5. Phase space description of quantum states (of light)

Let us focus on single mode states

A general state (in the Fock basis)

$$\varrho = \sum_{n,m} |n\rangle\langle n|\varrho|m\rangle\langle m| = \sum_{n,m} \underbrace{\langle n|\varrho|m\rangle}_{c_{n,m}} |n\rangle\langle m|$$

We can also write ϱ in coherent state „basis":

$$\varrho = \underbrace{\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|}_{\uparrow} \varrho \int \frac{d^2\beta}{\pi}|\beta\rangle\langle\beta| = \int \frac{d^2\alpha\, d^2\beta}{\pi^2} \langle\alpha|\varrho|\beta\rangle\, |\alpha\rangle\langle\beta|$$

$$R(\alpha^*, \beta) := \langle\alpha|\varrho|\beta\rangle e^{\frac{|\alpha|^2}{2}+\frac{|\beta|^2}{2}}$$ – R-representation

↑ not very illuminating, too many variables; we want a distribution in phase space $P(\underset{\uparrow}{\alpha})$ just one variable $\int dz\, d\omega$

single coordinate probability distribution

$\hat{q}$ – position operator $\quad \hat{q} = \int dq\, q\, |q\rangle\langle q|, \quad \langle q|q'\rangle = \delta(q-q')$

$$P(q) = \langle q|\varrho|q\rangle = \mathrm{Tr}(\varrho\, |q\rangle\langle q|)$$

Note that: $\quad |q\rangle\langle q| = \delta(q-\hat{q}) := \frac{1}{2\pi}\int dz\, e^{i(q-\hat{q})z}$

Check: $\quad \langle q'|\, |q\rangle\langle q|\, |q''\rangle \overset{?}{=} \langle q'|\delta(q-\hat{q})|q''\rangle$

$$\delta(q'-q)\,\delta(q-q'') \qquad\qquad \langle q'|\delta(q-q'')|q''\rangle$$

$$= \delta(q'-q'')\,\delta(q-q'') = \delta(q'-q)\,\delta(q-q'')$$

$$P(q) = \mathrm{Tr}[\varrho\, \delta(q-\hat{q})]$$

We want to define prob. distribution in phase space $P(q,p) = ?$

Let us try :

$$\mathcal{P}'(q,p) \overset{?}{:=} \mathrm{Tr}\left[\rho\, \delta(q-\hat{q})\, \delta(p-\hat{p})\right]$$

if $\hat{q}$ and $\hat{p}$ commuted this would be the only reasonable choice.

$$\mathcal{P}'(q,p) \overset{\text{up to ordering of operators}}{:=} \mathrm{Tr}\left[\rho\, \delta(q-\hat{q})\, \delta(p-\hat{p})\right]$$

We move to complex variable $\alpha = \dfrac{q+ip}{\sqrt{2}}$, $\mathcal{P}(\alpha)$

$$\int d^2\alpha\, \mathcal{P}(\alpha) = \underbrace{d\,\mathrm{Re}\,\alpha\, d\,\mathrm{Im}\,\alpha}_{\frac{dq\,dp}{2}}\, \mathcal{P}(\alpha) = \underbrace{\tfrac{1}{2}\mathcal{P}(\alpha)}_{\mathcal{P}'(q,p)}\, dq\, dp$$

$$\mathcal{P}(\alpha) = 2\mathcal{P}'(q,p), \quad q = \frac{\alpha+\alpha^*}{\sqrt{2}}, \quad p = \frac{\alpha-\alpha^*}{i\sqrt{2}}, \quad \hat{q} = \frac{\hat{a}+\hat{a}^+}{\sqrt{2}}, \quad \hat{p} = \frac{\hat{a}-\hat{a}^+}{i\sqrt{2}}$$

$$\mathcal{P}(\alpha) \overset{\text{ordering}}{=} 2\cdot\frac{1}{(2\pi)^2}\,\underline{\mathrm{Tr}\left[\rho \int dz\,dw\, e^{i\left(\frac{\alpha+\alpha^*-\hat{a}-\hat{a}^+}{\sqrt{2}}\right)z}\, e^{i\left(\frac{\alpha-\alpha^*-\hat{a}+\hat{a}^+}{i\sqrt{2}}\right)w}\right]} =$$

$$\beta = \frac{w-iz}{\sqrt{2}}$$

$$\overset{\text{ordering}}{=} \mathrm{Tr}\left[\rho \int \frac{d^2\beta}{\pi^2}\, e^{(\alpha-\hat{a})\beta^* - (\alpha^*-\hat{a}^+)\beta}\right]$$

(we put all operators in a single exponent)

$$\left\{\delta^{(2)}(\alpha-\hat{a})\right\}_\Omega := \left\{\int \frac{d^2\beta}{\pi^2}\, e^{(\alpha-\hat{a})\beta^* - (\alpha^*-\hat{a}^+)\beta}\right\}_{\underset{\Omega\text{ ordering}}{\Omega}} =$$

$$= \int \frac{d^2\beta}{\pi^2}\, e^{\alpha\beta^* - \alpha^*\beta} \left\{\underbrace{e^{-\beta^*\hat{a} + \beta\hat{a}^+}}_{D(\beta)}\right\}_\Omega$$

$$\left\{ w = \frac{\beta+\beta^*}{\sqrt{2}}, \quad z = \frac{\beta^*-\beta}{i\sqrt{2}} \right.$$

Choice of ordering:

a) normal order (all creation operators are on the left)

$$:D(\beta): = e^{\beta \hat{a}^{+}} e^{-\beta^{*} \hat{a}}$$

b) anti-normal ordering. (all anihilation operators on the left)

$$\vdots D(\beta) \vdots = e^{-\beta^{*} \hat{a}} e^{\beta \hat{a}^{+}}$$

c) Wigner-Weyl ordering (symmetric ordering)

$$D(\beta) = e^{\beta \hat{a}^{+} - \beta^{*} \hat{a}}$$

Note that thanks to BCH theorem:

$$\{D(\beta)\}_{s} := e^{\frac{s}{2}|\beta|^{2}} D(\beta) \overset{BCH}{=} \begin{cases} : \ : , & s=1 \\ \text{Wigner-Weyl}, & s=0 \\ \vdots \ \vdots , & s=-1 \end{cases}$$

$-1 \leq s \leq 1$
(ordering parameter)

$$\{\delta^{(2)}(\alpha-\hat{a})\}_{Q} = \left\{\iint \frac{d^{2}\beta}{\pi^{2}} e^{(\alpha-\hat{a})\beta^{*} - (\alpha^{*}-\hat{a}^{+})\beta}\right\}_{Q}$$

Glauber P-representation

$$P(\alpha) := \mathrm{Tr}(\rho : \delta^{(2)}(\alpha-\hat{a}) :)$$

Fact If $\rho = \int d^{2}\alpha \, \tilde{P}(\alpha) |\alpha\rangle\langle\alpha|$ then

$$\tilde{P}(\alpha) = P(\alpha) = \mathrm{Tr}(\rho : \delta^{(2)}(\alpha-\hat{a}) :)$$

Proof:

$$\mathrm{Tr}(\rho : \delta^{(2)}(\alpha-\hat{a}) :) = \mathrm{Tr}\left[\int d^{2}\alpha' \, \tilde{P}(\alpha') |\alpha'\rangle\langle\alpha'| \int \frac{d^{2}\beta}{\pi^{2}} e^{-(\alpha^{*}-\hat{a}^{+})\beta} \cdot e^{(\alpha-\hat{a})\beta^{*}}\right]$$

$$= \mathrm{Tr}\left[\int d^{2}\alpha' \, \tilde{P}(\alpha') |\alpha'\rangle\langle\alpha'| \, \delta^{(2)}(\alpha-\alpha')\right] = \tilde{P}(\alpha)$$

Fact (not very rigorous...)

Every ρ can be written as $\rho = \int d^2\alpha\, P(\alpha)\, |\alpha\rangle\langle\alpha|$

but $P(\alpha)$ may be negative or very singular ....

Proof:

Every state can be decomposed as $\rho = \sum_{n,m} \langle n|\rho|m\rangle \underbrace{|n\rangle\langle m|}$

Let us calculate: $\{ d^2\alpha \to r\,dr\,d\theta$ $\quad \alpha = re^{i\theta}$

$$\frac{\partial^{n+m}}{\partial r}\left[\int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{r^2 + i(m-n)\theta} \underbrace{|re^{i\theta}\rangle\langle re^{i\theta}|}_{=\, e^{-r^2}\sum_{n',m'} \frac{(re^{i\theta})^{n'}}{\sqrt{n'!}} \frac{(re^{-i\theta})^{m'}}{\sqrt{m'!}} |n'\rangle\langle m'|}\right]\Bigg|_{r=0} =$$

$$= \sum_{n',m'} \frac{\partial^{n+m}}{\partial r}\left[\underbrace{\int \frac{d\theta}{2\pi}\, e^{i(m-n)\theta - i(m'-n')\theta}}_{\delta_{m-n,\, m'-n'}} \cdot \frac{r^{n'+m'}}{\sqrt{n'!}\sqrt{m'!}} |n'\rangle\langle m'|\right]\Bigg|_{r=0}$$

the only non-zero contribution is when $n+m = n'+m'$

$n' = n$
$m' = m$

$$= \frac{(n+m)!}{\sqrt{n!}\sqrt{m!}} |n\rangle\langle m| \Rightarrow |n\rangle\langle m| = \int dr\, \delta(r) \cdot \frac{\sqrt{n!\,m!}}{(n+m)!} \frac{\partial^{n+m}}{\partial r} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{r^2 + i(m-n)\theta} \cdot |re^{i\theta}\rangle\langle re^{i\theta}|$$

We integrate by parts: (! Problematic !)

$$|n\rangle\langle m| = \frac{\sqrt{n!\,m!}}{(n+m)!} \int_0^\infty dr \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{r^2 + i(m-n)\theta}\, |re^{i\theta}\rangle\langle re^{i\theta}| \left(-\frac{\partial}{\partial r}\right)^{n+m} \delta(r)$$

So $|n\rangle\langle m| = \int d^2\alpha\, P_{n,m}(\alpha) |\alpha\rangle\langle\alpha|$, where $P_{n,m}(\alpha) = \frac{1}{2\pi r} \frac{\sqrt{n!\,m!}}{(n+m)!} e^{r^2 + i(m-n)\theta} \left(-\frac{\partial}{\partial r}\right)^{n+m} \delta(r)$

Fact

If you are given a normally-ordered observable

$$\hat{A}_N(\hat{a}) = \sum_{n,m} c_{n,m} (\hat{a}^\dagger)^n (\hat{a})^m$$

$$\mathrm{Tr}(\rho \hat{A}_N) = \int d^2\alpha\, P(\alpha)\, A_N(\alpha)$$

Proof

$$\mathrm{Tr}\left(\int d^2\alpha\, P(\alpha) |\alpha\rangle\langle\alpha| \cdot \sum_{n,m} c_{n,m} (\hat{a}^\dagger)^n (\hat{a})^m\right) =$$

$$= \mathrm{Tr}\left(\int d^2\alpha\, P(\alpha) \sum_{n,m} (\alpha^*)^m (\alpha)^n |\alpha\rangle\langle\alpha|\right) = \int d^2\alpha\, P(\alpha)\, A_N(\alpha)$$

Q-representation (Hussimi representation)

$$Q(\alpha) := \mathrm{Tr}\left(\rho \; :\delta^{(2)}(\alpha - \hat{\alpha}):\right) =$$

$$= \mathrm{Tr}\left(\rho \int \frac{d^2\beta}{\pi^2} e^{(\alpha - \underset{\Downarrow \gamma}{\hat{\alpha}})\beta^*} \, e^{-(\alpha^* - \underset{\downarrow \gamma^*}{\hat{\alpha}^\dagger})\beta}\right) =$$

$$\int \frac{d^2\gamma}{\pi} |\gamma\rangle\langle\gamma|$$

$$= \mathrm{Tr}\left(\rho \int \frac{d^2\gamma}{\pi} |\gamma\rangle\langle\gamma| \, \delta^{(2)}(\alpha - \gamma)\right) = \frac{1}{\pi} \mathrm{Tr}(\rho\, |\alpha\rangle\langle\alpha|) =$$

$$= \frac{1}{\pi} \langle\alpha| \rho |\alpha\rangle,$$

$$\int Q(\alpha)\, d^2\alpha = 1, \qquad Q(\alpha) \geq 0.$$

Wigner-Weyl distribution

$$W(\alpha) := \mathrm{Tr}\left(\rho \; \delta^{(2)}(\alpha - \hat{\alpha})\right) =$$

$$= \mathrm{Tr}\left(\rho \int \frac{d^2\beta}{\pi^2} e^{-(\alpha^* - \hat{\alpha}^\dagger)\beta + (\alpha - \hat{\alpha})\beta^*}\right)$$

$\int d^2\alpha \, W(\alpha) = 1$, $W(\alpha)$ - can be negative......

What is so special about Wigner distribution?
It gives proper marginal distributions

$$\int dp\, W(q,p) = \langle q|\rho|q\rangle$$

$$\int dq\, W(q,p) = \langle p|\rho|p\rangle$$

Relation between distributions: (convolution with a Gaussian)

$$W(\alpha) = \frac{2}{\pi}\int d^2\gamma\, e^{-2|\gamma-\alpha|^2} P(\gamma)$$

$$Q(\alpha) = \frac{2}{\pi}\int d^2\gamma\, e^{-2|\gamma-\alpha|^2} W(\gamma)$$

$$Q(\alpha) = \frac{1}{\pi}\int d^2\gamma\, e^{-|\gamma-\alpha|^2} P(\gamma)$$

Summary:

	$P(\alpha)$	$W(\alpha)$	$Q(\alpha)$
Definition	$\mathrm{Tr}(\rho :\delta^{(2)}(\alpha-\hat{a}):)$	$\mathrm{Tr}(\rho\, \delta^{(2)}(\alpha-\hat{a}))$	$\mathrm{Tr}(\rho \vdots \delta^{(2)}(\alpha-\hat{a}) \vdots)$
Positive	−	−	+
Gives correct marginals	−	+	−
Its positivity implies "classicality" of states	+	−	−