2, (ii) $|\nwarrow\searrow\rangle$, $|\swarrow\nearrow\rangle$ with probabilities 1/2. If, however, distiguishability of these 4 state was possible B would know which state he received and hence know what bit A wanted to send. Similarly if perfect cloning was possible B could first produce arbitrary large number of clones for each of these state and hence make them distinguishable.

7.3 Teleportation

We have learned in previous chapters that if given a single copy of an unknown state it is impossible to determine it precisely. We can neither clone nor measure it in a way that will allow us to learn what state we were given.

The problem of teleportation is the following. Consider two parties A and B that a share an entangled two qubit state. Additionally one of the parties A has an unknown qubit state $|\phi\rangle = a|0\rangle + b|1\rangle$ which she intends to send to B. She does not want to send it physically however, but rather use an entangled state they share and possibly classical communication channel in order to make the state $|\phi\rangle$ to "emerge" in B laboratory. Amazingly this is indeed possible, and even though A does not know what state she teleports, the state "reaches" B without any disturbance (in theory of course).

Let A and B share an entangled state $|\Psi^+\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$. Together with the qubit to be teleported the full state reads:

$$|\phi\rangle \otimes |\Psi^+\rangle = (a|0\rangle + b|1\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \tag{7.14}$$

Notice, however, that the above state can be equivalently written as:

$$\frac{1}{2\sqrt{2}}[(|01\rangle + |10\rangle) \otimes (a|0\rangle + b|1\rangle) + (|01\rangle - |10\rangle) \otimes (a|0\rangle - b|1\rangle) + (|00\rangle + |11\rangle) \otimes (a|1\rangle + b|0\rangle) + (|00\rangle - |11\rangle) \otimes (a|1\rangle - b|0\rangle)] \tag{7.15}$$

Hence if A performs a joint measurement on her two qubits in basis $|\Psi^+\rangle$, $|\Psi^-\rangle$, $|\Phi^+\rangle$, $|\Phi^-\rangle$, (Bell measurement), with probability 1/4, it will project B qubit on one of four states:

$$a|0\rangle + b|1\rangle, \quad a|0\rangle - b|1\rangle, \quad a|1\rangle + b|0\rangle, \quad a|1\rangle - b|0\rangle. \tag{7.16}$$

If no further information is transmitted from A to B, then the qubit of B is effectively in a completely mixed state (as it should be, since it was before A performed the measurement and no super-luminal communication is possible). If, however, A informs B, which measurement result she has obtained (which she can do via a classical channel), then depending on the result obtained: $|\Psi^+\rangle$, $|\Psi^-\rangle$, $|\Phi^+\rangle$ or $|\Phi^-\rangle$, B applies to his qubit a respetive unitary operation: $\mathbb{1}$, σ_z, σ_x or σ_y. As a result B always ends with his qubit in $|\phi\rangle$ state.

In a concise form we can write the teleportation as a 3 qubit to 3 qubit channel:

$$\sum_{i=0}^{3} |\Psi_i\rangle\langle\Psi_i| \otimes \sigma_i \left(|\phi\rangle\langle\phi| \otimes |\Psi^+\rangle\langle\Psi^+|\right) |\Psi_i\rangle\langle\Psi_i| \otimes \sigma_i^\dagger = \frac{1}{4}\mathbb{1} \otimes |\phi\rangle\langle\phi|. \tag{7.17}$$

where $|\Psi_i\rangle$ denote 4 Bell states, σ_i Pauli matrices, where $\sigma_0 = \mathbb{1}$.

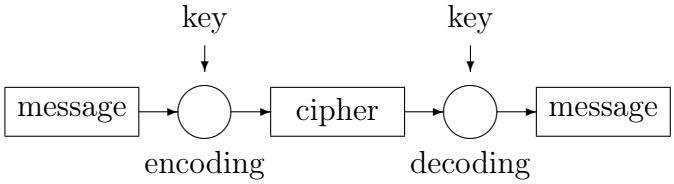
Chapter 8

Quantum eavesdropping - Quantum->Quantum->Classical channel

8.1 Cryptography

The science of cryptography is about transmitting a messages in the way that no illegitimate party can learn its meaning. One of the earliest cryptographic method was *Ceasar cipher* in which a letter in a message was replaced by a letter k places further in the alphabet. If we took $k = 3$ then CEASAR would be encoded as FADVDU. Such a code can be broken easily once one knows that the message was encoded using Ceasar cipher. One simply has to check all possible values of k, which is the number of letters in the alphabet – 26, which is not a great amount of work.

The general scheme in cryptography can be depicted as follows:



In case of the Ceasar cipher the message is CEASAR, the cipher is FADVDU and the key is $k = 3$.

A more general cipher is the substitution cipher, where each letter in mapped onto another letter. The Ceasar cipher is an example of substitution cipher. In a general substitution cipher we have 26! possibilities. Hence, such a cipher can not be broken by checking all possible letter substitutions, and thus is more secure than the Ceasar cipher. Nevertheless, it can be broken by letter frequency analysis, since each language has its particular letter frequency pattern, and one can quickly find out which letter was substituted to which one by investigating frequencies in which they appear.

Actually almost all used ciphers can theoretically be broken, and they strength stems from the practical difficulties of doing so. Nevertheless, if one really want to

use a cipher which is *proven* to be secure then there is such a cipher: *the one time pad*. Write your message in the binary form, take the key which is the a completely random sequence of 0 and 1 of the same length as the message and perform bitwise XOR operations to obtain the cipher

message	0	1	1	0	1	1	0
key	0	0	0	1	0	1	1
cipher = message $\oplus$ key	0	1	1	1	1	0	1

Notice that since the key is completely random so is the message. More formally let K be the random variable associated with the key. Let the key have length n. Complete randomness means that all binary sequences are equally probable: $p(K) = 1/2^n$. The cipher is obtained as $C = M \oplus K$, where M is the message. What means perfect security? It means that the cipher carries no information about the message for someone who does not know the key. This corresponds to the statement that mutual information between M and C is zero: $I(M : C) = 0$.

Proof security of one time pad. Let $p(M)$ be probability distribution of messages transmitted. The conditional probability $p(C|M)$ that a cipher C is obtained from message M reads:

$$p(C|M) = \sum_K p(K)\delta_{C,M\oplus K} = 1/2^n. \tag{8.1}$$

Hence obviously we have $p(C) = 1/2^n$. This means the cipher is completely random. The mutual information $I(M : C)$ reads:

$$I(M : C) = H(C) - H(C|M) = n - \sum_{C,M} p(C|M)p(M)\log_2 p(C|M) = 0. \tag{8.2}$$

Hence the one time pad is secure.

The one time pad has one drawback which makes it impractical for real life communication: it must be the same length as the transmitted message, otherwise the mutual information $I(M : C)$ is not zero and the cipher could in principle be broken. The main obstacle is thus the effective distribution of the random key to legitimate parties. One of the most promising techniques for doing this is *quantum key distribution* (QKD).

8.2 Quantum key distribution

We have observed in previous chapters that nonorthogonal quantum states cannot be distinguished nor cloned perfectly. This *inaccessibility* of quantum states which seems only a nuisance at a first glance proves to be the key to secure information transmission.

8.2.1 BB84 protocol

Let us describe here the most famous protocol proposed by Bennet and Brassard in 1984 (BB84). Consider two parties A, B, which are connected by a quantum channel

allowing for transmission od qubits (e.g. an optical fiber in through which single photons are sent), and a classical channel (e.g. telephone). We assume that noth channels are insecure and can be subjected to eavesdropping. We only assume that classical channel is authenticated i.e. A and B know that they talk to each other and their classical messages although potentially tapped cannot be altered.

A and B will use photon polarization for qubits transmitted via the quantum channel. A will send to B one of four states: $|\leftrightarrow\rangle, |\updownarrow\rangle, |\nearrow\rangle, |\nwarrow\rangle$ randomly with equal probabilities. We will say that the first two states form basis 1, and the last two basis 2. A and B assign logical values to this states as follows:

basis 1		basis 2	
$\|\leftrightarrow\rangle$	$\|\updownarrow\rangle$	$\|\nearrow\rangle$	$\|\nwarrow\rangle$
0	1	0	1

(8.3)

B measures the polarization state of an incoming photon randomly in one of two basis. If he measures in the correct basis his results should be perfectly correlated with bits sent, whereas when he measures in the incorrect basis his results will be completely uncorrelated with that of A. After the transmission took place B communicates to A via the classical channel in which basis he performed the measurement in a given run. He does not reveal, however, the actual results obtained. After this communication A i B keep only bits measured in compatible basis (approximately half). We call this a sifting stage. In ideal situation A and B should have perfectly correlated bits.

A	$\leftrightarrow$	$\nearrow$	$\nearrow$	$\nwarrow$	$\updownarrow$	$\updownarrow$	$\leftrightarrow$	$\nearrow$
B	$+$	$\times$	$+$	$+$	$+$	$\times$	$\times$	$\times$
compatible?	✓	✓			✓			✓
key	0	0			1			0

Now the quantum features enter the game. How A and B can be sure that they share bits that nobody else know about – i.e. that they have a one time pad. Put in one sentence this can be stated as follows:

> You can not distinguish perfectly between 4 states used in BB84, and moreover you cannot learn anything about their identity without introducing disturbance.

Hence, A and B can make themselves sure that nobody have eavesdropped on their communication, by revealing part of their bits on classical channel (e.g. 100 bits), and checking whether they all agree. If there are no errors they can be sure with high degree of confidence (the higher the more bits they have revealed) that nobody had eavesdropped. If all bits agree, the revealed bits are of course discarded, while the remaining ones are kept and constitute the one time pad. If there are errors in bits reveled, however, A and B it suggest a presence of an eavesdropper and hence they abort communication and try again.

The above scenario is oversimplified, and also impractical. In reality there will always be errors in communication even if there is no eavesdropped, but which result

from noise in the channel, imperfect detectors etc. Thus we need a more sophisticated approach: What is the tolerable error rate below which we can in some way distill a one time pad that will have no errors and will be secure i.e. no third party will have any information on it. This can be done using classical methods of error correction and privacy amplification.

8.3 Classical tools

8.3.1 Error correction

Asymptotic considerations. Imagine A and B share correlated bit strings of length n, where $p(A, B)$ is the probability distribution of the strings. We assume that $p(A, B) = p(a_1, b_1) \cdot \cdots \cdot p(a_n, b_n)$, which means each pair of bits is independently identically distributed. From chapter 2 we know that the mutual information $I(A : B) = nI(a : b)$ where by $I(a : b)$ we mean the mutual information corresponding to single bit random variables. Error correction is the process in which we allow A and B to exchange additional m bits of information in order to correct all errors and have perfectly correlated strings i.e. $I'(A : B) = n$. If strings are long we can addapt Shannon typical sequences technique for proving capacity of noisy channels. If $I(A : B) = nI(a : b)$ it means there is approximately $2^{n(1-I(a:b))}$ typical sequences of B that could in principle have been created from a given sequence od A and vice versa. Hence in order to identify uniquely a sequence in B with a sequence in A one needs to send

$$m = n(1 - I(a : b)). \tag{8.4}$$

In other words one needs $n(1 - I(a : b))$ additional bits in order to correct all errors. This is of course a theoretical bound. Real schemes will perform usually worse, yet the longer is the sequence the closer they can achieve the bound. More explicitly, if q is the probability of single bit error, then $I(a : b) = 1 - (-q \log_2 q)$ and hence the number of bits needed to be communicated is:

$$m = -nq \log_2 q. \tag{8.5}$$

Error correction in practice. One grasp the intuition of error correction by considering the simplest example when $n = 2$. Let p be the probability that a given bit of A is the same as the bit of B. Let A take her two bits, calculate XOR function on them and communicate the result to B. B checks whether the XOR function of A bits agrees with XOR function of his bits. If this is true they keep their bits unchanged and if not A send the value of the first bit to B – hence she effectively has sent all two bits. Notice that the only possibility that bits of A and B disagree is that they were error on both bits. Hence Hence after this error correction probability that A and B bits agree is $p' = 1 - (1 - p)^2 = p(2 - p) \geq p$ (for $p \in [0, 1]$). In this operations A has to send to B on average: $m = 1 + 2p(1 - p)$ bits. This is usually than the theoretical bound.

For larger n one could also use a strategy using pairs of bits but this strategy is not very efficient – usually to many bits have to be communicated (at least $n/2$). The

following strategy is more efficient. Let us denote by q the approximate fraction of errors between strings of A and B. In QKD e is estimated from the revealed part of shared bits. Usually before error correction A and B apply a common random permutation of their bits in order not to distinguish any of them. After that they choose a block length k such that $kq < 1$ i.e. such that it will be a very rare case that there is more than one error in the block, and divide their sequences in blocks of length k. A transmits the XOR function of bits in each block and communicate this to B. If XOR values of a given block agrees in A and B sequences they keep them intact. If they differ, they divide the block by half and A transmits XOR value calculated on subblocks. The subblock in which XOR values agree they leave it intact and divide by half the subblock in which XOR values differ, and continue this procedure until errors are localized. Doing so, sending approximately $(n/k)\log_2 k$ bits they can correct all errors provided there were no more than one error in a block of length k. Notice that when $kq = 1$ the number of bits communicated in this phase equals the theoretical bound in Eq. 8.5. However, there still may be errors remaining,due to the fact that there could have been more than one error in each block. Hence one has to repeat the procedure but this time with larger blocks k', since now the probability of an error is smaller $q' < q$. A few repetitions and checking XOR functions should eliminate most errors. When only a few errors are left dividing into blocks is not particulary effective we simply, take a random subset and calculate its XOR, if for e.g. 20 random subsets XOR values agree we have $2^(- 20)$ probability that an error is still there.

8.3.2 Privacy amplification

In cryptography what we really need to consider is the three party probability distribution $p(A, B, E)$, where E represents the data acquired by an eavesdropper. After the error correction procedure A and B have the same sequences i.e. $I(A : B) = n$, but most probably E also share some knowledge on them. If, however, E knowledge is not perfect i.e. $I(A : E) < n$, and $I(B : E) < n$ then A and B can perform so called privacy amplification procedure reducing their number of bits from n to n' but making them completely unknown for E.

Let us start with the simplest example when $n = 2$. A and B have two identical bit sequences of length 2. Let q be the probability of making an error for E when trying to deduce the value of a bit of A or B. A and B can perform XOR operation on their bits and keep its value, but unlike in error correction procedure they do not announce it. In effect they have shortened their sequence to one bit. What is the error probability q', E will make when guessing this value. She will not make an error only if she knew correct values of both bits or made errors in predicting value of both of them, hence: $q' = 1 - q^2 - (1 - q)^2 = 2q(1 - q) \geq q$ (for $q \in [0, 1/2]$). Hence her knowledge about bits of A and B will decrease.

In practice when n is large the above strategy will be applied to large blocks. Namely after assessing q, we take k such that ????$k(1-e)$??? $\simeq 1$, and apply a hashing function $f : \{0,1\}^n \mapsto \{0,1\}^k$ (the hashing function takes a binary sequence of length n and returns a binary sequence of length k, where a bit in the output sequence is a XOR

function of a random subset of input sequence) to bits of A and B. In this way their sequences will be shortened from n to k bits but the sequences will become completely unknown to E. The important question is how large k one can take and still be sure that E has no information on sequences of A and B. Intuitively of course the larger is the E information on A and B the smaller has to be k. This intuition is formalized in the Csiszár-Körner theorem, which combines both privacy amplification and error correction considerations.

8.3.3 Csiszár-Körner theorem

Using one way error-correction and privacy amplification, the number of secret bits k, A and B can distill is bounded:

$$k \leq \max\{I(A:B) - I(A:E), I(A:B) - I(B:E)\} \tag{8.6}$$

hence provided that E is less correlated with either A or B than they are with each other distillation of secret key is possible.

8.4 Attacks on the QKD

In order to apply error correction and privacy amplification we need to know how much E could have possibly learned about bits of A and B. Judging by the qubit error rate estimated from the revealed part of the bits A and B should find out what is the optimal attack E could have performed which allowed her to gain largest possible amount of information.

At the moment we will restrict ourselves to a simple class of attacks called intercept and resend attacks, which are not optimal, and hence considering only them does not guarantee full security, but are often considered since they are the only realistic attacks under present technology.

8.4.1 Intercept and resend attacks on BB84

In general, in intercept and resend attacks (IRA), E first measures incoming qubit in some basis and after learning result of the measurement and prepares a corresponding state which she sends to B. Ideally (for E of course) she would like to learn what state was sent and resend exactly the same state to B in order not to be detected.

In BB84, two basis are used for communication, basis 1:$|\leftrightarrow\rangle, |\updownarrow\rangle$, and basis 2: $|\nearrow\rangle, |\nwarrow\rangle$. During transmission E does not know which basis she should measure in since this is revealed only after all qubits has been sent. Consider two strategies she may choose:

1. Measurement in a randomly chosen basis – with probability 1/2, E measures either in $|\leftrightarrow\rangle, |\updownarrow\rangle$ or in a $|\nearrow\rangle, |\nwarrow\rangle$ basis

2. Measurement in an intermediate basis – every time E measures in $|22.5^\circ\rangle, |112.5^\circ\rangle$, which is an basis "in between" two basis used in BB84

Let us calculate what is the information gained by E in each of this attacks and what disturbance this attacks cause in the data of A and B.

Random basis In half of the cases E will measure in correct basis, hence will learn the state and transmit the state without any disturbance. In the second half, she will measure in the wrong basis. Since $|\langle\leftrightarrow|\nearrow\rangle|^2 = |\langle\leftrightarrow|\nwarrow\rangle|^2 = 1/2$ and $|\langle\updownarrow|\nearrow\rangle|^2 = |\langle\updownarrow|\nwarrow\rangle|^2 = 1/2$, she will obtain a correct measurement result with probability $1/2$. She will resend, a state in the wrong basis, however, and consequently B has $1/2$ probability of registering an error in communication even though his basis is set according with that of A. Summarizing B on average will observe qubit error rate (QBER) $QBER = 1/4$. Probability that E will measure an incorrect bit sent by A is $1/2 \cdot 1/2 = 1/4$, hence errors will be the same as between A and B (Notice also that E also with probability $1/4$ has an error on bit of B). Summarizing:

$$I(A:B) = 1 - h[1/4] \approx 0.189 \tag{8.7}$$

where $h[x_] = -x\log_2 x - (1-x)\log_2(1-x)$ is binary Shannon entropy, and the identical equation should hold for $I(A:E)$ and $I(B:E)$. However, in the above considerations we have neglected an important fact, namely after A and B announce basis they have used, E knows the cases when she measured in the correct basis. When she measured in the correct basis she has full knowledge on the bit, while when she measured on the wrong basis she learns nothing, consequently the true mutual information reads:

$$I(A:E) = I(B:E) = 1/2. \tag{8.8}$$

Obviously if A and B measure $QBER = 1/4$ they should abort their communication since E in principle could have gained more information than they. Let us now consider a more general situation in which E intercept only r fraction of incoming qubits. In this case, $QBER = r/4$, and consequently:

$$I(A:B) = 1 - h[r/4] \tag{8.9}$$

$$I(A:E) = I(B:E) = = r/2 \tag{8.10}$$

For $r \simeq 0.6821$ which $I(A:B) = I(A:E)$, this corresponds to $QBER = 0.171$. This tells us that if $QBER \geq 0.171$, we cannot distill any secret key since an eavesdropper could have obtained the same amount of information as we have using IRA in random basis. Taking a positive approach, if we assume an eavesdropper was restricted to perform IRA in random basis and we detect $QBER < 0.171$ we can distill some secret key which maximal length is given by Ciszár-Körner criterion and reads:

$$k \leq 1 - h[r/4] - r/2 = 1 - h[QBER] - 2QBER. \tag{8.11}$$

Intermediate basis attack Using intermediate basis, probability that E measures a wrong bit value is

$$q = |\langle 22.5^\circ| \updownarrow\rangle|^2 = 1/4(2-\sqrt{2}) \simeq 0.146. \tag{8.12}$$

Notice that this error is smaller than average error in random basis attack. Such an attack induces $QBER = 2q(1-q) = 1/4$:

$$I(A:B) = 1 - h[1/4] \approx 0.189 \tag{8.13}$$

Unlike in random basis attack, learning what basis was used in a given run does not provide E with any additional information, hence

$$I(A:E) = I(B:E) = 1 - h[q] \simeq 0.399, \tag{8.14}$$

which is smaller than in random basis attack. Even though average probability of error is smaller for E in intermediate basis attack the mutual information is smaller due to lack of certainty in which cases bits were correct and in which they were useless. Assume again that E intercept only a fraction r of incoming qubits, we get:

$$I(A:B) = 1 - h[r/4] \tag{8.15}$$
$$I(A:E) = I(B:E) = r(1-h[q]), \tag{8.16}$$

When $r \simeq 0.7548$, $I(A:B) = I(A:E) = I(B:E)$. This corresponds to $QBER = 0.189$, so considering intermediate basis attacks we get a bit higher QBER rate thresholds above which we cannot distill secret key. This is due to the fact that intermediate attacks are less efficient from the point of view of an eavesdropper than random basis attacks.

8.4.2 Optimal individual attack on BB84

Let us denote the four states used in BB84 protocols $|+z\rangle, |-z\rangle, |+x\rangle, |-x\rangle$, where $|\pm z\rangle$, $|\pm x\rangle$ are the eigenstates of σ_x and σ_z. A general individual attack on BB84 is a unitary operation U acting on the space which decribes states of the qubit send by A to B and the eavesdropper space:

$$U: \ |\pm k\rangle_B \otimes |0\rangle_E \mapsto \sqrt{F}|\pm k\rangle_B \otimes |\Psi_0^{\pm k}\rangle_E + \sqrt{D}|\mp k\rangle_B \otimes |\Psi_1^{\pm k}\rangle_E \tag{8.17}$$

where $k = x, z$ represents the basis chosen, D is equal to the QBER, and $F = 1 - D$. Since we just have four different input states to consider, we can without loosing generality limit the E space to 4 dimensions. Let us write the transformation when acting on the states from the z basis:

$$U: \ |+z\rangle_B \otimes |0\rangle_E \ \mapsto \ \sqrt{F}|+z\rangle_B \otimes |\Psi_0^{+z}\rangle_E + \sqrt{D}|-z\rangle_B \otimes |\Psi_1^{+z}\rangle_E \tag{8.18}$$
$$U: \ |-z\rangle_B \otimes |0\rangle_E \ \mapsto \ \sqrt{F}|-z\rangle_B \otimes |\Psi_0^{-z}\rangle_E + \sqrt{D}|+z\rangle_B \otimes |\Psi_1^{-z}\rangle_E \tag{8.19}$$

Unitarity implies the following constraint:

$$\langle \Psi_0^{+z} | \Psi_1^{-z} \rangle + \langle \Psi_1^{+z} | \Psi_0^{-z} \rangle = 0 \tag{8.20}$$

We assume the attack is symmetric which means that the output reduced density matrix has a Bloch vector which is shrinked but not rotated compared with the input one. Notice that this is not a limitation, since such rotation is useless as it increases the QBER while not providing any additional information for E, and can always be canceled by the eavesdropper. This implies that

$$\langle \Psi_0^{\pm z} | \Psi_1^{\pm z} \rangle = 0. \tag{8.21}$$

Now we look how the transformation acts on the state from the x basis. Recall that $|\pm x\rangle = (|+z\rangle \pm |-z\rangle)/\sqrt{2}$, hence the transformation reads:

$$\begin{aligned} U: \ |+x,0\rangle \mapsto & \sqrt{\frac{F}{2}}(|+z,\Psi_0^{+z}\rangle + |-z,\Psi_0^{-z}\rangle) + \sqrt{\frac{D}{2}}(|-z,\Psi_1^{+z}\rangle + |+z,\Psi_1^{-z}\rangle) = \\ & = \sqrt{F}|+x\rangle \otimes \frac{1}{2}(|\Psi_0^{+z}\rangle + \sqrt{\frac{D}{F}}|\Psi_1^{-z}\rangle + \sqrt{\frac{D}{F}}|\Psi_1^{+z}\rangle + |\Psi_0^{-z}\rangle) + \\ & + \sqrt{D}|-x\rangle \otimes \frac{1}{2}(\sqrt{\frac{F}{D}}|\Psi_0^{+z}\rangle + |\Psi_1^{-z}\rangle - |\Psi_1^{+z}\rangle - \sqrt{\frac{F}{D}}|\Psi_0^{-z}\rangle) = \\ & = \sqrt{F}|+x\rangle \otimes |\Psi_0^{+x}\rangle + \sqrt{D}|-x\rangle \otimes |\Psi_1^{+x}\rangle \end{aligned} \tag{8.22}$$

and analogously for $|-x\rangle$. If the attack must treat the x basis in the same way as the z basis, we must have

$$\langle \Psi_0^{+x} | \Psi_1^{+x} \rangle = 0 \tag{8.23}$$

This implies:

$$\mathrm{Re}\langle \Psi_0^{+z} | \Psi_1^{-z} \rangle - \mathrm{Re}\langle \Psi_1^{+z} | \Psi_0^{-z} \rangle = 0 \tag{8.24}$$

$$\sqrt{\frac{F}{D}}\mathrm{Im}\langle \Psi_0^{-z} | \Psi_0^{+z} \rangle + \sqrt{\frac{D}{F}}\mathrm{Im}\langle \Psi_1^{+z} | \Psi_1^{-z} \rangle = 0 \tag{8.25}$$

Without loosing generality, we can always redefine: $|\Psi_1^{+z}\rangle \to e^{i\varphi}|\Psi_1^{+z}\rangle$, such that $\mathrm{Im}\langle \Psi_1^{+z} | \Psi_1^{-z} \rangle = 0$. From Eq. (8.25) it follows that $\mathrm{Im}\langle \Psi_0^{+z} | \Psi_0^{-z} \rangle = 0$. Finally taking into account Eq. (8.24) together with Eq. (8.20) we arrive at:

$$\langle \Psi_0^{+z} | \Psi_1^{-z} \rangle = \langle \Psi_1^{+z} | \Psi_0^{-z} \rangle = 0. \tag{8.26}$$

As a result without losing generality we can parameterize the states as:

$$|\Psi_0^{+z}\rangle = [1,0,0,0] \tag{8.27}$$

$$|\Psi_1^{+z}\rangle = [0,1,0,0] \tag{8.28}$$

$$|\Psi_0^{-z}\rangle = [cos\alpha, 0, sin\alpha, 0] \tag{8.29}$$

$$|\Psi_1^{-z}\rangle = [0, cos\beta, 0, sin\beta] \tag{8.30}$$

Moreover, if the quality of the attack is to be the same in the x basis as in the z basis states $|\Psi_0^{\pm x}\rangle$, $|\Psi_1^{\pm x}\rangle$ need to be normalized:

$$\langle\Psi_0^{+x}|\Psi_0^{+x}\rangle = \langle\Psi_1^{+x}|\Psi_1^{+x}\rangle = 1 \quad (8.31)$$

$$\langle\Psi_0^{-x}|\Psi_0^{-x}\rangle = \langle\Psi_1^{-x}|\Psi_1^{-x}\rangle = 1 \quad (8.32)$$

$$(8.33)$$

This implies

$$1 + \cos\alpha + \frac{D}{F}(\cos\beta + 1) = 2 \quad (8.34)$$

And finally recalling that $D = 1 - F$, the fidelity F reads:

$$F = \frac{1 + \cos\beta}{2 + \cos\beta - \cos\alpha} \quad (8.35)$$

The attack is thus parameterized with two real parameters α, β.

Let us assume that the z basis was used in a given run of the protocol. The goal of the eavesdropper is to infer the value of the bit. Notice that the space spanned by $|\Psi_1^{+z}\rangle, |\Psi_1^{-z}\rangle$ is orthogonal to the one spanned by $|\Psi_0^{+z}\rangle, |\Psi_0^{-z}\rangle$. This means that by projecting on one of these subspaces E knows for sure whether he inflicted an error in the transmission or not. After projecting on the subspaces E has to distinguish between $|\Psi_1^{+z}\rangle$ and $|\Psi_1^{-z}\rangle$ or between $|\Psi_0^{+z}\rangle$ and $|\Psi_0^{-z}\rangle$. For this purpose E uses the optimal discrimination of two non-orthogonal states strategy from Sec.5.1.1. The information she gains can therefore be written in the form

$$I(A : E) = F\left(1 - h\left[\frac{1 + \sin\alpha}{2}\right]\right) + D\left(1 - h\left[\frac{1 + \sin\beta}{2}\right]\right). \quad (8.36)$$

Fixing F (i.e. fixing the QBER) the above information is maximal for $\alpha = \beta$. Finally we get that depending on α the QBER reads:

$$QBER = \frac{1 - \cos\alpha}{2}, \quad (8.37)$$

while the mutual informations:

$$I(A : B) = 1 - h[QBER] \quad (8.38)$$

$$I(A : E) = I(B : E) = 1 - h\left[\frac{1 + \sin\alpha}{2}\right] \quad (8.39)$$

Notice that $I(A : E) = I(B : E)$ this follows simply from the fact that E always knows whether he inflicted an error in communication from A to B. Looking for the QBER for which $I(A : B) = I(A : E)$ we obtain the QBER threshold:

$$QBER_{th} = \frac{1 - 1/\sqrt{2}}{2} = \approx 14.6\%. \quad (8.40)$$

If the QBER is above this threshold the BB84 protocol is not safe. If it is below, the protocol is safe against individual attacks.

8.5 Other QKD protocols

8.5.1 Six state protocol

A natural generalization of BB84 is to use also circularly polarized basis $|\circlearrowright\rangle, |\circlearrowleft\rangle$. A send with probability 1/6 one of six states $|\leftrightarrow\rangle, |\updownarrow\rangle, |\nwarrow\rangle, |\nearrow\rangle, |\circlearrowright\rangle, |\circlearrowleft\rangle$, while B measures randomly in one of three basis. On average 2/3 of the bits will be discarded in the sifting phase. This protocol seems more secure than BB84, since it uses 3 different basis instead of 2, and hence make it harder for an eavesdropper it gain information under a given QBER. Nevertheless the fact that only 1/3 of bits is kept make this protocol less useful when one what to achieve higher transmission rates.

Intercept and resend attack on on 6S protocol

If the eavesdropper attack r fraction of the qubits, measures them randomly in one of three basis and resends the measured state the QBER he inflicts reads: $QBER = r/3$. The mutual informations read:

$$I(A:B) = 1 - h[QBER] \tag{8.41}$$
$$I(A:E) = r/3 \tag{8.42}$$

The QBER threshold corresponding to the situation when $I(A:B) = I(A:E)$ reads: $QBER = 22.7\%$.

Individual attack on 6S using the optimal universal asymmetric cloning

The 6S protocol is easier to investigate in terms of security thanks it its higher symmetry, than that of BB84. Apart from "measurement" attacks in which the eavesdropper measure qubits on-the-fly, a more sophisticated attack appear to be more powerful. Instead of measuring the qubit the eavesdropper can perfrom optimal $1 \rightarrow 2$ cloning operation send one copy to B and keep one for himself. After basis reconciliation process has taken place the eavesdropper can measure his clone in order to gain information on transmitted bits.

Let the eavesdropper perform the optimal universal asymmetric cloning described in Sec. 6.2.1, parameterized by two real parameters a, b subject to normalization constraint $a^2 + b^2 + ab = 1$ ($b = (-a + \sqrt{4 - 3a^2})/2$). He keeps the first clone for himself and send the second one to B. Given that A had sent the state $|\psi\rangle$, B and E obtain the following reduced states:

$$\rho_E = \left[\frac{(a+b)^2}{2} + \frac{a^2}{2}\right]|\psi\rangle\langle\psi| + \frac{b^2}{2}|\psi_\perp\rangle\langle\psi_\perp| \tag{8.43}$$
$$\rho_B = \left[\frac{(a+b)^2}{2} + \frac{b^2}{2}\right]|\psi\rangle\langle\psi| + \frac{a^2}{2}|\psi_\perp\rangle\langle\psi_\perp| \tag{8.44}$$

This implies that the QBER equals $a^2/2$. The eavesdropped wait until basis are announced and measures his copy in the correct basis. His probability of error equals

$b^2/2$. The mutual informations $I(A:B)$, $I(A:E)$ read:

$$I(A:E) = 1 - h[a^2/2] \tag{8.45}$$
$$I(A:B) = 1 - h[b^2/2] \tag{8.46}$$

where $h[x] = -x\log_2 x$. The QBER threshold corresponds to the symmetric case $a = b = 1/\sqrt{3}$, which yields $QBER = 1/6 = 16.7\%$, and $I(A:B) = I(A:E) = 0.35$. One can see that this attack is much more powerful than prepare and resend strategy.

There is a subtle issue to be mentioned. In what was said above we have concentrated only on $I(A:B)$, $I(A:E)$ quantities, ignoring completely $I(B:E)$, which according to the Csiszar-Korner theorem plays equivalent role to $I(A:E)$. As a result of the optimal asymmetric cloning attack the two clones are in the state:

$$\rho_{EB} = \frac{1}{2}\left[(a+b)^2|\psi,\psi\rangle\langle\psi,\psi| + (a|\psi,\psi_\perp\rangle + b|\psi_\perp,\psi\rangle)(a\langle\psi,\psi_\perp| + b\langle\psi_\perp,\psi|)\right] \tag{8.47}$$

After measuring in the correct basis (i.e., $|\psi\rangle$, $|\psi_\perp\rangle$) the mutual information $I(B:E)$ reads:

$$I(B:E) = 1 - h[(a+b)^2/2] \tag{8.48}$$

One can check that this information is always less then $I(A:E)$, hence strictly speaking the attack is not that powerful as it might had seemed. Nevertheless it is possible to modify the cloning procedure in such a way to make it "symmetric" (do not confuse with symmetric cloning) with respect to $I(A:E)$ and $I(B:E)$, without compromising clones quality.

8.5.2 Optimal individual attack on 6S protocol

We can repeat the derivation of the optimal individual attack on BB84 protocol from Sec. 8.4.2. The only difference is the higher symmetry of the problem which requires that we need to consider also the attack on $|\pm y\rangle$ states. This additional constraint leads to setting the parameter $\beta = \pi/2$. Hence the QBER as a function of α reads:

$$QBER = \frac{1-\cos\alpha}{2-\cos\alpha} \tag{8.49}$$

And the mutual informations:

$$I(A:B) = 1 - h[QBER] \tag{8.50}$$
$$I(A:E) = I(B:E) = 1 - \frac{1}{2-\cos\alpha}h\left[\frac{1+\sin\alpha}{2}\right] \tag{8.51}$$

Notice that because $\beta = \pi/2$ then if E learned that he inflicted a mistake in the transmission he knows for sure what is the bit value ($|\Psi_1^{+z}\rangle$ is orthogonal to $|\Psi_1^{-z}\rangle$). Otherwise he performs the optimal two state discrimination. The QBER threshold corresponding to $\alpha = 0.618686$, equals

$$QBER_{th} = 15.6\% \tag{8.52}$$

Obviously it is larger that the one for BB84, since the 6 state protocol is more difficult to eavesdrop due to more states present.

Relation to the optimal asymmetric cloning

Notice that the result seems to indicate that the previously consider optimal universal symmetric cloning attack is not optimal, as it yielded higher QBER threshold. The reason for this is that we chose the symmetric cloning as the one where we expect the mutual information $I(A:E)$ and $I(A:B)$ to be equal. However, we only equalized probabilities of errors i.e. clones qualities. From the present analysis we see that comparing error probabilities and informations is not equivalent, since there are two cases, either error was inflicted in A to B transmission and E knows the bit perfectly, or E has to perform discrimination.

Let us take the optimal asymmetric universal cloning machine parameterized with a, b ($b = (-a + \sqrt{4-3a^2})/2$). Using such a cloning we can obtain the following mutual informations

$$I(A:E) = \frac{b^2}{2} + \left(1 - \frac{b^2}{2}\right)\left(1 - h\left[\frac{1 - a^2/2 - b^2/2}{1 - b^2/2}\right]\right) \tag{8.53}$$

$$I(A:B) = 1 - h[b^2/2] \tag{8.54}$$

where $\frac{b^2}{2}$ is the probability of E inflicting an error in A to B transmission (QBER), in which case E learns the bit perfectly otherwise with probability $(1 - b^2/2)$ he learns the correct bit only with probability $\frac{1-a^2/2-b^2/2}{1-b^2/2}$ — on average the success probability of guessing the value of the bit is $1 - a^2/2$ which is in agreement with the cloning fidelity. Looking for the QBER threshold when $I(A:B) = I(A:E)$ we get: $a = 0.595275$, $b = 0.559238$. Notice that this is asymmetric cloning — we need to give B a copy of a bit higher quality, in order that out informations are equal. The corresponding QBER

$$QBER_{th} = 1 - \frac{b^2}{2} = 15.6\%, \tag{8.55}$$

Which proves that using the optimal universal cloning in a proper way is equivalent to the optimal eavesdropping.

8.5.3 B92

A natural question arises, if two nonorthogonal states cannot be perfectly distinguished, then maybe one can construct a QKD protocol using only two states instead of four as used in BB84. Amazingly this is indeed possible. A sends either $|\leftrightarrow\rangle$, or $|\swarrow\rangle$. B measures either in $|\leftrightarrow\rangle, |\updownarrow\rangle$ or in $|\swarrow\rangle, |\nwarrow\rangle$ basis. Unlike in BB84 he does not communicate the basis he used, but rather informs A about the cases in which he measured $|\updownarrow\rangle$ or $|\nwarrow\rangle$ (without specifying which of them). This is an information that tells A that in this run B had a basis incompatible with the one she used. Hence if she denotes by 0 and 1 the cases when she sends $|\leftrightarrow\rangle$ and $|\swarrow\rangle$ respectively, and B denotes by 0 and 1 the cases when he used basis $|\swarrow\rangle, |\nwarrow\rangle$ and $|\leftrightarrow\rangle, |\updownarrow\rangle$, they will have perfectly correlated bits at this positions. Notice also that there was no information revealed to an eavesdropper when B informed A about positions at which he measured $|\updownarrow\rangle$ or $|\nwarrow\rangle$. Moreover non perfect distinguishability of nonorthogonal states forces E

to induce errors whenever she wants to learn something and thus makes the protocol secure.

Chapter 9

Great unification

1. Asymptotic cloning is state estimation 2. Optimal individual eavesdropping is cloning + optimal discrimination