07 No-cloning, teleport, EPR & Bell inequality

No-cloning Theorem

Motivation (lecture "distinguishability")

- Ambiguous distinction
  $|\psi\rangle, |\phi\rangle$

$$P_{error} = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi|\phi\rangle|^2}\right)$$

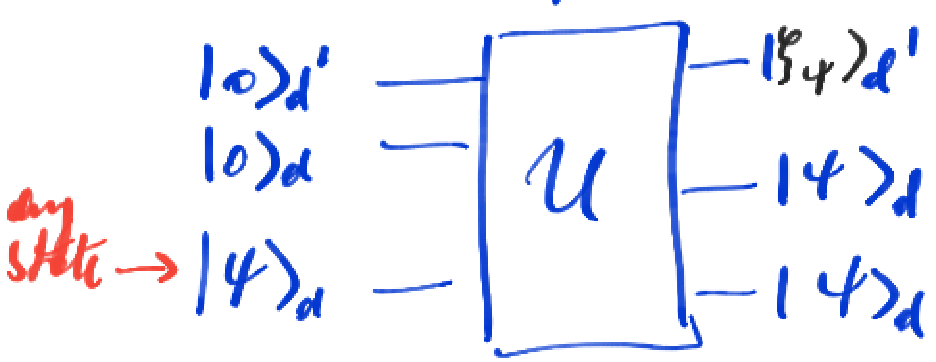
$$P_{error} = 0 \iff |\psi\rangle \perp |\phi\rangle$$

- N copies: $|\psi\rangle^N, |\phi\rangle^N$

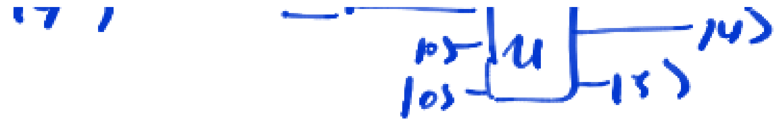
$$P^N_{error} = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi|\phi\rangle|^{2N}}\right)$$

$$|\langle\psi|\phi\rangle| \neq 1 \Rightarrow \lim_{N\to\infty} p^N_{error} = 0$$

⇒ If I had "universal" cloning machine (for any state $|\psi\rangle \in \mathcal{H}_d$) then could distinguish non-orthogonal states!



then:



$$|\psi\rangle \to \boxed{\mathcal{U}^{\otimes N}} \to |\psi\rangle^{\otimes N}$$

Consequences

1) $|\psi\rangle, |\phi\rangle \xRightarrow{\text{clone}} |\psi\rangle^{\otimes N}, |\phi\rangle^{\otimes N} \Rightarrow$ distinguish $P^N_{\text{error}} \xrightarrow[N\to\infty]{} 0$

2) superluminal! communication

$$\boxed{A} \sim\sim\sim \psi_- \sim\sim\sim \boxed{B}$$

Ex. "entangled"

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

RSP protocol (+ cloning)

Alice measures		Bob has
"0": $\{|0\rangle, |1\rangle\}$	$\longrightarrow$	$|1\rangle$ or $|0\rangle$
"1": $\{|+\rangle, |-\rangle\}$	$\longrightarrow$	$|-\rangle$ or $|+\rangle$

$\Rightarrow$ Bob clones his qubit N times

"0": $|1\rangle^N$ or $|0\rangle^N$

"1": $|-\rangle^N$ or $|+\rangle^N$

$\Rightarrow$ Bob measures in $\{|0\rangle, |1\rangle\}$ basis

"0" $\to$ all outcomes "0" or all "1"

"1" $\to \approx \frac{N}{2}$ outcomes "0" and $\approx \frac{N}{2}$ outcomes "1"

Bob can distinguish the cases $\Longrightarrow$ SUPERLUMINAL SENDING OF BIT.

No cloning theorem

"ad absurdum"

Let $|\phi\rangle, |\psi\rangle$ s.t. $0 < |\langle\psi|\phi\rangle| < 1$

and there exists

a universal cloning machine

i.e.:

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle \xrightarrow{\mathcal{U}} |\psi\rangle \otimes |\psi\rangle \otimes |\xi_\psi\rangle$$

$$|\phi\rangle \otimes |0\rangle \otimes |0\rangle \longrightarrow |\phi\rangle \otimes |\phi\rangle \otimes |\xi_\phi\rangle$$

for $|\tilde{\Psi}\rangle = \mathcal{U}|\Psi\rangle \quad \langle\tilde{\Psi}|\tilde{\Psi}\rangle = \langle\Psi|\Psi\rangle$

$$\langle\psi|\phi\rangle = \langle\psi|\phi\rangle\langle\psi|\phi\rangle\langle\xi_\psi|\xi_\phi\rangle$$

$$\underbrace{\langle\psi|\phi\rangle}_{\neq 0}\Big(1 - \underbrace{\langle\psi|\phi\rangle\langle\xi_\psi|\xi_\phi\rangle}_{|\dots| < 1}\Big) = 0$$

$\Rightarrow$ contradiction! $\blacksquare$

N.B. does not apply when $\langle\phi|\psi\rangle = 0$

$\Rightarrow$ can create UCM that works for any pair of $|\psi\rangle \perp |\phi\rangle$!

Teleportation

$|\psi\rangle$ $\boxed{A}$ ~~~ Ψ_- ~~~ $\boxed{B}$

qubit state to "teleport"

$|\psi\rangle_1 \; |\psi\rangle_{23}$ — 3 qubits $\underbrace{1,2}_{\text{Alice}}, \underbrace{3}_{\text{Bob}}$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|\psi\rangle_1 |\psi\rangle_{23} = (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) =$$

$$= \frac{1}{\sqrt{2}}(\alpha|001\rangle - \alpha|010\rangle + \beta|101\rangle - \beta|110\rangle)$$

$$= \frac{1}{2}\Big[-|\psi_-\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) +$$
$$- |\psi_+\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) +$$
$$+ |\phi_-\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) +$$
$$+ |\phi_+\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)\Big]$$

1) Alice measures in Bell basis qubits 1&2

$$\{|\psi_+\rangle, |\psi_-\rangle, |\phi_+\rangle, |\phi_-\rangle\}$$

outcomes: "1" "2" "3" "4"

2) Alice communicates her outcome to Bob

3) Bob applies unitary on his qubit U_i

$i =$ "1" :	$\mathbb{1}$	his	$\alpha|0\rangle + \beta|1\rangle$	✓
"2" :	σ_z	final :	$\sigma_z(\alpha|0\rangle - \beta|1\rangle)$	✓
"3" :	σ_x	state	$\sigma_x(\alpha|1\rangle + \beta|0\rangle)$	✓
"4" :	$i\sigma_y$		$i\sigma_y(\alpha|1\rangle - \beta|0\rangle)$	✓

$\overset{\parallel}{\alpha|0\rangle + \beta|1\rangle}$

N.B.

(i) concise notation

$$|\tilde{\psi}(i)\rangle_3 = {}_{12}\langle \psi_i | \,|\psi\rangle_1 \otimes |\psi_-\rangle_{23}$$

$$\hookrightarrow \{ |\psi_+\rangle, |\psi_-\rangle, |\Phi_+\rangle, |\Phi_-\rangle \}$$

Conditional state (unnormalised)

$$= \left\{ \tfrac{1}{2}(\alpha|0\rangle + \beta|1\rangle), \tfrac{1}{2}(\alpha|0\rangle - \beta|1\rangle), \dots \right\}$$

<u>result of teleport$^{\underline{n}}$</u>:

$$\sum_{i=1}^{4} U_i |\tilde{\psi}(i)\rangle_3 = |\psi\rangle_3$$

(ii) what if Bob is <u>ignorant</u> about Alice's result

his state

$$\rho_3 = \sum_{i=1}^{4} |\tilde{\psi}(i)\rangle\langle\tilde{\psi}(i)| =$$

$$= \frac{1}{4}\Big[(\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|) + (\alpha|0\rangle - \beta|1\rangle)(\alpha^*\langle 0| - \beta^*\langle 1|) + (\alpha|1\rangle + \beta|0\rangle)(\alpha^*\langle 1| + \beta^*\langle 0|) + (\alpha|1\rangle - \beta|0\rangle)(\alpha^*\langle 1| - \beta^*\langle 1|) \Big] =$$

$$= \frac{1}{4}\left[2(|\alpha|^2 + |\beta|^2)|0\rangle\langle 0| + 2(|\alpha|^2 + |\beta|^2)|1\rangle\langle 1| \right] =$$

$$= \frac{1}{2}\,\mathbb{1} \quad \text{(}\underline{\text{maximally mixed state}}\text{!)}$$

Bell non-locality

Motivation $\rightarrow$ EPR paradox

Einstein-Podolsky-Rosen 1935

SG$_{1A}$

SG...

$i = \{x, y, z\}$

$$\Updownarrow$$

$$\boxed{A} \sim\sim\sim\sim \psi_- \sim\sim\sim \boxed{B}$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

<u>Violation of local realism</u> { "everything predetermined" }

<u>RSP with observables</u> $\Rightarrow$ "elements of physical reality"

(a) Alice measures: $\hat{\sigma}_z^A$ — spin in "z" direction

$\{|0\rangle, |1\rangle\}$: $|0\rangle_A \to |1\rangle_B$, $|1\rangle_A \to |0\rangle_B$

Bob measures: $\hat{\sigma}_z^B$ Bob's spin must have been "$-\frac{1}{2}$" or "$\frac{1}{2}$" in "z" with <u>certainty</u>!

(b) Alice measures: $\hat{\sigma}_x^A$ — spin in "x" direction

$\{|+\rangle, |-\rangle\}$: $|+\rangle_A \to |-\rangle_B$, $|-\rangle_A \to |+\rangle_B$

Bob measures: $\hat{\sigma}_z^B$ Bob's spin is "$-\frac{1}{2}$" or "$\frac{1}{2}$" with <u>uncertainty</u>! probability $\frac{1}{2}$

$\hat{\sigma}_z^B \Rightarrow$ <u>cannot</u>! be element of physical reality on its own!

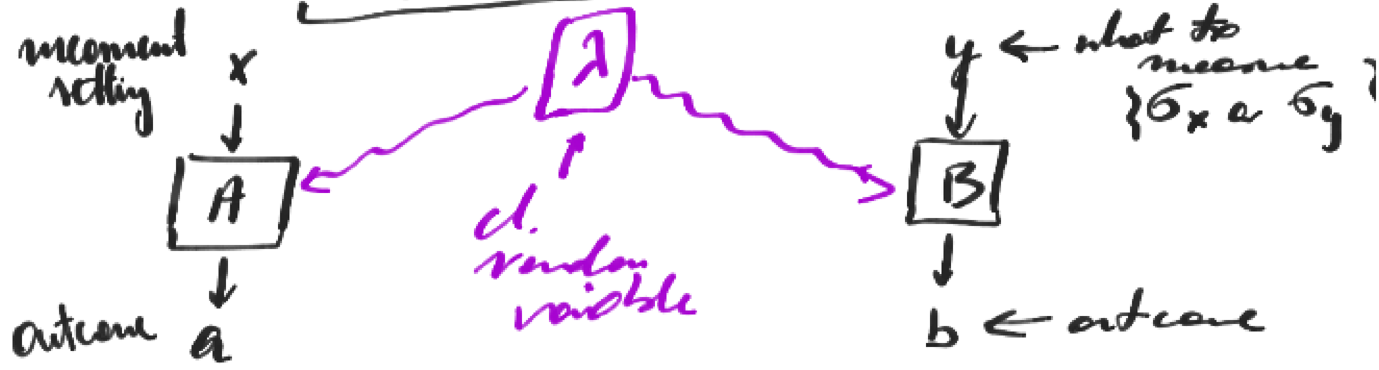
$\Rightarrow$ its <u>statistics</u> depend on Alice!

"spooky action at a distance" A. Einstein

A. Einstein

Solution?

Local hidden Variables (LHV)



joint probability distribution

$$p(a\, b / x\, y)$$

shared by Alice & Bob

|Local Hidden Variables| (LHVs)

$$p(ab|xy) = \sum_\lambda p(\lambda)\, p(a|x\lambda)\, p(b|y\lambda)$$

Correlations via the LHV

Example: EPR (above)

$x = \{0, 1\}$ $\qquad$ $y = \{1\}$

A measures: $\hat{\sigma}_x$ $\hat{\sigma}_z$ [2 settings] $\qquad$ B measures: $\hat{\sigma}_z$ [1 setting]

$\hat{\sigma}_x^A$ $\quad p(ab / x=0)$

b \ a	0	1
0	1/4	1/4
1	1/4	1/4

$\hat{\sigma}_z^A$ $\quad p(ab / x=1)$

b \ a	0	1
0	0	1/2
1	1/2	0

• LHV: Coin → "head"/"tails"

$\lambda = \{0, 1\}$ $\qquad p(\lambda=0) = p(\lambda=1) = 1/2$

$p(a|x,\lambda)$: $\lambda=0$

$x \backslash a$	0	1	
0	1/2	1/2	← random
1	1	0	← always "0"

$\lambda=1$

$x \backslash a$	0	1	
0	1/2	1/2	← random
1	0	1	← always "1"

$p(b|\lambda)$:

$\lambda \backslash b$	0	1
0	0	1
1	1	0

$$p(ab|x=0) = \sum_{\lambda=0}^{1} p(\lambda)\; p(a|x=0)\, p(b|\lambda)$$

$$p(ab|x=1) = \sum_{\lambda=0}^{1} p(\lambda)\; p(a|x=1)\, p(b|\lambda)$$

(Exercise 1) $\Rightarrow$ they reconstruct correlations!

But do LHVs resolve everything??

No!! $\Rightarrow$ Bell (non-local) Correlations

Bell inequalities

that apply to all LHV models but are violated! by quantum mechanics.

[3 correlators

original 1964 — 3 measurement settings, 2 outcomes

$\Downarrow$

CHSH 1969 [Clauser, Horne, Shimony, Holt]

D. Kaiser "How hippies saved physics?"

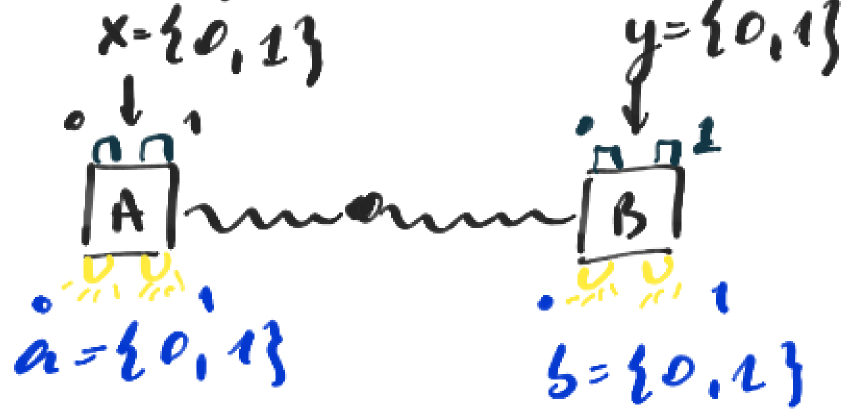
Fundamental Fysiks Group [Capra, Clauser, Eberhard, Herbert]

(history of no-cloning theorem:)

1981 $\Rightarrow$ Herbert (superluminal comm$^{\underline{n}}$)

$\Rightarrow$ Wootters, Zurek, Dieks
establish NCT

CHSH inequality



[2 measurement settings
2 outcomes

$\boxed{p(ab|xy)}$ ← joint prob. distribution shared by Alice and Bob

Define random variables (observables) for outcomes

Alice: $A = \begin{cases} -1, & a=0 \\ 1, & a=1 \end{cases}$ Bob: $B = \begin{cases} -1, & b=0 \\ 1, & b=1 \end{cases}$

$\boxed{\text{Bell (CHSH) operator}}$
4 correlations

$$\mathcal{B} := \underbrace{A_0 B_0} + \underbrace{A_0 B_1} + \underbrace{A_1 B_0} - \underbrace{A_1 B_1}$$

$$= A_0 (B_0 + B_1) + A_1 (B_0 - B_1)$$

any <u>correlation</u> ± 1 $|\mathcal{B}| \leq 2$!
for any instance

LHV models

$$p(AB|xy) = \int d\lambda \, p(\lambda) \, p(A|x\lambda) \, p(B|y\lambda)$$

$\Rightarrow$ we can define: $\bar{A}_x(\lambda) = \sum_{\pm 1} A \, p(A|x\lambda)$

Consider <u>average value (also over the LHV)</u>:

$$\langle \mathcal{B} \rangle = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

$$= \int d\lambda \, p(\lambda) \Big(\bar{A}_0(\lambda) \bar{B}_0(\lambda) + \bar{A}_0(\lambda) \bar{B}_1(\lambda) + \bar{A}_1(\lambda) \bar{B}_0(\lambda) - \bar{A}_1(\lambda) \bar{B}_1(\lambda) \Big)$$

$\forall \lambda$: previous argument applies

$$\Rightarrow \quad |\langle \mathcal{B} \rangle| \leq 2$$

for any LHV model!

Assumptions

- reality (EPR): measurements just reveal preexisting quantities $A_x(\lambda)$, $B_y(\lambda)$
- locality: measurements (settings) of Alice do not affect measurements (outcomes) of Bob

Does quantum mechanics fulfil these assumptions?

IMPORTANT!!!

In any case, quantum mechanics must NOT allow for superluminal signalling!

In short, we say that any physical theory must be NON-SIGNALLING:

$$\sum_a p(ab|xy) = p(b|xy) \equiv p(b|y)$$

a ← outcomes of Alice (and similarly for Bob)

Otherwise, Alice can send information

faster than light to 13.05 : $x \to b$
A m. setting → B outcome

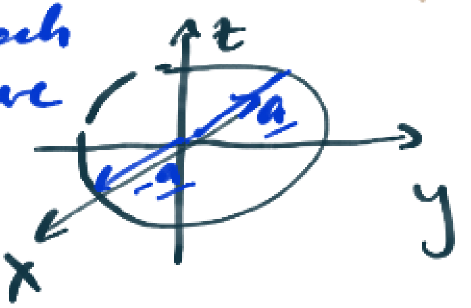
[N.B.:] To verify reality/locality assumptions A & B must come together (or call one-another) and determine the full shared correlations : $p(ab|xy)$

Quantum Scenario

· Alice and Bob share maximally entangled pair

$$A = \begin{cases} +1, & |\underline{a}\rangle \text{ or } |\underline{a}'\rangle \\ -1, & |-\underline{a}\rangle \text{ or } |-\underline{a}'\rangle \end{cases} \qquad B = \begin{cases} +1, & |\underline{b}\rangle \text{ or } |\underline{b}'\rangle \\ -1, & |-\underline{b}\rangle \text{ or } |-\underline{b}'\rangle \end{cases}$$

Bloch sphere



$$|\underline{a}\rangle\langle\underline{a}| = \frac{1}{2}(\mathbb{1} + \underline{a}\cdot\hat{\underline{\sigma}})$$

with $|\underline{a}|=1$

$$\langle\underline{a}|-\underline{a}\rangle = 0$$

orthogonal !!!

POVMs

Alice:

$x=0$: $\{\hat{M}^0_A\}_A = \{ \underbrace{|\underline{a}\rangle\langle\underline{a}|}_{A=1}, \underbrace{|-\underline{a}\rangle\langle-\underline{a}|}_{A=-1} \}$

$x=1$: $\{\hat{M}^1_A\}_A = \{ \underbrace{|\underline{a}'\rangle\langle\underline{a}'|}_{A=1}, \underbrace{|-\underline{a}'\rangle\langle-\underline{a}'|}_{A=-1} \}$

Bob:

$x=0$: $\{\hat{M}^0_B\}_B = \{ |\underline{b}\rangle\langle\underline{b}|, |-\underline{b}\rangle\langle-\underline{b}| \}$

$x=1$: $\{\hat{M}^1_B\}_B = \{ \underbrace{|\underline{b}'\rangle\langle\underline{b}'|}_{B=1}, \underbrace{|-\underline{b}'\rangle\langle-\underline{b}'|}_{B=-1} \}$

joint probability distr^n shared:

$$p(AB|xy) = \mathrm{Tr}\{ |\Psi^-\rangle\langle\Psi^-|_{AB}\, \hat{M}^x_A \otimes \hat{M}^y_B \}$$

Notation:

$$p(AB|xy) \equiv p(\underset{A}{\pm}\underline{a} \text{ or } \underset{A}{\pm}\underline{a}', \underset{B}{\pm}\underline{b} \text{ or } \underset{B}{\pm}\underline{b}')$$

x y

e.g. $p(A=1, B=1 \mid x=0, y=0) =$

$= p(\underline{a}, \underline{b}) \Leftarrow$ Alice measured $|\underline{a}\rangle$
Bob measured $|\underline{b}\rangle$

$\{ p(-\underline{a}, \underline{b}') = p(A=-1, B=1 \mid x=0, y=1)$

} etc.

$$p(\underline{a},\underline{b}) = \left| {}_A\langle \underline{a}| \otimes {}_B\langle \underline{b}| \, |\Psi^-\rangle_{AB} \right|^2$$

<u>Exercise</u> $= \frac{1}{4}\left(1 - \underline{a}\cdot\underline{b}\right)$

p(Alice measures $\underline{a}$):

$$p(\underline{a}) = p(\underline{a}|\underline{b}) + p(\underline{a}|-\underline{b}) = \frac{1}{2}$$

p(Alice measures $\underline{a}$ given Bob measured $\underline{b}$)

$$p(\underline{a}|\underline{b}) \underset{\text{Bayes}}{=} \frac{p(\underline{a},\underline{b})}{p(\underline{b})} = \frac{1}{2}\left(1 - \underline{a}\cdot\underline{b}\right)$$

<u>Correlators (of observables)</u>

$$\langle A_x B_y \rangle = \sum_{AB} AB \; p(AB|xy)$$

$$\langle A_0 B_0 \rangle = \sum_{AB} AB \; p(AB|x=0,y=0)$$

$$= p(\underline{a},\underline{b}) - p(-\underline{a},\underline{b}) - p(\underline{a},-\underline{b}) + p(-\underline{a},-\underline{b})$$

$$= \frac{1}{4}(\cancel{1} - \underline{a}\cdot\underline{b}) - \frac{1}{4}(\cancel{1} + \underline{a}\cdot\underline{b}) - \frac{1}{4}(\cancel{1} + \underline{a}\cdot\underline{b})$$

$$+ \frac{1}{4}(\cancel{1} - \underline{a}\cdot\underline{b}) = -\underline{a}\cdot\underline{b}$$

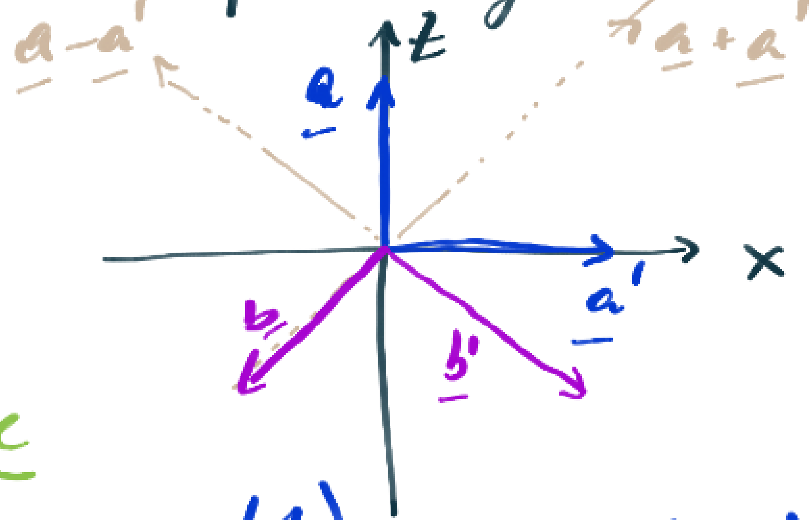
Bell (CHSH) Correlations

$$\langle \mathcal{B} \rangle = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

$$= -\underline{a}\cdot\underline{b} - \underline{a}\cdot\underline{b}' - \underline{a}'\cdot\underline{b} + \underline{a}'\cdot\underline{b}'$$

$$= -\underline{b}\,(\underline{a}+\underline{a}') - \underline{b}'\cdot(\underline{a}-\underline{a}')$$

take: XZ plane of Bloch sphere



Exercise

$$\underline{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad \underline{a}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \underline{b} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix},\quad \underline{b}' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\langle \mathcal{B} \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

Quantum Mechanics

is not a local realistic theory

$\Rightarrow$ need to drop either or .

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