- Generalized measurements [Later]

idea: we do not measure the quantum system directly, but first let it interact with the "measurement device" and then "read-out" the measurement outcome

$$\rho_S \otimes \underset{M}{|0\rangle\langle 0|} \longrightarrow \underbrace{U_{SM}\,\rho_S \otimes |0\rangle\langle 0|\, U_{SM}^\dagger}_{\rho_{SM}}$$

↑ initial state of the measurement device

measure M in some basis $\{|i\rangle_M\}$

$$p_i = \mathrm{Tr}\left(U_{SM}\, \rho_S \otimes |0\rangle\langle 0|\, U_{SM}^\dagger \cdot \mathbb{1} \otimes \underset{M}{|i\rangle\langle i|} \right) =$$

$$= \left(\rho_S \cdot \langle 0| U_{SM}^\dagger \cdot \mathbb{1} \otimes |i\rangle\langle i|\, U_{SM} |0\rangle \right) =$$

$$= \left(\rho_S \underbrace{\langle 0| U_{SM}^\dagger |i\rangle \langle i| U_{SM} |0\rangle}_{M_i} \right)$$

notice that $M_i = K_i^\dagger K_i$ if we treated M as the environment

$$p_i = \mathrm{Tr}(\rho_S M_i), \quad M_i \geq 0 \quad \{ M_i = K_i^\dagger K_i \geq 0$$
$$\sum_i M_i = \mathbb{1}_S$$

We call any set of $\{M_i\}$ such that $M_i \geq 0$, $\sum_i M_i = \mathbb{1}$ a generalized measurement:

$$p_i = \mathrm{Tr}(\rho_S M_i)$$

Similarly as for the case of evolution we can always find $\mathcal{U}_{SM}$ and $\{|i\rangle_M\}$ such that

$$\mathrm{Tr}(\rho_S M_i) = \mathrm{Tr}(\mathcal{U}_{SM}\rho_S \otimes |0\rangle\langle 0| \,\mathcal{U}_{SM}^\dagger \cdot \mathbb{1}\otimes |i\rangle\langle i|)$$

analogously,
$$\mathcal{U}_{SM}|\psi\rangle\otimes|0\rangle = \sum_i \sqrt{M_i}\,|\psi\rangle\otimes|i\rangle.$$

- Remark:

given $\rho_S' = \sum_i K_i^\dagger \rho K_i$ we can interpret

$\rho_S^{(i)} = K_i \rho_S K_i^\dagger$ as a post measurement conditional state (unnormalized) such that $\mathrm{Tr}(\rho_S^{(i)}) = p_i$.

Decoherence can be viewed as a measurement by the environment to which we do not have access.

{ note that if $K_i \to U_i K_i \Rightarrow M_i \to M_i$
so that it does not affect measurement probabilities as it just correspond to a change of post-measurement state.

- Quantum channels

we want to abstract from the physical context and understand better the mathematical structure of q. evolution.

$$\rho' = \Lambda(\rho) = \sum_i K_i \rho K_i^\dagger \quad , \quad \sum_i K_i^\dagger K_i = \mathbb{1}$$

$\rho \in \mathcal{L}(\mathcal{H})$ - linear operators on $\mathcal{H}$ $\quad (\rho \geq 0, \mathrm{Tr}\rho = 1)$

a) $\Lambda : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ – linear operator (super operator)

$$\rho'^{\,i_2}_{\;j_2} = \sum_k \sum_{i_1 j_1} (K_k)^{i_2}_{\;i_1} \rho^{i_1}_{\;j_1} (K_k)^{\dagger\, j_1}_{\;\;j_2}$$

We can „vectorize" ρ:

$$\rho^{i_2}_{\;j_2} = \underbrace{\left(\sum_k (K_k)^{i_2}_{\;i_1} (K_k)^{\dagger\, j_1}_{\;\;j_2} \right)}_{\Phi_{\Lambda\, i_2 j_2}^{\;i_1 j_1}} \rho^{i_1}_{\;j_1}$$

vectorization of ρ

$$\rho'_{i_2 j_2} = \sum_{i_1 j_1} \Phi_{\Lambda\, i_2 j_2}^{\;i_1 j_1} \rho_{i_1 j_1} \qquad [\rho'] = [\Phi_\Lambda][\rho]$$

$$|\rho'\rangle\rangle = \Phi_\Lambda |\rho\rangle\rangle$$

b) Λ – transforms positive matrices to positive

– positive map

but there is more:

$\Lambda_S \otimes I_R$ is also positive $\equiv$ completely positive (CP)

$$\sum_i K_i \otimes \mathbb{1} \; \rho_R \; K_i^\dagger \otimes \mathbb{1} \geq 0$$

Mathematical theorem: Every CP map has the form $\sum_i K_i \rho K_i^\dagger = \Lambda(\rho)$

c)

$$\mathrm{Tr}(\Lambda(\rho)) = \mathrm{Tr}(\rho \underbrace{\sum_i K_i^\dagger K_i}_{=\mathbb{1}}) = \mathrm{Tr}(\rho) = 1$$

trace preservation

$$\boxed{\text{quantum channels} = \text{completely positive trace preserving maps (CPTP)}}$$

In general $\dim \mathcal{H}_1 \neq \dim \mathcal{H}_2$.

- Choi – Jamiołkowski Isomorphism

Consider $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle_1 \otimes |i\rangle_2 \in \mathcal{H}_1 \otimes \mathcal{H}_1$

Define: $D_\Lambda = \Lambda \otimes \mathcal{I} (|\psi\rangle\langle\psi|)$ $\qquad |\psi\rangle \; \boxed{\Lambda} \; D_\Lambda$

$D_\Lambda \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1)$

↑ dynamical matrix (Choi – Jamiołkowski operator)

$$D_\Lambda = \sum_k K_k \otimes \mathbb{1} \; \frac{1}{d} \Big(\sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j| \Big) K_k^\dagger \otimes \mathbb{1} =$$

$$= \frac{1}{d} \sum_k \sum_{ij} K_k |i\rangle\langle j| K_k^\dagger \otimes |i\rangle\langle j|$$

$$(D_\Lambda)^{i_2 i_1}_{j_2 j_1} = \langle i_2| \otimes \langle i_1| \; D_\Lambda \; |j_2\rangle \otimes |j_1\rangle =$$

$$= \frac{1}{d} \sum_k \underbrace{\langle i_2 | K_k | i_1 \rangle}_{|K_k\rangle\rangle_{i_2 i_1}} \underbrace{\langle j_2 | K_k^\dagger | j_1 \rangle}_{\langle\langle K_k|_{j_2 j_1}}$$

$$D_\Lambda = \frac{1}{d} \sum_k |K_k\rangle\rangle\langle\langle K_k|$$

↑ op Krausa zapisany jako wektor w $\mathcal{H}_2 \otimes \mathcal{H}_1$

$$D_\Lambda \geq 0 \quad \mathrm{Tr}_{\mathcal{H}_2} D_\Lambda = \frac{1}{d} \sum_k \sum_{ij} \mathrm{Tr}\big(\underbrace{K_k |i\rangle\langle j| K_k^\dagger}_{} \big) \otimes |i\rangle\langle j| = \frac{1}{d} \mathbb{1}_{\mathcal{H}_1}$$

Notice that $\left(D_1\right)^{i_2 i_1}_{j_2 j_1} = \frac{1}{d} \sum_k \left(K_k\right)^{i_2}_{i_1} \left(K_k\right)^{\dagger\, j_1}_{j_2} = \Phi_1{}^{i_2 j_2}_{i_1 j_1}$

reshuffling

$$D_1 = \begin{bmatrix} \Phi^{00}_{00} & \Phi^{00}_{01} & \Phi^{00}_{10} & \Phi^{00}_{11} \\ \Phi^{01}_{00} & \Phi^{01}_{01} & \Phi^{01}_{10} & \Phi^{01}_{11} \\ \Phi^{10}_{00} & \Phi^{10}_{01} & \Phi^{10}_{10} & \Phi^{10}_{11} \\ \Phi^{11}_{00} & \Phi^{11}_{01} & \Phi^{11}_{10} & \Phi^{11}_{11} \end{bmatrix}$$

For every $D_1 \geq 0$, $\mathrm{Tr}_{\mathcal{H}_2} D_1 = \frac{1}{d}\mathbb{1}$ we have a corresponding CPTP map:

$$\Lambda(\rho) := d \,\mathrm{Tr}_{\mathcal{H}_1}\left(D_1 \cdot \mathbb{1}_{\mathcal{H}_2} \otimes \rho^T_{\mathcal{H}_1} \right) =$$

$$= \mathrm{Tr}_{\mathcal{H}_1}\left[\sum_k \sum_{ij} K_k |i\rangle\langle j| K_k^\dagger \otimes |i\rangle\langle j| \cdot \mathbb{1}_{\mathcal{H}_2} \otimes \sum_{i_1 j_1} \rho^{T\, i_1}_{\ j_1} |i_1\rangle\langle j_1| \right]$$

$$= \sum_{k, i_1 j_1} \rho^{T\, i_1}_{\ j_1} K_k |j_1\rangle\langle i_1| K_k^\dagger = \sum_k K_k \rho K_k^\dagger$$

$$\boxed{\begin{array}{lcl} \Lambda : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2) & & D_1 \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1) \\ \text{CPTP Map} & \Longleftrightarrow & D_1 \geq 0 \\ \Lambda(\rho) = \sum_k K_k \rho K_k^\dagger & & D_1 = \frac{1}{d} \sum_k |K_k\rangle\rangle\langle\langle K_k| \\ \sum_k K_k^\dagger K_k = \mathbb{1} & & \mathrm{Tr}_{\mathcal{H}_2} D_1 = \frac{1}{d} \mathbb{1}_{\mathcal{H}_1} \end{array}}$$

When it is useful: when we have linear transformation of ϱ and want to find canonical Kraus representation

$$\Phi \longrightarrow D_\Lambda \longrightarrow \text{diagonalize } D_\Lambda = \frac{1}{d}\sum_k |K_k\rangle\rangle\langle\langle K_k|$$

eigenvectors of D_Λ correspond to Kraus operators

- Remark: Kraus representation is not unique

$$\Lambda(\varrho) = \sum_k K_k \varrho K_k^\dagger, \qquad K_k' = \sum_l u_{kl} K_l$$

$$\Lambda'(\varrho) = \sum_k \sum_l u_{kl} K_l \varrho \left(\sum_{l'} u_{kl'} K_{l'}\right)^\dagger =$$

$$= \sum_{ll'} \underbrace{\sum_k u_{kl} u_{kl'}^*}_{(uu^\dagger)_{ll'}} K_l \varrho K_{l'}^\dagger \overset{\text{if } uu^\dagger = 1}{=} \sum_k K_k \varrho K_k^\dagger$$

Kraus representation corresponding to eigendecomposition of D_Λ - canonical Kraus representation (orthogonal Kraus matrices)

We see that arbitrary CP map on $\mathcal{L}(\mathcal{H})$ where $\dim \mathcal{H} = d$ can be expressed using d^2 Kraus operators.