10-q. communication
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1. Quantum Compression

Recall Shannon coding theorem:

$$\overset{n}{M} \longrightarrow \boxed{\mathcal{E}} \xrightarrow{\;X^k\;} \boxed{D} \longrightarrow M'^n$$

message $\uparrow$ n i.i.d. $\{m, p(m)\}$ — encoding — decoding

for faithful transmission we need $k = n H(M)$ bits. (we need to encode typical sequences

We wants to formulate Quantum coding theorem – faithful compression of quantum states.

Let our „messages" be pure states $|\psi_m\rangle$ ~~prepared~~ with probability $p(m)$. Let us combine them into

$\rho = \sum_m p(m)\, |\psi_m\rangle\langle\psi_m|$. So by analogy we can draw our problem:

$$\rho^{\otimes n} \longrightarrow \boxed{\mathcal{E}} \xrightarrow{\;\sigma_k\;} \boxed{D} \longrightarrow \rho'_n$$

encoding CP map — state of k qubits — to represent faithfully transmitted states

Remark It is not enough to require that $F(\rho^{\otimes n}, \rho'_n) = 1$ ~~Since we~~ know ρ we could just reprepare it at the output without using any channel. We need something more. – we need to „transmit" quantum state – notice that similar issue arises in classical setting

$$\rho^{\otimes n} = \sum_{m_1 \ldots m_n} \underbrace{p(m_1) \cdot \ldots \cdot p(m_n)}_{\sum_{m^n} \cancel{p(m^n)}} \underbrace{|\psi_{m_1}\rangle\langle\psi_{m_1}| \otimes \ldots |\psi_{m_n}\rangle\langle\psi_{m_n}|}_{|\Psi_{m^n}\rangle\langle\Psi_{m^n}|}$$

$$|\Psi_{m^n}\rangle\langle\Psi_{m^n}| \longrightarrow \rho'_{m^n} = D\left(\mathcal{E}\left(|\Psi_{m^n}\rangle\langle\Psi_{m^n}|\right)\right)$$

We want ρ'_{m^n} to be as close as possible to $|\Psi_{m^n}\rangle\langle\Psi_{m^n}|$

Average fidelity:

$$F = \sum_{m^n} p(m^n)\, \langle\Psi_{m^n}|\rho'_{m^n}|\Psi_{m^n}\rangle = \sum_{m^n} p(m^n)\, \mathrm{Tr}\left(|\Psi_{m^n}\rangle\langle\Psi_{m^n}|\rho'_{m^n}\right)$$

Reliable compression with rate R:

$$\forall_{\varepsilon}\ \exists_{n_0}\ \forall_{n \geq n_0}\ F \geq 1-\varepsilon \quad \text{where } k = n \cdot R$$

$\nwarrow$ number of qubits used for encoding

Definition Von Neuman entropy

$$S(\rho) = -\mathrm{Tr}\, \rho \log \rho$$

Let $\rho = \sum_x p_x |x\rangle\langle x|$ be eigenvalue decomposition

$$S(\rho) = -\sum_x p_x \log p_x = H(X)$$

If we generate n i.i.d states according to $\{p(m), |\psi_m\rangle\}$ on average we have $\rho^{\otimes n}$. Eigenvalue decomposition

$$\rho^{\otimes n} = \sum_{x_1 \dots x_n} p_{x_1} \cdots p_{x_n} |x_1\rangle\langle x_1| \otimes \dots |x_n\rangle\langle x_n|$$

With high probability we will have only states from "typical subspace":

$$P = \sum_{\{x\}\text{-typical sequen}} |x_1\rangle\langle x_1| \otimes \dots \otimes |x_k\rangle\langle x_k|$$

P - rzut na podprzestrzeń $2^{n \cdot S(\rho)}$ wymiarową

Czyli wystarczy $n \cdot S(\rho)$ qubitów do kodowania

$$\mathrm{Tr}(\rho^{\otimes N} P) \approx 1 - \delta$$

$$\mathcal{E}(\rho) = P \rho P + \mathcal{E}_\perp(\rho)$$

everything outside the typical subspace goes to a fixed state..

What is the fidelity?

$$F = \sum_m p(m) \left[\mathrm{Tr}\left(|\psi_m\rangle\langle\psi_m| P(n,\varepsilon) |\psi_m\rangle\langle\psi_m| P(n,\varepsilon) \right) + \right.$$
$$\left. + \mathrm{Tr}\left(|\psi_m\rangle\langle\psi_m| \, |0\rangle\langle 0| \right) \cdot \mathrm{Tr}\left(|\psi_m\rangle\langle\psi_m| \cdot P^\perp(n,\varepsilon) \right) \right] \geq$$

$$\geq \sum_m p(m) \underbrace{\mathrm{Tr}\left[|\psi_m\rangle\langle\psi_m| P(n,\varepsilon) |\psi_m\rangle\langle\psi_m| P(n,\varepsilon) \right]}_{A}$$

We know that $\mathrm{Tr}(\rho^{\otimes n} P(n,\varepsilon)) \geq 1 - \delta$

Let $|\psi_m\rangle = \lambda_m |\varphi_m\rangle + \mu_m |\eta_m\rangle \qquad |\varphi\rangle \in T(n,\varepsilon) \quad |\eta\rangle \in T^\perp(n,\varepsilon)$

$$= \sum_m p(m) |\lambda_m|^4 = \sum_m p(m) (1 - |\mu_m|^2)^2 \geq 1 - 2 \sum_m p(m) |\mu_m|^2$$

Notice however that

$$\mathrm{Tr}(\rho^{\otimes n} P(n,\varepsilon)) = \sum_m p(m) \mathrm{Tr}(|\psi_m\rangle\langle\psi_m| P(n,\varepsilon)) = 1 - \sum_m p(m) |\mu_m|^2$$

$$\geq 1 - \delta \quad \Rightarrow \quad \sum_m p(m) |\mu_m|^2 \leq \delta \quad \Rightarrow$$

$$F \geq 1 - 2\delta$$

What is the fidelity?

$$F = \sum_{m^n} p(m^n) \Big[\mathrm{Tr}\big(|\psi_{m^n}\rangle\langle\psi^m_{m}| \, P(m,\varepsilon) |\psi_{m^n}\rangle\langle\psi_{m^n}| P(m,\varepsilon) \big) +$$

$$+ \mathrm{Tr}\big(|\psi_{m^n}\rangle\langle\psi^m_m| \, |0\rangle\langle 0| \big) \cdot \mathrm{Tr}\big(|\psi_{m^n}\rangle\langle\psi_{m^n}| \cdot P^{\perp}(m,\varepsilon) \big) \geq$$

$$\geq \sum_{m^n} p(m^n) \, \mathrm{Tr}\underbrace{\Big[|\psi_{m^n}\rangle\langle\psi^m_m| \, P(m,\varepsilon) \, |\psi_{m^n}\rangle\langle\psi_{m^n}| \, P(m,\varepsilon) \Big]}_{A}$$

We know that $\mathrm{Tr}(\rho^{\otimes m} P(m,\varepsilon)) \geq 1-\delta$

Let $|\psi_{m^n}\rangle = \lambda_{m^n} |\varphi_{m^n}\rangle + \mu_{m^n} |\eta_{m^n}\rangle \qquad |\varphi\rangle \in T(m,\varepsilon) \quad |\eta\rangle \in T^{\perp}(m,\varepsilon)$

$$= \sum_{m^n} p(m^n) |\lambda_{m^n}|^4 = \sum_{m^n} p(m^n) (1 - |\mu_{m^n}|^2)^2 \geq 1 - 2\sum_{m^n} p(m^n) |\mu_{m^n}|^2$$

Notice however that

$$\mathrm{Tr}(\rho^{\otimes m} P(m,\varepsilon)) = \sum_{m^n} p(m^n) \, \mathrm{Tr}\big(|\psi_{m^n}\rangle\langle\psi_{m^n}| \, P(m,\varepsilon) \big) = 1 - \sum_{m^n} p(m^n) |\mu_{m^n}|^2$$

$$\geq 1-\delta \quad \Rightarrow \quad \sum_{m^n} p(m^n) |\mu_{m^n}|^2 \leq \delta \quad \Rightarrow$$

$$F \geq 1 - 2\delta$$

2. Quantum channel capacity

- Holevo bound

Let X, $p(x)$ be classical random variable, according to which we prepare states $\rho(x)$. How much information we can learn about X by measuring $\rho(x)$

$$X \longrightarrow \boxed{\text{prepare state}} \xrightarrow{\rho(x)} \boxed{\text{measurement } \{\Pi(y)\}} \longrightarrow Y \qquad \int \Pi(y)\, dy = \mathbb{1}$$

$$p(x,y) = p(x) \cdot \mathrm{Tr}(\Pi(y) \rho(x))$$

<u>Theorem:</u> $I(X:Y) \leq \underbrace{S(\rho) - \sum_x p_x S(\rho_x)}_{X\text{-Holevo quantity}}$ where

$\rho = \sum_x p_x \rho_x$

(bound on accessible information .

<u>Lemma</u>: strong subadditivity (without a proof)

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$$

to czmi to

$$S(A:B) \leq S(A:BC)$$

gdzie $S(A:B) = S(A) + S(B) - S(A,B)$

gdzie $S(A:B) = S(A)+S(B) - S(A,B)$

$$S(A)+S(B)-S(A,B) \leq S(A)+S(B,C) - S(A,B,C)$$

Wszystkie fizyczne kwantowe wcale nie,

<u>Lemma 2</u> $\quad \begin{matrix} A \\ B \end{matrix} \xrightarrow{\Lambda} \begin{matrix} A \\ B' \end{matrix} \quad \geq \quad \begin{matrix} A \\ B \\ E \end{matrix} \xrightarrow{U} \begin{matrix} A \\ B' \\ E' \end{matrix}$

jeśli na podukładzie B działa odwzorowanie CP to

$$S(A:B) \geq S(A:B')$$

$$\Lambda(\rho_B) = \mathrm{Tr}_E\left(U_{BE}\, \rho_B \otimes |0\rangle\langle 0|\, U_{BE}^{\dagger}\right) = \sum_i K_i \rho_B K_i^{\dagger}$$

$$S(A:B) = S(A)+S(B) - S(A,B) =$$

$$= S(A) + S(BE) - S(A,B,E) = S(A)+S(B'E') - S(A,B',E')$$

$$\overset{S.S.}{\geq} S(A)+S(B') - S(A,B') = S(A:B')$$

<u>Proof</u>

Consider a state $\rho_{ABE} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x \otimes |0\rangle\langle 0|$

$|x\rangle$ orthonormal basis

Niech B wykonuje pomiar na B. Można to zapisać jako odwzorowanie:

Consider the following CP map

$$\rho'_{ABC} = \mathcal{E}(\rho_{ABC}) = \sum_x p_x |x\rangle\langle x| \otimes \sum_y \sqrt{\Pi_y}\, \rho_x \sqrt{\Pi_y} \otimes \underbrace{U_y |0\rangle\langle 0| U_y^{\dagger}}_{|y\rangle\langle y|}$$

$$\text{bo } S(A:BC) \geq S(A:B'C') \geq S(A:C') = I(X:Y)$$

$$H(X) + S(\rho) - \sum_x p_x S(\rho_x) - H(X) \qquad \rho_{AC'} = \sum_{x,y} p_x \mathrm{Tr}(\Pi_y \rho_x)\, |x\rangle\langle x| \otimes |y\rangle\langle y|$$

$$S(\rho) - \sum_x p_x S(\rho_x) \geq I(X:Y)$$

<u>Fact</u>: N qubit channel cannot transmit more than N bits of classical information

clearly $I(X:Y) \leq S(\rho) \leq N$

<u>Capacity of a noisy quantum channel</u>

(Holevo - Schumacher - Westmoreland theorem)

M ... channel ... $\Pi(m')$ ... M'

$m \in M$ — $\boxed{\text{encoding}}$ — $\rho_m(m)$ — $\boxed{\overset{\text{channel}}{\Lambda^{\otimes n}}}$ — $\rho'_m(m)$ — $\boxed{\overset{\Pi(m')}{\text{measurement}}}$ — M'

Ile logicznych bitów można przesłać na użycie kanału: (asymptotycznie przy dużym n)

Jeśli używać tylko stanów produktowych ρ_m :

$$C^{(1)}(\Lambda) = \chi(\Lambda) = \max_{\{p_x, \rho_x\}} \Big[\overbrace{S\Big(\underbrace{\Lambda\big(\textstyle\sum_x p_x \rho_x\big)}_{\rho'}\Big)}^{\text{q. equivalent of } H(Y)} - \overbrace{\underbrace{\textstyle\sum_x p_x S(\Lambda(\rho_x))}_{\bar{S}}}^{\text{q. equivalent of } H(Y|X)} \Big]$$

$\{p_x, \rho_x\}$ encoding

capacity if we restrict to encoding into product states

Wciąż przez lata pytano czy $C \overset{?}{=} C^{(1)}$.