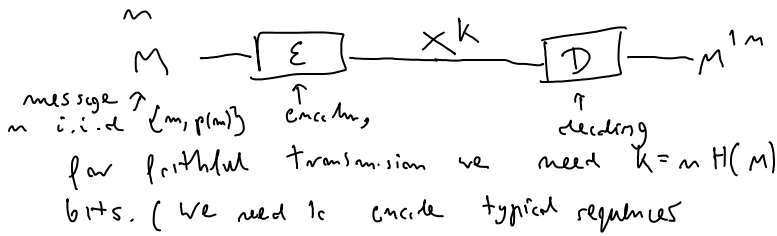


1. Quantum Compression

Recall Shannon Coding theorem:

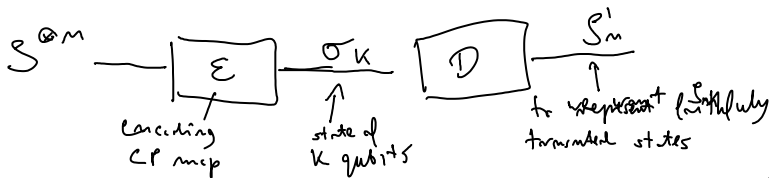


We want to formulate Quantum Coding theorem
 - faithful compression of quantum states.

Let our "messages" be pure states $|\psi_m\rangle$ with probability $p(m)$. Let us combine them into

$$S = \sum_m p(m) |\psi_m\rangle\langle\psi_m|$$

So by analogy we can draw our problem:



Remark It is not enough to require that $F(S^{\otimes n}, S'_n) = 1$
 since we know S we could just reproduce it at the output without using any channel. We need something more - we need to "transmit" quantum state
 - notice that similar issue arises in classical setting

$$S^{\otimes n} = \sum_{m_1, \dots, m_n} p(m_1) \dots p(m_n) |\psi_{m_1}\rangle\langle\psi_{m_1}| \otimes \dots \otimes |\psi_{m_n}\rangle\langle\psi_{m_n}|$$

$$|\psi_{m^n}\rangle\langle\psi_{m^n}| \rightarrow S'_{m^n} = D(E(|\psi_{m^n}\rangle\langle\psi_{m^n}|))$$

We want S'_{m^n} to be as close as possible to $|\psi_{m^n}\rangle\langle\psi_{m^n}|$

Average fidelity:

$$F = \sum_{m^n} p(m^n) \langle\psi_{m^n}| S'_{m^n} |\psi_{m^n}\rangle = \sum_{m^n} p(m^n) \text{Tr}(|\psi_{m^n}\rangle\langle\psi_{m^n}| S'_{m^n})$$

Reliable compression with rate R :

$$\forall \epsilon \exists n \forall m^n \exists M^k F \geq 1 - \epsilon \quad \text{where } k = n \cdot R$$

\leftarrow number of qubits used for encoding

Definition Von Neumann entropy

$$S(S) = -\text{Tr} S \log S$$

Let $S = \sum_x p_x |x\rangle\langle x|$ be eigenvalue decomposition

$$S(S) = -\sum_x p_x \log p_x = H(X)$$

If we generate n i.i.d. states according to $\{p(x), |\psi(x)\rangle\}$
 on average we have $S^{\otimes n}$. Eigenvalue decomposition

$$S^{\otimes n} = \sum_{x_1 \dots x_n} p_{x_1} \dots p_{x_n} |x_1\rangle\langle x_1| \otimes \dots \otimes |x_n\rangle\langle x_n|$$

With high probability we will have only states
 from "typical subspace":

$$P = \sum_{\{x\}\text{-typical seqs}} |x_1\rangle\langle x_1| \otimes \dots \otimes |x_n\rangle\langle x_n|$$

P - most nr. of states $2^{n \cdot S(S)}$ significant

(typical history) $n \cdot S(S)$ qubit of the transfer

$$\text{Tr}(S^{\otimes n} P) \approx 1 - \delta$$

$$S(S) = P S P + \sum_{\perp}(S)$$

everything outside the typical
 subspace goes to a fixed
 state...

What is the fidelity?

$$F = \sum_{m,m'} p(m) \left[\text{Tr}(|\psi_m\rangle\langle\psi_m| \rho(m, \epsilon) |\psi_{m'}\rangle\langle\psi_{m'}| \rho(m, \epsilon)) + \text{Tr}(|\psi_m\rangle\langle\psi_m| |0\rangle\langle 0|) \cdot \text{Tr}(|\psi_{m'}\rangle\langle\psi_{m'}| \cdot \rho^{\perp}(m, \epsilon)) \right]$$

$$\geq \sum_{m,m'} p(m) \underbrace{\text{Tr}(|\psi_m\rangle\langle\psi_m| \rho(m, \epsilon) |\psi_{m'}\rangle\langle\psi_{m'}| \rho(m, \epsilon))}_A$$

We know that $\text{Tr}(S^{\otimes n} P(m, \epsilon)) \geq 1 - \delta$

$$\text{Let } |\psi_m\rangle = \lambda_m |\psi_m^0\rangle + \mu_m |\psi_m^1\rangle \quad (|\psi^0\rangle \in T(m, \epsilon) \quad |\psi^1\rangle \in T^{\perp}(m, \epsilon))$$

$$= \sum_{m,m'} p(m) |\lambda_m \lambda_{m'}|^2 = \sum_{m,m'} p(m) (1 - \mu_m^2)^2 \geq 1 - 2 \sum_{m,m'} p(m) |\mu_m|^2$$

Notice however that

$$\text{Tr}(S^{\otimes n} P(m, \epsilon)) = \sum_{m,m'} p(m) \text{Tr}(|\psi_m\rangle\langle\psi_m| \rho(m, \epsilon)) = 1 - \sum_{m,m'} p(m) |\mu_m|^2$$

$$\geq 1 - \delta \quad \Rightarrow \quad \sum_{m,m'} p(m) |\mu_m|^2 \leq \delta \quad \Rightarrow$$

$$F \geq 1 - 2\delta$$

What is the fidelity?

$$F = \sum_{m,m'} p(m) \left[\text{Tr}(|\psi_{m,m'}\rangle\langle\psi_{m,m'}| \rho(m, \epsilon) |\psi_{m,m'}\rangle\langle\psi_{m,m'}| \rho(m', \epsilon)) + \text{Tr}(|\psi_{m,m'}\rangle\langle\psi_{m,m'}| |0\rangle\langle 0|) \cdot \text{Tr}(|\psi_{m,m'}\rangle\langle\psi_{m,m'}| \cdot \rho^\perp(m, \epsilon)) \right] \geq$$

$$\geq \sum_{m,m'} p(m) \underbrace{\left[\text{Tr}(|\psi_{m,m'}\rangle\langle\psi_{m,m'}| \rho(m, \epsilon) |\psi_{m,m'}\rangle\langle\psi_{m,m'}| \rho(m', \epsilon)) \right]}_A$$

We know that $\text{Tr}(S^{\otimes m} \rho(m, \epsilon)) \geq 1 - \delta$

$$\text{Let } |\psi_{m,m'}\rangle = \lambda_{m,m'} |\psi_m\rangle + \mu_{m,m'} |\psi_{m'}\rangle \quad (|\psi\rangle \in T(m, \epsilon) \quad |\psi'\rangle \in T^{\perp}(m, \epsilon))$$

$$= \sum_{m,m'} p(m) |\lambda_{m,m'}|^4 = \sum_{m,m'} p(m) (1 - |\mu_{m,m'}|^2)^2 \geq 1 - 2 \sum_{m,m'} p(m) |\mu_{m,m'}|^2$$

Notice however that

$$\text{Tr}(S^{\otimes m} \rho(m, \epsilon)) = \sum_{m,m'} p(m) \text{Tr}(|\psi_{m,m'}\rangle\langle\psi_{m,m'}| \rho(m, \epsilon)) = 1 - \sum_{m,m'} p(m) |\mu_{m,m'}|^2$$

$$\geq 1 - \delta \quad \Rightarrow \quad \sum_{m,m'} p(m) |\mu_{m,m'}|^2 \leq \delta \quad \Rightarrow$$

$$F \geq 1 - 2\delta$$

2. Quantum channel capacity

• Holevo bound

Let $X, p(x)$ be classical random variable, according to which we prepare states $S(x)$. How much information we can learn about X by measuring $S(x)$



$$p(x, y) = p(x) \cdot \text{Tr}(\Pi(y) S(x))$$

Theorem: $I(X:Y) \leq \underbrace{S(S) - \sum_x p_x S(S_x)}_{X\text{-Holevo quantity}}$ where

$$S = \sum_x p_x S_x$$

(bound on accessible information)

Lemma: strong subadditivity (without a proof)

$$S(S_{ABC}) + S(S_B) \leq S(S_{AB}) + S(S_{BC})$$

to conclude

$$S(A:B) \leq S(A:BC)$$

genu $S(A:B) = S(A) + S(B) - S(A,B)$

$$m \in M \rightarrow \boxed{\text{encoding}} \rightarrow S_m(m) \rightarrow \boxed{\text{channel}} \rightarrow S'_m(m) \rightarrow \boxed{\text{decoding}} \rightarrow M'$$

16 bitowy ciąg bitów można przetworzyć na ciąg

kantów: (zgrupować parę długości n)

Jeśli uwzględnić tylko stany produkcyjne S_m :

$$C(\lambda) = \chi(\lambda) = \max \left[\overbrace{S\left(\lambda \left(\sum_x p_x S_x\right)\right)}^{\text{q. equivalent of } H(Y)} - \overbrace{\sum_x p_x S(\lambda(S_x))}^{\text{q. equivalent of } H(Y|X)} \right]$$

copy if we restrict to encoding into product states

Wzrost pytania pyta o $C \stackrel{?}{=} C^{(1)}$