01 Entanglement (lecture)

Notation

$|\psi\rangle \in \mathcal{H}_d$

$\rho \in \mathcal{B}(\mathcal{H}_d), \mathcal{L}(\mathcal{H}_d)$

Qubit: $|\psi\rangle \in \mathbb{C}_2$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad , |\alpha|^2 + |\beta|^2 = 1$

$\rho \in \mathcal{B}(\mathbb{C}_2) \qquad \rho = \frac{1}{2}(\mathbb{1} + \underline{r} \cdot \underline{\sigma})$

density operator (mixed state)

- $\hat{A} = \hat{A}^\dagger \leftarrow$ observable , e.g. $\{\hat{x}, \hat{p}\}$
- $\{p_i, |\psi_i\rangle\} \leftarrow$ quantum ensemble

$$A_\psi = \langle\psi|\hat{A}|\psi\rangle = \mathrm{Tr}\{\boxed{|\psi\rangle\langle\psi|}\,\hat{A}\}$$

$$\Downarrow$$

$$A = \sum_i p_i A_{\psi_i} = \sum_i p_i \langle\psi_i|\hat{A}|\psi_i\rangle$$

$$= \mathrm{Tr}\{\boxed{\sum_i p_i |\psi_i\rangle\langle\psi_i|}\,\hat{A}\}$$

$$\rho = \begin{cases} |\psi\rangle\langle\psi| , & \text{pure} \\ \sum_i p_i |\psi_i\rangle\langle\psi_i| , & \text{mixed} \end{cases}$$

N.B. $\langle\psi_i|\psi_j\rangle \neq \delta_{ij}$ not necessarily orthogonal

🏠 Prove: $\rho^\dagger = \rho$, $\mathrm{tr}\{\rho\} = 1$, $\underset{\sim}{\rho \geq 0}$

positive semidefinite matrix

spectral decomp^n:

$$\rho = \sum_{i,j=1}^{d} \rho_{ij} |i\rangle\langle j| = \sum_{k=1}^{\text{rank}} \lambda_k |e_k\rangle\langle e_k|$$

} eigendecomposition

$\langle e_k | e_l \rangle = \delta_{kl}$ rank-1 $\Updownarrow$ pure state

Def. "purity"

$$\mathrm{Tr}\{\rho^2\} = \mathrm{Tr}\{\rho\rho\} = \mathrm{Tr}\{\sum_k \lambda_k^2 |e_k\rangle\langle e_k|\}$$

$$= \sum_k \lambda_k^2 \langle e_k | e_k \rangle = \sum_k \lambda_k^2$$

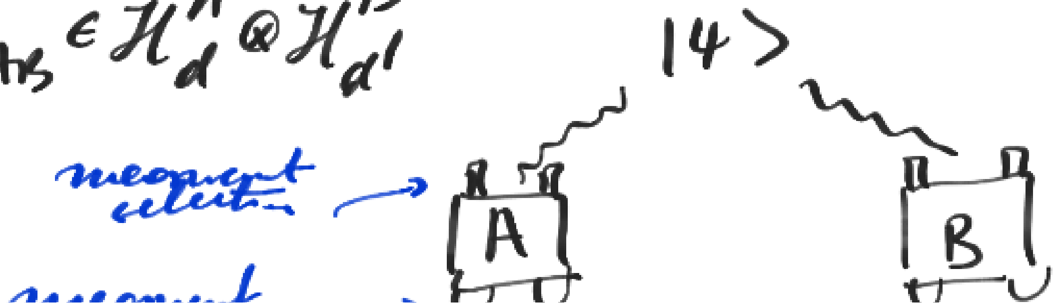
$$\frac{1}{d} \leq \mathrm{Tr}\{\rho^2\} = \sum_k \lambda_k^2 \leq \sum_k \lambda_k = 1$$

"=" max. mixed "=" pure state

Bipartite states

Alice + Bob share a quantum state

$|\psi\rangle_{AB} \in \mathcal{H}_d^A \otimes \mathcal{H}_{d'}^B$



outcomes →

$$|\psi\rangle_{AB} = \sum_{i,j} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B \; \{ \equiv |i,j\rangle = |ij\rangle \}$$

with $\langle i|j\rangle = \delta_{ij}$

$$\dim(\mathcal{H}^{AB}) = dd'$$

$d = d' = 2$ (2 qubits) $\Rightarrow \dim(\mathbb{C}_2 \otimes \mathbb{C}_2) = 4$

basis: (computational) $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$

Bell basis

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|\updownarrow\leftrightarrow\rangle - |\leftrightarrow\updownarrow\rangle)$$

{SPDC process}

form a basis, as

$$\langle ij|kl\rangle = (\langle i|_A \otimes \langle j|_B)(|k\rangle_A \otimes |l\rangle_B) =$$

$$= \langle i|k\rangle_A \langle j|l\rangle_B = \delta_{ik}\delta_{jl}$$

density matrices

$$\rho \in B(\mathcal{H}_d^A \otimes \mathcal{H}_{d'}^B)$$

$$\rho = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i| \qquad |\psi_i\rangle \in \mathcal{H}_d^A \otimes \mathcal{H}_d^B$$

with $|\psi_i\rangle = \sum_{kl} \alpha_{kl}^{(i)} |kl\rangle$

$$\rho = \sum \; p_i \sum \alpha_{kl}^{(i)} |kl\rangle (\alpha_{k'l'}^{(i)})^* \langle k'l'|$$

$$= \sum_{\substack{kl\\k'l'}} \underbrace{\left(\sum_i p_i \, \alpha_{kl}^{(i)} \alpha_{k'l'}^{(i)*}\right)}_{\rho^{kl}_{k'l'}} |kl\rangle\langle k'l'|$$

2 qubits: $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

e.g. $\rho^{01}_{11} |01\rangle\langle 11| = \rho^{01}_{11} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} (0\,0\,0\,1) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&\rho^{01}_{11} \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$

$$\Rightarrow \quad \rho = \begin{array}{c|c} \begin{matrix} \langle 00| & \langle 01| & \langle 10| & \langle 11| \end{matrix} & \\ \begin{pmatrix} \rho^{00}_{00} & \rho^{00}_{01} & \rho^{00}_{10} & \rho^{00}_{11} \\ \rho^{01}_{00} & \rho^{01}_{01} & \rho^{01}_{10} & \rho^{01}_{11} \\ \rho^{10}_{00} & \rho^{10}_{01} & \rho^{10}_{10} & \rho^{10}_{11} \\ \rho^{11}_{00} & \rho^{11}_{01} & \rho^{11}_{10} & \rho^{11}_{11} \end{pmatrix} & \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} \end{array}$$

Definition notation!

1) Pure states: $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

is product iff $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$

$= |\psi\rangle_A \otimes |\chi\rangle_B$

2) Mixed states: $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$

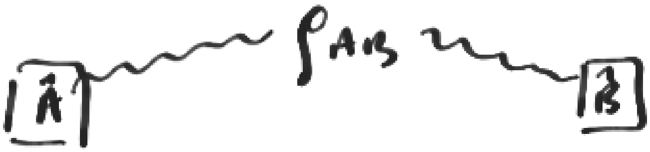
is separable iff "mixture of product states"

$$\rho = \sum_i p_i \, |\phi_i\rangle_A\langle\phi_i| \otimes |\chi_i\rangle_B\langle\chi_i|$$

{ N.B. product state is separable with
TRIVIAL $p_i = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases}$

3) Any state that is not separable is entangled

Motivation: classical correlation



observable $\hat{A} \otimes \hat{B}$ {e.g. $\hat{X} \otimes \hat{P}$}

correlation:

$$\langle AB \rangle = \mathrm{Tr}\{\rho_{AB} \, \hat{A} \otimes \hat{B}\}$$

$$\rho_{AB} \in SEP \Rightarrow \rho_{AB} = \sum_i p_i \, |\phi_i\rangle_A\langle\phi_i| \otimes |\chi_i\rangle_B\langle\chi_i$$

$$\langle AB \rangle_{SEP} = \sum_i \underset{\nwarrow}{p_i} \, \langle \hat{A} \rangle_{\phi_i} \, \langle \hat{B} \rangle_{\chi_i}$$

classical correlations

- Convex mixture of separable states

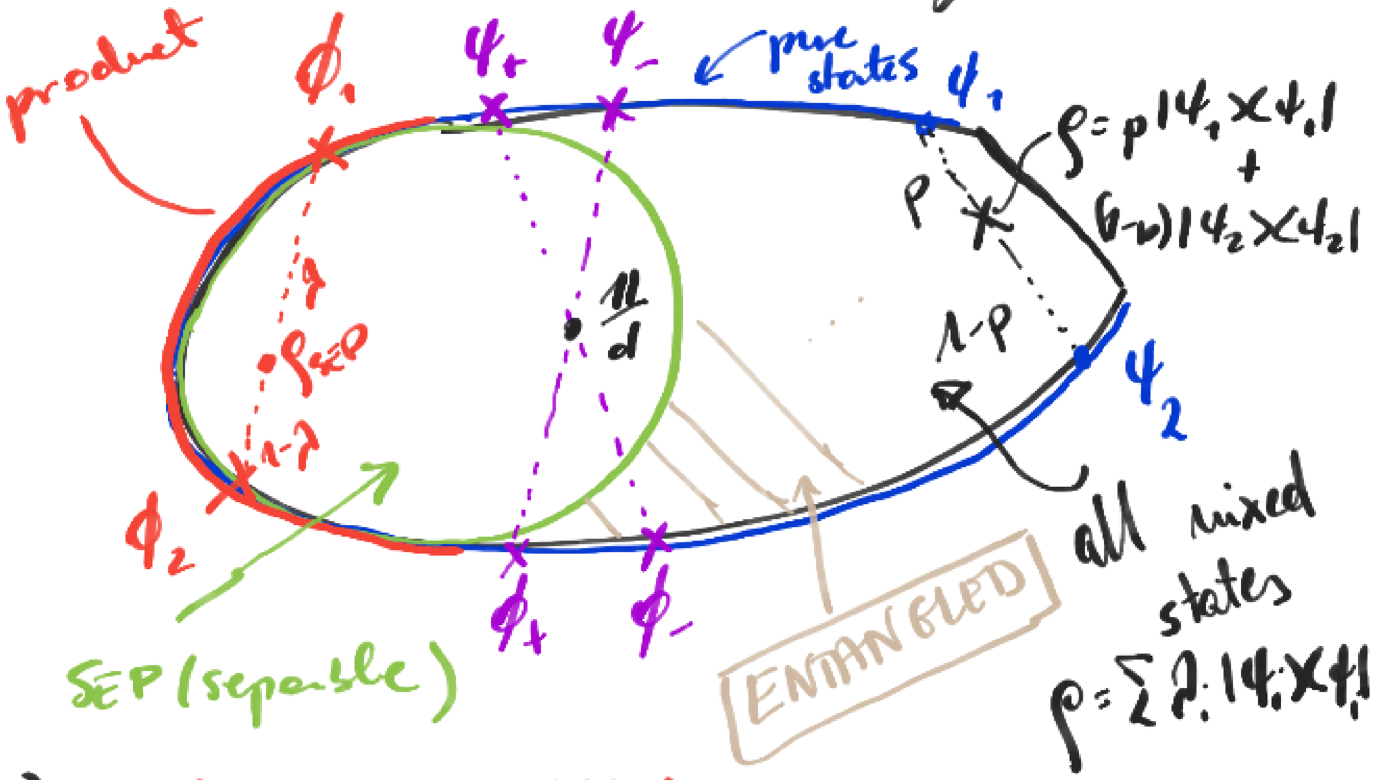
is separable

$\rho_1, \rho_2 \in SEP \Rightarrow \lambda\rho_1 + (1-\lambda)\rho_2 \in SEP$ for $0 \leq \lambda \leq 1$

(Proof)

- ok but all mixed states $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B) \geq 0$ form also a convex set

$\forall \rho, \sigma \geq 0 : \lambda\rho + (1-\lambda)\sigma \geq 0$ for $0 \leq \lambda \leq 1$



1) $|\phi_1\rangle = |\chi_1\rangle_A |\xi_1\rangle_B$

$|\phi_2\rangle = |\chi_2\rangle_A |\xi_2\rangle_B$

$\rho_{SEP} = \lambda |\phi_1\rangle\langle\phi_1| + (1-\lambda)|\phi_2\rangle\langle\phi_2|$

2) Max. mixed state

$\mathbb{1} = \frac{1}{} \sum |ij\rangle\langle ij| = \sum \frac{1}{} |i\rangle\langle i| \otimes |j\rangle\langle j|$

$d \quad d \quad ij \qquad\qquad ij \quad d$

mixture of product states

but! for 2 qubits

$$\frac{\mathbb{1}}{4} = \frac{1}{4}\left(|\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| + |\phi_+\rangle\langle\phi_+| + |\phi_-\rangle\langle\phi_-|\right)$$

Bell states (entangled!) form an orthonormal basis!

$\Rightarrow$ by mixing (noise) entangled states we can obtain separable states

Entanglement verification

1) Pure states

Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \alpha_{ij} \, |i\rangle_A \, |j\rangle_B$$

Singular value decomposition (SVD) of a complex matrix

$$\alpha_{ij} = \sum_{k=1}^{r} u_{ik}^{*} \lambda_k V_{kj}$$

$\alpha : d_A \times d_B$

$$d_A\left[\underbrace{\quad \alpha \quad}_{d_B}\right] = d_A\left[\underbrace{U^*}_{r}\right]\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_r\end{bmatrix}\left[\underbrace{\quad V \quad}_{d_B}\right]\Big\}r$$

$r \to$ "Schmidt rank"

$$|\psi\rangle_{AB} = \sum_{ij}\sum_{k} U^*_{ik}\lambda_k V_{kj}\ |i\rangle|j\rangle =$$

$$= \sum_{k=1}^{r}\lambda_k \underbrace{\left(\sum_{i=1}^{d_A} U^\dagger_{ki}|i\rangle\right)}\left(\sum_{i=1}^{d_B} V_{kj}|j\rangle\right)$$

$$= \sum_{k=1}^{r}\lambda_k\ |e_k\rangle_A \otimes |f_k\rangle_B$$

"Schmidt coefficients"

<u>SVD</u> : $\Rightarrow$ $1 \leq r\left(\substack{\text{Schmidt}\\\text{rank}}\right) \leq \min\{d_A, d_B\}$

- $r = 1 \rightarrow |\psi\rangle_{AB} = |e\rangle_A \otimes |f\rangle_B$

  <u>product</u> state (separable)

- $r > 1 \Rightarrow$ <u>entangled</u> state

- $r = \min\{d_A, d_B\} \ \wedge \ \underset{k}{\forall}: |\lambda_k| = \frac{1}{\sqrt{r}}$

(equal weights)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{r}} \sum_k e^{i\phi_k} |e_k\rangle_A \otimes |f_k\rangle_B$$

$\Rightarrow$ maximally entangled state

2 qubits:

- Maximally entangled

$$|\psi_{ME}\rangle = \frac{1}{\sqrt{2}}\left(|\phi\rangle|\chi\rangle + e^{i\varphi}|\phi_\perp\rangle|\chi_\perp\rangle\right)$$

e.g.: $|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

and so $|\psi_+\rangle, |\phi_\pm\rangle \Rightarrow$ Bell states are max. entangled

- Partially entangled

e.g. $|\psi_+(\lambda)\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\lambda}\,|01\rangle + \sqrt{1-\lambda}\,|10\rangle\right)$

with $$\underset{\text{product}}{0} < \lambda < \underset{\text{max. ent.}}{\frac{1}{2}} < \lambda < \underset{\text{product}}{1}$$

Mixed states

- no universal way!
- sufficient condition:

• [illegible] partial transposition

negativity under partial transposition (PT)

suff. and ~~necessary~~ only for $d_A \times d_B$

2×2

2×3

Transposition

$$\rho^T = \sum_{\substack{ij\\kl}} \rho^{ij}_{kl} \left(|ij\rangle\langle kl|_{AB}\right)^T = \sum_{\substack{ij\\kl}} \rho^{ij}_{kl} \left(|i\rangle\langle k|_A\right)^T \otimes \left(|j\rangle\langle l|_B\right)^T$$

$$= \sum_{\substack{ij\\kl}} \rho^{ij}_{kl} \, |k\rangle\langle i| \otimes |l\rangle\langle j| = \sum_{\substack{ij\\kl}} \rho^{ij}_{kl} \, |kl\rangle\langle ij|$$

$$= \sum_{\substack{ij\\kl}} \rho^{kl}_{ij} \, |ij\rangle\langle kl|$$

Partial transposition

$$\rho^{T_A} = \sum_{\substack{ij\\kl}} \rho^{ij}_{kl} \left(|i\rangle\langle k|_A\right)^T \otimes |j\rangle\langle l|_B$$

$$= \sum_{\substack{ij\\kl}} \rho^{kj}_{il} \, |ij\rangle\langle kl|$$

similarly $\rho^{T_B} = \sum_{\substack{ij\\kl}} \rho^{il}_{kj} \, |ij\rangle\langle kl|$

For separable states

$$\rho \in SEP \Rightarrow \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

$$\Rightarrow \rho^{T_A} = \sum_i p_i \left(|\psi_i\rangle\langle\psi_i|\right)^T \otimes |\phi_i\rangle\langle\phi_i|$$

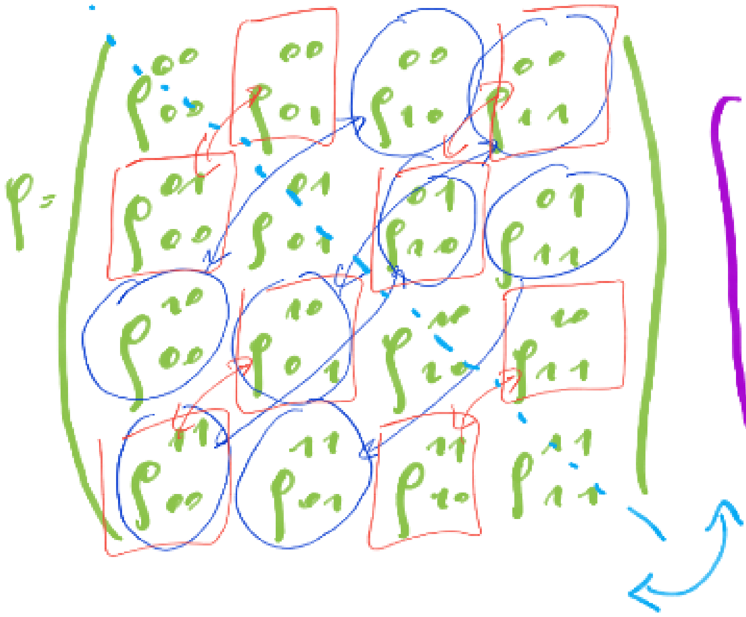
$$\geq 0$$ <u>also a state</u>

similarly: $\rho^{T_B} \geq 0$ $\{$ <u>N.B.</u> $(\rho^{T_A})^T = \rho^{T_B}$

<u>Hence</u>: $\rho \in SEP \Rightarrow \left(\rho^{T_A} \geq 0 \Leftrightarrow \rho^{T_B} \geq 0\right)$

<u>Hence</u>: $\left(\rho^{T_A} < 0 \Leftrightarrow \rho^{T_B} < 0\right) \Rightarrow \rho$ is entangled

2 qubits $(d_A = d_B = 2)$



$$\rho = \begin{pmatrix} \rho^{00}_{00} & \rho^{00}_{01} & \rho^{00}_{10} & \rho^{00}_{11} \\ \rho^{01}_{00} & \rho^{01}_{01} & \rho^{01}_{10} & \rho^{01}_{11} \\ \rho^{10}_{00} & \rho^{10}_{01} & \rho^{10}_{10} & \rho^{10}_{11} \\ \rho^{11}_{00} & \rho^{11}_{01} & \rho^{11}_{10} & \rho^{11}_{11} \end{pmatrix}$$

$\rho^T, \rho^{T_A}, \rho^{T_B}$

[State is entangled <u>if and only if</u> for 2x2 (2x3) $\rho^{T_A} < 0, \rho^{T_B} < 0$]