II. Open quantum systems

• subsystem description (reduced density matrix)

$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ 

ρ_{SE}

We can only access system S

$$\mathrm{Tr}(\rho_{SE} \cdot A \otimes \mathbb{1}) = \mathrm{Tr}\Big(\sum_{\substack{i_S i_E \\ j_S j_E}} (\rho_{SE})^{i_S i_E}_{j_S j_E} \, |i_S\rangle\langle j_S| \otimes |i_E\rangle\langle j_E| \cdot A \otimes \mathbb{1} \Big) =$$

$$= \mathrm{Tr}_S \Big(\underbrace{\Big[\sum_{i_E} (\rho_{SE})^{i_S i_E}_{j_S i_E} \Big]}_{(\rho_S)^{i_S}_{j_S}} |i_S\rangle\langle j_S| \cdot A \Big)$$

$\rho_S = \mathrm{Tr}_E(\rho_{SE})$ - reduced density matrix

full description of what is accessible from system S alone

If

$\rho_{SE} = |\psi_{SE}\rangle\langle\psi_{SE}|$ is pure

ρ_S may be mixed (mixed state as a result of tracing out the environment)

• purification (we can always interpret ρ_S as a reduced state of pure state on larger space)

$$\rho_S = \sum_k q_k |\psi_k\rangle\langle\psi_k| \longrightarrow |\Phi\rangle_{SE} = \sum_k \sqrt{q_k}\, |\psi_k\rangle \otimes |k\rangle$$

• evolution of a subsystem

$$\rho_{SE}(0) = \rho_S(0) \otimes \rho_E \xrightarrow{U_{SE}} \underbrace{U_{SE}\, \rho_S(0) \otimes \rho_E\, U_{SE}^{+}}_{\rho_{SE}(t)}$$

initially S uncorrelated with E

$$\rho_S(t) = \mathrm{Tr}_E\left(\rho_{SE}(t)\right)$$

Let $|i\rangle_E$ - be o.n. basis in E. Without loss of generality $\rho_E = |0\rangle\langle 0|$ (we can always enlarge space E to purify ρ_E)

$$\rho_S(t) = \sum_i {}_E\langle i|\, U_{SE}\, \rho_S(0) \otimes |0\rangle_E\langle 0|\, U_{SE}^{+}\, |i\rangle_E$$

$$\left\{ \begin{array}{l} U_{SE} = \sum U_{SE}{}^{i_S i_E}_{j_S j_E}\, |i_S\rangle\langle j_S| \otimes |i_E\rangle\langle j_E| \\ {}_E\langle i|\, U\, |0\rangle_E = \sum_{i_S, j_S} U^{i_S\, i}_{j_S\, 0}\, |i_S\rangle\langle j_S| \\ \text{operator on } S \end{array} \right.$$

$$= \sum_i \underbrace{{}_E\langle i|U_{SE}|0\rangle_E}_{K_i}\, \rho_S(0)\, \underbrace{{}_E\langle 0|U_{SE}^{+}|i\rangle_E}_{K_i^{+}} =$$

K_i ← Kraus operators { not necessarily hermitian nor unitary

$$\sum_i K_i^{+} K_i = \sum_i \langle 0|\, U_{SE}^{+}\, |i\rangle\langle i|\, U_{SE}\, |0\rangle = \mathbb{1}_S$$

General evolution on open quantum system:

$$\rho' = \Lambda(\rho) = \sum_i K_i \rho K_i^{+}, \quad \sum_i K_i^{+} K_i = \mathbb{1}$$

Does any set $\{K_i\}$ satisfying $\sum_i K_i^\dagger K_i = \mathbb{1}$ corresponds to a physically possible evolution?

Define U_{SE} such that:

$$\forall_{|\psi\rangle} U_{SE} |\psi\rangle_S \otimes |0\rangle_E = \sum_i K_i |\psi\rangle_S \otimes |i\rangle$$

this leads to correct transformation:

$$\varrho_S = \sum_k q_k |\psi_k\rangle\langle\psi_k|$$

$$\Lambda(\varrho) = \mathrm{Tr}_E\, U_{SE}(\varrho_S \otimes |0\rangle_E\langle 0|)\, U_{SE}^\dagger =$$

$$= \mathrm{Tr}_E\left(\sum_{ij}\sum_k q_k K_i |\psi\rangle\langle\psi| K_j \otimes |i\rangle\langle j| \right) = \sum_i K_i \varrho K_i^\dagger$$

Check if U_{SE} is unitary:

$$|\psi\rangle \otimes |0\rangle \xrightarrow{U_{SE}} \sum_i K_i |\psi\rangle \otimes |i\rangle$$

$$|\psi'\rangle \otimes |0\rangle \qquad\qquad \sum_i K_i |\psi'\rangle \otimes |i\rangle$$

take scalar product:

$$\langle\psi|\psi'\rangle = \langle\psi| \underbrace{\sum_i K_i^\dagger K_i}_{= \mathbb{1}} |\psi'\rangle = \langle\psi|\psi'\rangle \quad \text{ok}$$

$$U_{SE} = \begin{array}{cc} |\psi\rangle\otimes|0\rangle & \\ \left[\begin{array}{cc} \vdots & ? \\ \vdots & \\ \vdots & \end{array}\right] & \end{array}$$

these are legitimate unitary matrix columns (orthonormal vectors)

we can always complete the matrix to become a proper unitary matrix

$\Lambda(\rho)$ is a legitimate transformation of a density matrix of an open quantum system (that is initially uncorrelated with the environment) iff:

$$\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger , \quad \sum_i K_i^\dagger K_i = \mathbb{1}$$