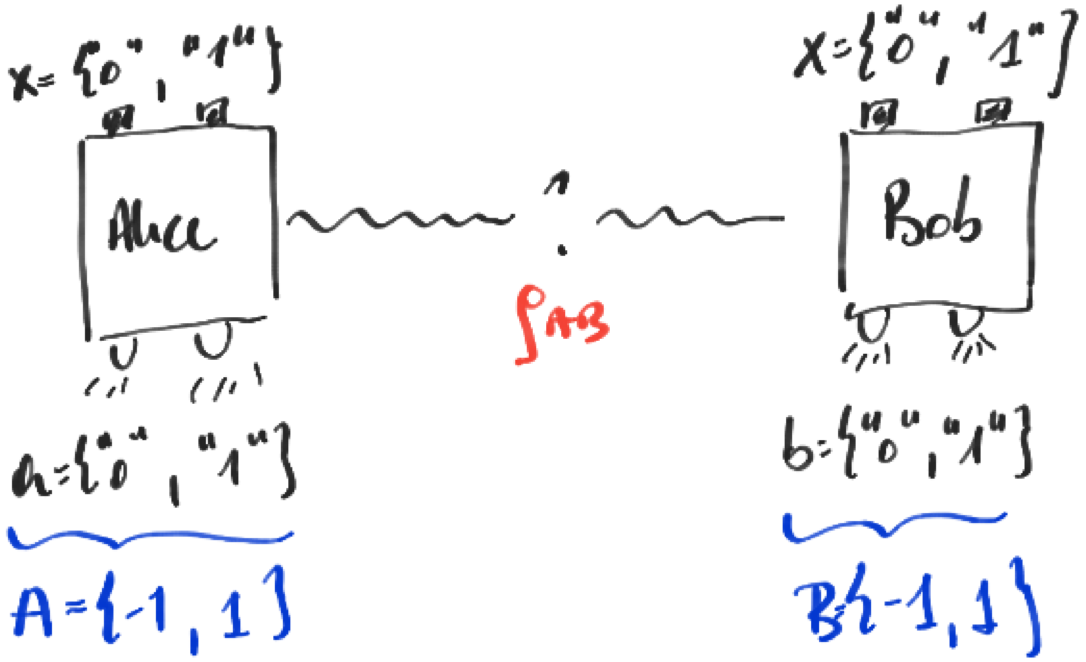
Bell non-locality (reminder)

Scenario with:

[2 measurement settings
2 outcomes]



According to quantum mechanics:

$$p(AB|xy) = \mathrm{Tr}\{\rho_{AB}\, \hat{M}_A^{(x)} \otimes \hat{M}_B\}$$

Expectation values of random var's & correlators

$$\langle A_x \rangle = \sum_{A=\pm 1} A\, p(A|x) \qquad \langle B_y \rangle = \sum_{B=\pm 1} B\, p(B|y)$$

$$\langle A_x B_y \rangle = \sum_{A,B=\pm 1} AB\, p(AB|xy)$$

CHSH inequality

Bell (CHSH) correlator:

$$\langle \beta \rangle := \langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle$$

LHVs $\left[p(ab|xy) = \sum_\lambda p(\lambda)\, p(a|x\lambda)\, p(b|y\lambda) \right]$

$$|\langle \beta \rangle| \leq 2$$

QMs $|\langle \beta \rangle| = 2\sqrt{2}$

$\Rightarrow$ QM is NOT! local realistic theory

NB: QM does NOT! allow for superluminal signalling

NON-SIGNALLING conditions

$$\sum_a p(ab|xy) = p(b|y)$$

{and similarly for $\sum_b$} "x"-independent

Proof:

$$\sum_a p(ab|xy) = \sum_a \mathrm{Tr}\{\rho_{AB}\, \hat{M}_a^{(x)} \otimes \hat{M}_b^{(y)}\} = \ldots$$

$\left\{ \text{for any POVM: } \sum_a M_a^{(x)} = \mathbb{1} \right\} \Rightarrow \ldots = \mathrm{Tr}\{\rho_{AB}\, \mathbb{1} \otimes \hat{M}_b^{(y)}\}$

$$=: p(b|y)$$

QED.

always ensured for QM!

QM:

$$\langle A_x B_y \rangle = \sum_{A,B=\pm 1} AB \, \mathrm{Tr}\{\rho_{AB} \, \hat{M}_A^{(x)} \otimes \hat{M}_B^{(y)}\}$$

$$= \mathrm{Tr}\Big\{\rho_{AB} \underbrace{\Big(\sum_{A=\pm 1} A \, \hat{M}_A^{(x)}\Big)}_{\hat{\sigma}_{\underline{a}}^{(x)}} \otimes \underbrace{\Big(\sum_{B=\pm 1} B \, \hat{M}_B^{(y)}\Big)}_{\hat{\sigma}_{\underline{b}}^{(x)}}\Big\}$$

$x=0$:

$$\hat{\sigma}_{\underline{a}} = (+1)\,|\underline{a}\rangle\langle\underline{a}| + (-1)\,|-\underline{a}\rangle\langle-\underline{a}|$$

$$= \frac{1}{2}\left(\mathbb{1} + \underline{a}\cdot\hat{\underline{\sigma}}\right) - \frac{1}{2}\left(\mathbb{1} - \underline{a}\cdot\hat{\underline{\sigma}}\right) =$$

$$= \underline{a}\cdot\hat{\underline{\sigma}}$$

$x=1$: $\hat{\sigma}_{\underline{a}'} = \underline{a}'\cdot\hat{\underline{\sigma}}$

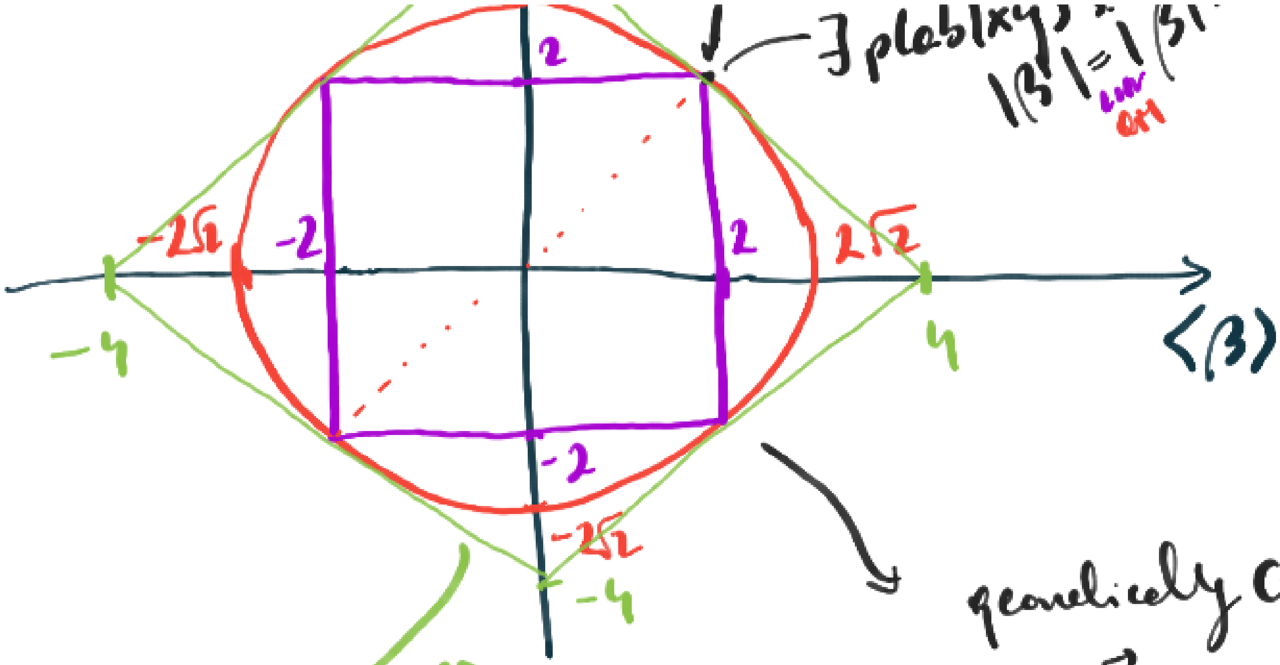
$$\Rightarrow \langle\beta\rangle = \mathrm{Tr}\{\rho_{AB} \, \overbrace{\hat{\beta}(\underline{a},\underline{a}',\underline{b},\underline{b}')}^{\text{CHSH operator!}}\} \quad (**)$$

$$\hat{\beta} := \hat{\sigma}_{\underline{a}} \otimes \hat{\sigma}_{\underline{b}} + \hat{\sigma}_{\underline{a}'} \otimes \hat{\sigma}_{\underline{b}} + \hat{\sigma}_{\underline{a}} \otimes \hat{\sigma}_{\underline{b}'} - \hat{\sigma}_{\underline{a}'} \otimes \hat{\sigma}_{\underline{b}'}$$

2D picture of Bell-violation

$$\langle\beta'\rangle = \langle A_1 B_1\rangle + \langle A_0 B_1\rangle + \langle A_1 B_0\rangle - \langle A_0 B_0\rangle$$

local



PR-boxes (non-signaling)

geometrically CHSH

$$|\mathbf{B}\cdot\vec{\beta}| \underset{LHV}{\le} 2$$

$$\left|\sum_{\substack{ab\\xy}} p(ab|xy)\,\beta(ab|xy)\right|$$

Non-signaling

$|\langle\beta\rangle| \underset{NS}{\le} 4$ [Popescu-Rohrlich 1994] "PR box"

∃ theories that have stronger correlations than quantum mechanics $2\sqrt{2} < |\beta| \le 4$

but are still! non-signalling!

Bell non-locality of Werner states

· CHSH value is linear (convex):

(**) $\langle \hat\beta \rangle_\rho = \mathrm{Tr}\{\rho_{AB}\, \hat\beta(\underline{a}, \underline{a}', \underline{b}, \underline{b}')\}$

Proof:

consider $\rho' = \lambda\rho + (1-\lambda)\sigma$

$$\langle \hat\beta \rangle_{\rho'} = \mathrm{Tr}\{(\lambda\rho + (1-\lambda)\sigma)\,\hat\beta\} =$$

$$= \lambda\, \mathrm{Tr}\{\rho\hat\beta\} + (1-\lambda)\,\mathrm{Tr}\{\sigma\hat\beta\}$$

$$= \lambda \langle \hat\beta \rangle_\rho + (1-\lambda)\langle \hat\beta \rangle_\sigma$$

• <u>2 qubit Werner state</u>:

$$\rho_p = p\,|\Psi_-\rangle\langle\Psi_-| + (1-p)\frac{\mathbb{1}}{4}$$

$$\langle \hat\beta \rangle_{\rho_p} = p\,\langle \hat\beta \rangle_{\Psi_-} + (1-p)\langle \hat\beta \rangle_{\frac{\mathbb{1}}{4}}$$

• $\langle \hat\beta \rangle_{\frac{\mathbb{1}}{4}} = \frac{1}{4}\mathrm{Tr}\{\cancelto{0}{\hat\sigma_{\underline{a}}\otimes\hat\sigma_{\underline{b}}} + \cancelto{0}{\hat\sigma_{\underline{a}'}\otimes\hat\sigma_{\underline{b}}} + \cancelto{0}{\hat\sigma_{\underline{a}}\otimes\hat\sigma_{\underline{b}'}} - \cancelto{0}{\hat\sigma_{\underline{a}'}\otimes\hat\sigma_{\underline{b}'}}\}$

$= 0 \Rightarrow$ no matter what choice of $\underline{a}, \underline{b}, \underline{a}', \underline{b}'$!

$\Rightarrow$ choose optimally for $|\Psi_-\rangle$:

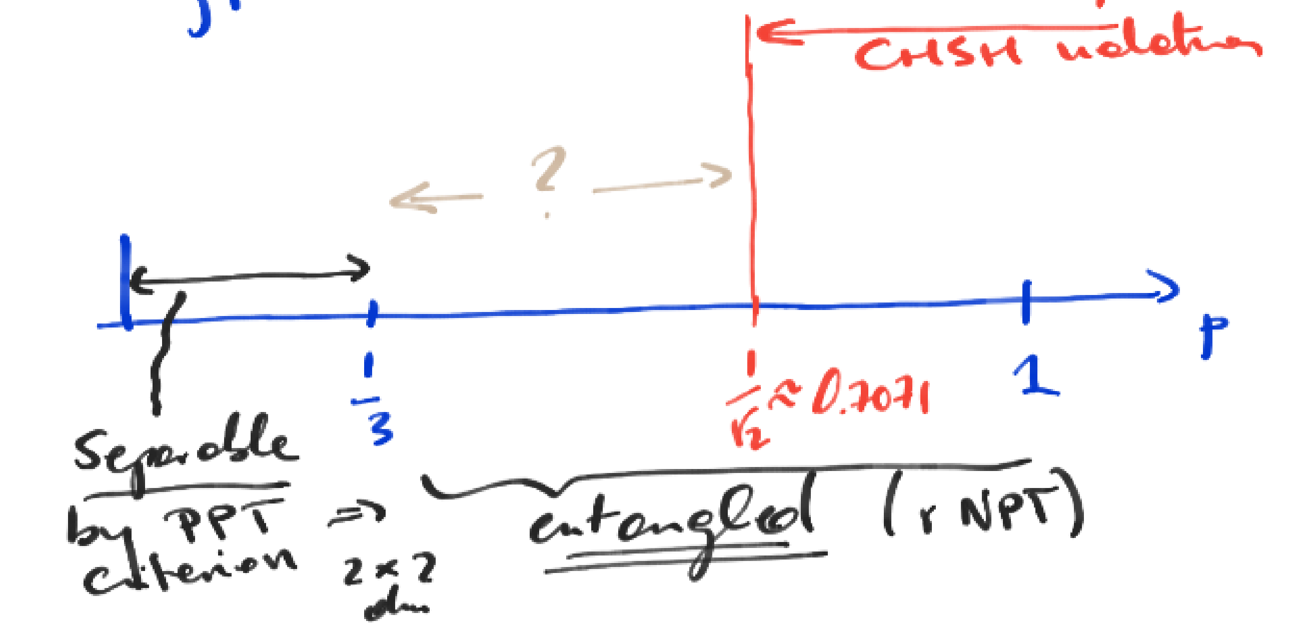
$$\langle \hat\beta \rangle_{\rho_p} = p\,\langle \hat\beta \rangle_{\Psi_-} = p\,2\sqrt{2}$$

Bell violation of CHSH fn:

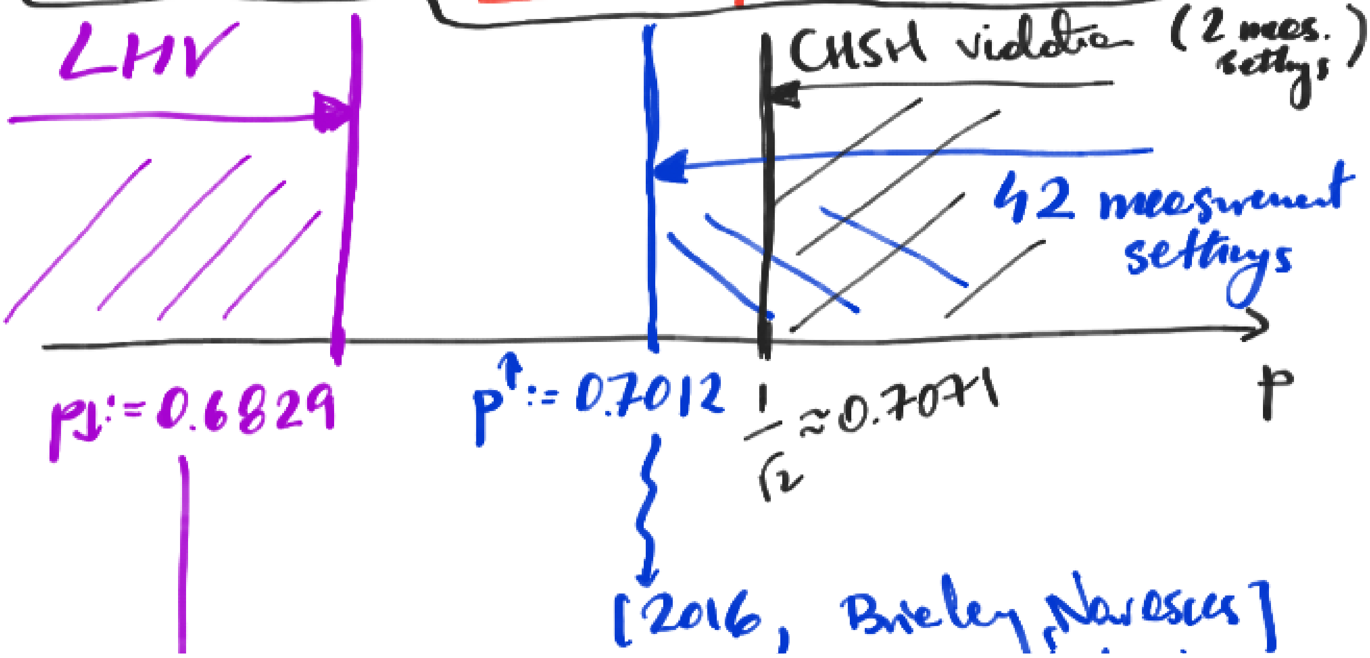
$$p\,2\sqrt{2} > \underset{\text{LHV}}{2} \Rightarrow p > \frac{1}{\sqrt{2}}$$

CHSH violation

$$\Rightarrow \quad \rho_p = p|\Psi_-\rangle\langle\Psi_-| + \frac{1}{4}\mathbb{1}$$



What about LHV models and other Bell inequalities?



& Vertesi

[2017, Hirsch et al, with projective measurements]

⇒ Problem of determining Grothendieck constant of order 3

$$1.4261 \approx \frac{1}{p^{\uparrow}} \leq K_G(3) \leq \frac{1}{p_{\downarrow}} \approx 1.4644$$

[see Alexander Grothendieck Alg. Geometry
"greatest Mathematician of XXth century"
Fields medal 1966]

Non-locality without inequalities

Greenberger, Horne and Zeilinger 1989

GHZ Paradox with GHZ states

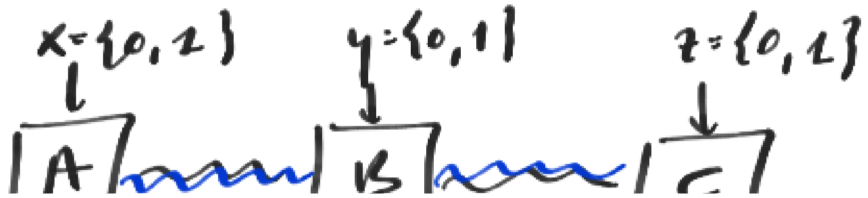
GHZ state of 3 qubits:

$$|\psi^3_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

easily generalised to N qubits:

$$|\psi^N_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$$

3partite scheme



share ∃ GHZ

state

$A_x = \pm 1 \qquad B_y = \pm 1 \qquad C_z = \pm 1$

each measure : $\{ \hat{\sigma}_x \text{ or } \hat{\sigma}_y \}$

$x, y, z = \{0, 1\}$

single outcome

(*) $A_0 B_0 C_0 = \begin{cases} +1, +1, +1 & \text{with } p = \mathrm{Tr}\{\psi_{GHZ} P_{+} \otimes P_{+} \otimes P_{+}\} \\ -1, -1, 1 & \vdots \\ -1, 1, -1 & \\ 1, -1, -1 & \end{cases}$

$= +1$ (not average!!!)

but $\langle A_0 B_0 C_0 \rangle = \langle \hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x \rangle_{GHZ} = 1$ ✓

implies all $A_0 B_0 C_0 = 1$

similarly:

$\langle A_0 B_1 C_1 \rangle = \langle \hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y \rangle = -1 \Rightarrow$ always $A_0 B_1 C_1 = -1$

$\langle A_1 B_1 C_0 \rangle = -1 \Rightarrow$ always $A_1 B_1 C_0 = -1$

$\langle A_1 B_0 C_1 \rangle = -1 \Rightarrow$ always $A_1 B_0 C_1 = -1$

LHVs:

$$p(ABC/xyz) = \sum_\lambda p(\lambda)\, p(A/x\lambda)\, p(B/y\lambda)\, p(C/z\lambda)$$

given $\lambda \Rightarrow$ definite value of $A_{0/1}$, $B_{0/1}$ and $C_{0/1}$

$\Rightarrow$ hence, I can compute a product:

(*) $(A_0 B_0 C_0)(A_0 B_1 C_1)(A_1 B_0 C_1)(A_1 B_1 C_0)$

$= A_0^2 B_0^2 C_0^2 A_1^2 B_1^2 C_1^2 = 1$

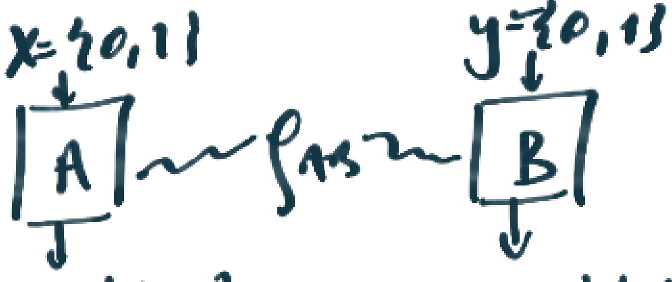
(*) R.H.S.: $(1)(-1)(-1)(-1) = -1$

contradiction!

$\Rightarrow$ LHV model cannot exist!

II Hardy's paradox

2 partite scheme (as in Bell, CHSH)



$A_x = \{\pm 1\}$ $\quad$ $B_y = \{\pm 1\}$

$$p(AB|xy)$$

Lucien Hardy 1993

(i) $p(+1\ +1\ |\ 0\ 0) = 0$

(ii) $p(+1\ -1\ |\ 1\ 0) = 0$

(iii) $p(\underset{A}{-1}\ \underset{B}{+1}\ |\ \underset{x}{0}\ \underset{y}{1}) = 0$

LHV: $\quad p(AB|xy) = \sum_\lambda p(\lambda)\, p(A|x\lambda)\, p(B|x\lambda)$

(iv) $p(\underset{A}{+1}\ \underset{B}{+1}\ |\ \underset{x}{1}\ \underset{y}{1}) =$

$$= \sum_\lambda p(\lambda)\, p(\underset{A}{+1}|\underset{x}{1},\lambda)\, p(\underset{B}{+1}|\underset{y}{1},\lambda)$$

(ii) $0 = \sum_\lambda p(\lambda)\, p_A(\underset{A}{+1}|\underset{x}{1},\lambda)\, p_B(\underset{B}{-1}|\underset{y}{0},\lambda)$

(iii) $0 = \sum_\lambda p(\lambda)\, p_A(\underset{A}{-1}|\underset{x}{0},\lambda)\, p_B(\underset{B}{1}|\underset{y}{1},\lambda)$

(i) $0 = \sum_\lambda p(\lambda)\, p_A(1|0,\lambda)\, p_B(1|0,\lambda)$

(A) $\Rightarrow$ $\boxed{p(+1+1|11)=0}$ LHV

because for each λ: either $p(A=1|x=1,\lambda)=0$

or $p(B=1|y=1,\lambda)=0$

$\left\{ \text{Remember that e.g.: } p_{A/B}(-1|0,\lambda)+p_{A/B}(1|0,\lambda)=1 \right\}$

Quantum mechanics

take

$$\underline{|\psi_{AB}\rangle = \alpha(|01\rangle + |10\rangle) + \beta|00\rangle}$$

NOT!! maximally entangled

(B) $\Rightarrow$ Schmidt rank ?

normalisation $\Rightarrow 2|\alpha|^2 + |\beta|^2 = 1$

A & B perform same! measurements

$x=\{0,1\}$ $y=\{0,1\}$

$$\hat{M}^{(0)}_{A/B} = \{\underset{-1}{|0\rangle\langle 0|}, \underset{+1}{|1\rangle\langle 1|}\}$$

$$\hat{M}^{(1)}_{A/B} = \{\underset{-1}{|\theta\rangle\langle\theta|}, \underset{+1}{|\theta_\perp\rangle\langle\theta_\perp|}\}$$

$|\theta\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

$|\theta_\perp\rangle = \sin\theta|0\rangle - \cos\theta|1\rangle$

(i) $p(+1,+1|00) = |\langle 1|\otimes\langle 1|\,|\psi_{AB}\rangle|^2 = 0$

(ii) $p(+1,-1\,|\,1\,0) = |\langle\theta_\perp| \otimes \langle 0|\; |\psi_{AB}\rangle|^2 =$

$= |(\sin\theta \langle 00| - \cos\theta \langle 10|)|\psi_{AB}\rangle|^2 =$

$= |\sin\theta\, \beta - \cos\theta\, \alpha|^2$

(iii) $p(-1,+1\,|\,0\,1) = |\langle 0| \otimes \langle\theta_\perp|\; \psi_{AB}\rangle|^2 =$

$= |(\sin\theta \langle 00| - \cos\theta \langle 01|)|\psi_{AB}\rangle|^2 =$

$= |\sin\theta\, \beta - \cos\theta\, \alpha|^2$

choose $\alpha = \tan\theta\, \beta \Rightarrow$ all (i)(ii) & (iii) vanish !!!

(iv) $p(+1+1\,|\,1\,1) = |\langle\theta_\perp| \otimes \langle\theta_\perp|\; |\psi_{AB}\rangle|^2 =$

$= |(\sin\theta\langle 0| - \cos\theta\langle 1|)^{\otimes 2}\; |\psi_{AB}\rangle|^2 =$

$= |(\sin^2\theta\langle 00| + \cos^2\theta\langle 11| - \sin\theta\cos\theta(\langle 01| + \langle 10|))\,|\psi_{AB}\rangle|^2$

$= |\sin^2\theta\,\beta - 2\sin\theta\cos\theta\,\alpha|^2$

$= |\sin^2\theta\,\beta - 2\sin\theta\cos\theta\tan\theta\,\beta|^2 =$

$= |\sin^2\theta\,\beta - 2\sin^2\theta\,\beta|^2 = \sin^4\theta\,|\beta|^2 > 0$

Normalisation:

$2|\alpha|^2 + |\beta|^2 = 1$ $\qquad \alpha_{opt} = \beta\tan\theta$

$(2\tan^2\theta + 1)|\beta|^2 = 1$

$\Rightarrow |\beta|^2 = \frac{1}{\ldots} \qquad 0 < |\beta| < 1$

$$\Rightarrow |\beta| = \frac{\quad}{2\tan^2\theta+1} \qquad 0 \le |\beta| \le 1$$

$$\uparrow_{\theta=\frac{\pi}{2}} \qquad \uparrow_{\theta=0}$$

$$\Rightarrow |\alpha|^2 = \frac{1-|\beta|^2}{2} = \frac{1}{2+\cot^2\theta} \qquad 0 \le |\alpha| \le \frac{1}{2}$$

$$\uparrow_{\theta=0} \qquad \uparrow_{\theta=\frac{\pi}{2}}$$

Hardy's

$$P_{(iv)} = \frac{\sin^4\theta}{2\tan^2\theta+1}$$

$$\max_{\theta}\{P_{(iv)}\} = \frac{5\sqrt{5}-11}{2} \quad \Big\}\ \text{at } \theta = -2\arctan\sqrt{\sqrt{5}+2}$$

$\approx$ 9% [Hardy 1993]

Exercise 1

CHSH Violation with partially entangled states

1)

$$|\Psi_\lambda\rangle_{AB} = \sqrt{\lambda}\,|01\rangle - \sqrt{1-\lambda}\,|10\rangle$$

state shared by Alice & Bob

2)

$$\langle\hat{B}\rangle_{\Psi_\lambda} = \langle\, \hat{\sigma}_{\underline{a}}\otimes\hat{\sigma}_{\underline{b}} + \hat{\sigma}_{\underline{a}'}\otimes\hat{\sigma}_{\underline{b}} + \hat{\sigma}_{\underline{a}}\otimes\hat{\sigma}_{\underline{b}'} - \hat{\sigma}_{\underline{a}'}\otimes\hat{\sigma}_{\underline{b}'}\rangle_{\Psi_\lambda}$$

$$\langle \vec{\sigma}_{\underline{a}} \otimes \vec{\sigma}_{\underline{b}} \rangle_{\psi_\lambda} = \langle \psi_\lambda |\; \underline{a} \cdot \underline{\hat{\sigma}} \otimes \underline{b} \cdot \underline{\hat{\sigma}} \;| \psi_\lambda \rangle =$$

$$= \lambda \langle 01 | a_i \hat{\sigma}_i \otimes b_j \hat{\sigma}_j | 01 \rangle +$$

$$- \sqrt{\lambda}\sqrt{1-\lambda} \left(\langle 01 | a_i \hat{\sigma}_i \otimes b_j \hat{\sigma}_j | 10 \rangle + c.c. \right) +$$

$$(1-\lambda) \langle 10 | a_i \hat{\sigma}_i \otimes b_j \hat{\sigma}_j | 10 \rangle$$

$$= -\lambda a_z b_z - 2\sqrt{\lambda}\sqrt{1-\lambda} \left(a_x b_x + a_y b_y \right) +$$

$$-(1-\lambda) a_z b_z =$$

$$= -a_z b_z - 2\sqrt{\lambda}\sqrt{1-\lambda} \left(a_x b_x + a_y b_y \right) =$$

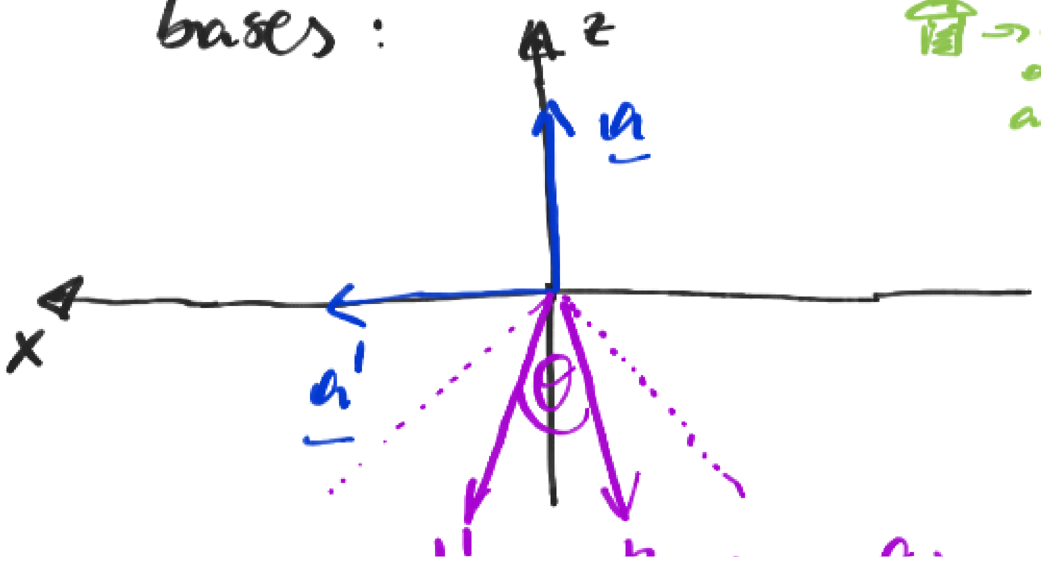
$$= -a_z b_z - 2\sqrt{\lambda}\sqrt{1-\lambda} \left(\underline{a}_\perp \cdot \underline{b}_\perp \right)$$

$$\langle \mathcal{B} \rangle_{\psi_\lambda} = -a_z b_z - a'_z b_z - a_z b'_z + a'_z b'_z +$$

$$-2\sqrt{\lambda}\sqrt{1-\lambda} \left(\underline{a}_\perp \cdot \underline{b}_\perp + \underline{a}'_\perp \cdot \underline{b}_\perp + \underline{a}_\perp \cdot \underline{b}'_\perp - \underline{a}'_\perp \cdot \underline{b}'_\perp \right)$$

3) choose Alice's & Bob's measurement bases:

→ optimise over all angles



$$\underline{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{a}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ 0 \\ -\cos\frac{\theta}{2} \end{pmatrix}, \quad \underline{b}' = \begin{pmatrix} \sin\frac{\theta}{2} \\ 0 \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

$$\langle \hat{\beta} \rangle_{\psi_\lambda} = \cos\frac{\theta}{2} + 0 + \cos\frac{\theta}{2} + 0 +$$

$$-2\sqrt{\lambda}\sqrt{1-\lambda}\left(0 - \sin\frac{\theta}{2} + 0 - \sin\frac{\theta}{2}\right) =$$

$$= 2\left(\cos\frac{\theta}{2} + 2\sqrt{\lambda}\sqrt{1-\lambda}\,\sin\frac{\theta}{2}\right)$$

maximise over θ:

$$c + 2\sqrt{\lambda}\sqrt{1-\lambda}\sqrt{1-c^2} \Rightarrow c_{max} = \sqrt{\frac{1}{1+4\lambda(1-\lambda)}}$$

$$\langle \hat{\beta} \rangle_{\psi_\lambda} = 2\left(\frac{1}{\sqrt{1+4\lambda(1-\lambda)}} + 2\sqrt{\lambda(1-\lambda)}\sqrt{\frac{4\lambda(1-\lambda)}{1+4\lambda(1-\lambda)}}\right)$$

$$= 2\,\frac{1+4\lambda(1-\lambda)}{\sqrt{1+4\lambda(1-\lambda)}} = 2\sqrt{1+4\lambda(1-\lambda)}$$

at $\theta = 2\arccos\sqrt{\frac{1}{1+4\lambda(1-\lambda)}}$

check:

$\lambda = \frac{1}{2}$ $\quad \theta = 2\arccos\sqrt{\frac{1}{1+1}} = 2\arccos\frac{\sqrt{2}}{2} = 2\,\frac{\pi}{4} = \frac{\pi}{2}$ ✓

$$\langle \hat{\beta} \rangle_{\lambda=\frac{1}{2}} = 2\sqrt{1+1} = 2\sqrt{2}$$ ✓

Exercise 2

Optimal (!?) cloning of qubits

$$|\psi\rangle_1 \otimes |0\rangle_2 \otimes |\xi\rangle_A \; \cancel{\xrightarrow{\mathcal{U}}} \; |\psi\rangle_1 |\psi\rangle_2 |\xi_\psi\rangle_A$$

NOT

$$|\psi\rangle_1 \otimes |0\rangle \otimes |\xi\rangle_A \xrightarrow{\mathcal{U}} |\Phi_\psi\rangle_{123}$$

$$\rho_1 = \mathrm{Tr}_{2A}\{|\Phi_\psi\rangle\langle\Phi_\psi|\}$$

$$\rho_2 = \mathrm{Tr}_{1A}\{|\Phi_\psi\rangle\langle\Phi_\psi|\}$$

Fidelity of cloning

← "overlap"

1st qubit: $F_1 = \langle\psi|\rho_1|\psi\rangle \Rightarrow = 1$ iff $\rho_1 = \psi$

2nd qubit: $F_2 = \langle\psi|\rho_2|\psi\rangle$

Consider transf^{n}:

$$|0\rangle_1|0\rangle_2|0\rangle_A \xrightarrow{\mathcal{U}} \sqrt{\tfrac{2}{3}}\,|000\rangle + \sqrt{\tfrac{1}{3}}\,|\psi_+\rangle|1\rangle$$

$$|1\rangle_1|0\rangle_2|0\rangle_A \xrightarrow{\mathcal{U}} \sqrt{\tfrac{1}{3}}\,|\psi_+\rangle|0\rangle + \sqrt{\tfrac{2}{3}}\,|111\rangle$$

$\Rightarrow$ for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$|\psi\rangle|0\rangle|0\rangle \xrightarrow{\mathcal{U}} \frac{1}{\sqrt{6}}\Big(2\alpha|000\rangle + \alpha|011\rangle + \alpha|101\rangle + \beta|010\rangle + \beta|100\rangle + 2\beta|111\rangle\Big)$$

$\beta|100\rangle + 2\beta|111\rangle\,)$

$$\rho_1 = \frac{1}{6}\begin{pmatrix} 4|\alpha|^2+1 & 4\alpha\beta^* \\ 4\alpha^*\beta & 4|\beta|^2+1 \end{pmatrix} = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{3}\frac{\mathbb{1}}{2}$$

$$\rho_2 = \rho_1$$

$$\Rightarrow \quad F = F_1 = F_2 = \frac{2}{3} + \frac{1}{6} = \frac{5}{6}$$

<u>N.B.</u> In fact, it is <u>optimal fidelity</u>.